INDUCTIVE LIMITS OF NORMED ALGEBRAS(1)

BY
SETH WARNER

For a Hausdorff locally convex space E with topological dual E', the following properties are equivalent: (B 1) Every bound linear transformation from E into any locally convex space is continuous; (B 2) No strictly stronger locally convex topology on E has the same bound sets; (B 3) Every bornivore set is a neighborhood of zero (a *bornivore* set is a convex, equilibrated set absorbing every bound set); (B 4) E is the (linear) inductive limit [7, pp. 61–66] of normed spaces $\{E_\alpha\}$ with respect to linear maps $g_\alpha: E_\alpha \rightarrow E$ such that $\bigcup_\alpha g_\alpha(E_\alpha) = E$; (B 5) The topology of E is $\tau(E, E')$ (the strongest locally convex topology on E yielding E' as topological dual), and every bound linear form on E is continuous. (We use the terminology of Bourbaki [7; 8; 10]; for the proofs see Theorem 8 of [15, p. 527], Proposition 5 of [10, p. 10], and Theorem 3 of [11, p. 328].) Such spaces are called *bornological*; they are the "boundedly closed, relatively strong" spaces of [15] and the "*GF* spaces" of [11]. Our purpose here is to formulate the analogous notion of the algebraic inductive limit of locally m-convex algebras and to study those locally m-convex algebras which are algebraic inductive limits of normed algebras.

In §1 we summarize briefly the basic definitions and some of the elementary results in the theory of locally m-convex algebras. In §2, after defining the notion of the algebraic inductive limit of locally m-convex algebras in an obvious way, we consider the following problem. If E is an algebra, $\{E_\alpha\}$ a family of locally m-convex algebras, $g_\alpha: E_\alpha \rightarrow E$ a homomorphism for all α, there exist on E both the algebraic inductive limit topology with respect to locally m-convex algebras E_α and homomorphisms g_α, and the linear inductive limit topology with respect to locally convex spaces E_α and linear maps g_α; when do they coincide? This question is important, for the topologies of certain important locally convex spaces (occurring, for example, in the theory of distributions and the theory of integration) are defined as linear inductive limits of locally convex spaces which are also locally m-convex algebras, with respect to linear maps which are also homomorphisms of the algebras considered. To establish that such topologies are actually locally m-convex, and thus fall within the purview of topological algebra, it is desirable to have general criteria insuring that the two inductive limit topologies coincide.

Presented to the Society, April 15, 1955; received by the editors May 14, 1955 and in revised form, January 12, 1956.

(1) Part of this material is drawn from a dissertation written under the supervision of Professor G. W. Mackey.

In §3 we formulate the algebraic analogues of the linear concepts "bound set," "bornivore set," "$\tau(E, E')$," "bound linear transformation," and we prove the equivalence of properties analogous to (B 1)–(B 5). Algebras possessing these properties, the analogues of bornological spaces, are called i-bornological algebras. In §4 we discuss the extent to which the property of being i-bornological is preserved under certain operations of algebra, such as the formation of quotient algebras, finite Cartesian products, ideals, etc.

In §5 we determine when the algebra of all continuous real-valued functions on a completely regular space, furnished with the compact-open topology, is i-bornological, and also we consider infinite Cartesian products of i-bornological algebras. A measure-theoretic problem of Ulam arises in this context, and our discussion includes a brief digression on its role in mathematics. In §6 we first consider a new class of locally m-convex algebras, called P-algebras, and in terms of these we give necessary and sufficient conditions for certain metrizable locally m-convex algebras to be i-bornological. In particular we show that an $\mathfrak{F}$-algebra E is the algebraic inductive limit of Banach algebras $\{E_\alpha\}$ with respect to homomorphisms $\{g_\alpha\}$ such that $E = \bigcup_\alpha g_\alpha(E_\alpha)$ if and only if E satisfies an apparently much weaker condition. The paper concludes with a list of unresolved questions.

Throughout, the field of complex numbers is denoted by "$\boldsymbol{C}$," the field of reals by "$\boldsymbol{R}$," and the scalar field of an algebra by "$\boldsymbol{K}$"; $\boldsymbol{K}$ is either $\boldsymbol{R}$ or $\boldsymbol{C}$.

1. **Introduction.** Let E be an algebra over $\boldsymbol{K}$. A subset A of E is called *idempotent* if $A^2 \subseteq A$. A *locally multiplicatively-convex topology* (hereafter abbreviated to "locally m-convex topology") $\mathfrak{J}$ on E is a locally convex topology on vector space E such that zero has a fundamental system of idempotent neighborhoods [16, Definition 2.1]. Equivalently, $\mathfrak{J}$ is defined by a family of pseudo-norms $\{p_\alpha\}$ each of which satisfies the multiplicative inequality $p_\alpha(xy) \leqq p_\alpha(x)p_\alpha(y)$ for all $x, y \in E$. Multiplication is then clearly continuous everywhere. E with such a topology is called a *locally m-convex algebra*. If A is an idempotent subset of E, so are its convex envelope, its equilibrated envelope, its closure, and any homomorphic or inverse homomorphic image; hence there exists a fundamental system of convex, equilibrated, idempotent neighborhoods of zero. (For proofs of these and other elementary properties of locally m-convex algebras, see §§1–5 of [16] or [2].) Conversely, using Proposition 5 of [7, p. 7], one may easily show that if $\mathfrak{V}$ is a filter base of convex, equilibrated, absorbing, idempotent subsets of E such that $V \in \mathfrak{V}$ and $|\lambda| \leqq 1$ imply $\lambda V \in \mathfrak{V}$, then there is a unique locally m-convex topology on E for which $\mathfrak{V}$ is a fundamental system of neighborhoods of zero. The intersection of a family of convex, idempotent sets is again convex and idempotent; hence the topology generated by a family of locally m-convex topologies on E is again locally m-convex.

For $x, y \in E$ we write "$x \circ y$" for "$x+y-xy$." $\circ$ is then an associative composition on E with identity zero. If x has an inverse x' for this composition, x is called *advertible*, x' the *adverse* of x. If F is any subset of E, we let $F \circ x = [z \circ x \mid z \in F]$; if $\mathfrak{F}$ is any family of subsets of E, we let $\mathfrak{F} \circ x = [F \circ x \mid F \in \mathfrak{F}]$, etc. E is called a *Q-algebra* (cf. [16, Definition E.1]) if the set of advertible elements is a neighborhood of zero. E is called *advertibly complete* [25, §3] if, whenever $\mathfrak{F}$ is a Cauchy filter on E such that for some x we have $\mathfrak{F} \circ x \to 0$ and $x \circ \mathfrak{F} \to 0$, then $\mathfrak{F}$ converges.

A locally convex space is *barrelled* if every barrel is a neighborhood of zero (a *barrel* is a closed, convex, equilibrated, absorbing set). A locally m-convex algebra is *idempotently barrelled* (hereafter abbreviated to *i-barrelled*) if every idempotent barrel is a neighborhood of zero. $\mathfrak{F}$-spaces (complete, metrizable, locally convex spaces) are barrelled [10, p. 6]; in particular $\mathfrak{F}$-algebras are i-barrelled (an *$\mathfrak{F}$-algebra* is a complete, metrizable, locally m-convex algebra [16, Definition 4.1]).

We list a few examples of locally m-convex algebras (others will be considered in the sequel):

EXAMPLE 1. Let T be a set, E a $\boldsymbol{K}$-algebra of $\boldsymbol{K}$-valued functions on T, $\mathfrak{S}$ a family of subsets of T such that each $x \in E$ is bound on each $A \in \mathfrak{S}$. The topology on E of uniform convergence on members of $\mathfrak{S}$ is then locally m-convex, defined by the pseudo-norms $\{p_A \mid A \in \mathfrak{S}\}$ where $p_A(x) = \sup [|x(t)| \mid t \in A]$. If T is a topological space, E the algebra $\mathcal{C}(T, \boldsymbol{K})$ of all continuous functions on T, and if each $t \in T$ is interior to some $A \in \mathfrak{S}$, then E is complete (Proposition 2 of [6, p. 14]). Also if T is a domain in $\mathbf{C}$, E the algebra of all analytic functions on T, then E with the compact-open topology is complete by a standard theorem of complex analysis.

EXAMPLE 2. Let E be an algebra of bound, continuous $\boldsymbol{R}$-valued functions on $\boldsymbol{R}$ which are of bounded variation on every compact interval. Let $p(x) = \sup [|x(t)| \mid t \in \boldsymbol{R}]$ and let $v_n(x)$ be the variation of x in $[-n, n]$. Then $v_n(xy) \leqq p(x)v_n(y) + v_n(x)p(y)$. Hence, if $W_{m,n} = [x \in E \mid p(x) \leqq 2^{-m}, v_n(x) \leqq 2^{-m}]$, $\{W_{m,n}\}_{m,n=1}^{\infty}$ forms a fundamental system of idempotent neighborhoods of zero for the topology on E defined by these pseudo-norms.

EXAMPLE 3. Let P be either an open subset or the closure of an open subset of $\boldsymbol{R}^m$, $\mathcal{E}_P$ the algebra of infinitely differentiable $\boldsymbol{K}$-valued functions on $\boldsymbol{R}^m$ vanishing outside P. The topology of Schwartz [19, p. 88] on $\mathcal{E}_P$ is defined as follows: For any compact subset A of P and any m-tuple p of non-negative integers (we use the notation of [19, p. 14]), let $N_A^p(x) = \sup [|(D^p x)(t)| \mid t \in A]$. If r is a positive integer and $0 < \epsilon \leqq 1$, let $V(A, r, \epsilon) = [x \in \mathcal{E}_P \mid N_A^p(x) \leqq 2^{-r}\epsilon$ for all m-tuples p such that $|p| \leqq r]$. Leibniz's rule for the derivative of a product shows that each $V(A, r, \epsilon)$ is idempotent. Since the family of all such $V(A, r, \epsilon)$ is a fundamental system of neighborhoods of zero for the topology defined by the pseudo-norms N_A^p, that topology is locally m-convex.

EXAMPLE 4. Let E be an algebra, E' a total subspace of the algebraic dual of E. If E' is generated by the multiplicative linear forms in E', the weak topology $\sigma(E, E')$ defined on E by E' is surely locally m-convex. (For a necessary and sufficient condition that $\sigma(E, E')$ be locally m-convex, see Theorem 1 of [26].)

EXAMPLE 5. If E is an algebra, the collection of all convex, equilibrated, idempotent, absorbing sets is a fundamental system of neighborhoods of zero for a locally m-convex topology on E, and this topology is clearly the strongest locally m-convex topology on E. For future purposes we examine this topology for the case where E is the subalgebra of $\boldsymbol{K}[X_1, \cdots, X_n]$ consisting of all polynomials in n indeterminants without constant term. If $(\alpha_1, \cdots, \alpha_n)$ is any n-tuple of positive real numbers, let $V(\alpha_1, \cdots, \alpha_n)$ be the convex equilibrated envelope of $[\alpha_1^{m_1} \cdots \alpha_n^{m_n} X_1^{m_1} \cdots X_n^{m_n} | (m_1, \cdots, m_n)$ any n-tuple of non-negative integers not all of which are zero$]$. It is immediate that the family of all such $V(\alpha_1, \cdots, \alpha_n)$, with the α_i rational and <1, is a fundamental system of neighborhoods of zero for the strongest locally m-convex topology on E. This topology is thus metrizable.

The subalgebra E of $\boldsymbol{K}[X]$ of all polynomials without constant term furnished with the topology of Example 5 is an example of an i-barrelled algebra which is not a barrelled space. E is clearly i-barrelled since any convex, equilibrated, absorbing, idempotent set is a neighborhood of zero. Let B be the convex equilibrated envelope of $\{2^{-n^2}X^n\}_{n=1}^{\infty}$. B is then clearly absorbing, so $\overline{B}$ is a barrel. If $\beta > 1$, $\beta 2^{-n^2}X^n \notin \overline{B}$, for if $0 < \alpha < 2^{-n}(\beta-1)^{1/n}$, an easy calculation shows that $[\beta 2^{-n^2}X^n + V(\alpha)] \cap B = \varnothing$. For any $\alpha > 0$, $\lim_{n\to\infty} 2^{-n^2}\alpha^{-n} = 0$, so for large n, $\alpha^n > 2^{-n^2}$, i.e., $\alpha^n = \beta_{\alpha,n}2^{-n^2}$ where $\beta_{\alpha,n} > 1$. Hence if $V(\alpha) \subseteq \overline{B}$, for suitably large n we have $\alpha^n X^n = \beta_{\alpha,n}2^{-n^2}X^n \in V(\alpha) \subseteq \overline{B}$, a contradiction. Hence $\overline{B}$ is not a neighborhood of zero.

2. **Algebraic inductive limits.** Let E be an algebra over $\boldsymbol{K}$, $\{E_\alpha\}$ a family of locally m-convex algebras, g_α a homomorphism from E_α into E. There is at least one locally m-convex topology on E for which all the homomorphisms g_α are continuous, namely, the topology whose only open sets are E and $\varnothing$. Hence there is a strongest such topology.

DEFINITION 1. The *algebraic inductive limit topology on E with respect to locally m-convex algebras $\{E_\alpha\}$ and homomorphisms $\{g_\alpha\}$* is the strongest locally m-convex topology on E for which all the homomorphisms g_α are continuous. E with this topology is called the *algebraic inductive limit of algebras $\{E_\alpha\}$ with respect to homomorphisms $\{g_\alpha\}$*.

Usually the algebras E_α will be subalgebras of the algebra (without topology) E and g_α the inclusion map from E_α into E. In this case we shall omit explicit reference to the g_α's.

PROPOSITION 1. *Let $\{E_\alpha\}$ be a family of locally m-convex algebras, g_α a homomorphism from E_α into algebra E for all α. If V is an idempotent, convex, equilibrated, absorbing subset of E, then V is a neighborhood of zero for the algebraic inductive limit topology if and only if $g_\alpha^{-1}(V)$ is a neighborhood of zero in E_α for all α. If F is a locally m-convex algebra and if f is a homomorphism* *from E equipped with the algebraic inductive limit topology into F, then f is continuous if and only if $f \circ g_\alpha$ is continuous for all α.*

Proof. The class $\mathfrak{V}$ of all idempotent, convex, equilibrated, absorbing sets V such that $g_\alpha^{-1}(V)$ is a neighborhood of zero in E_α for all α is clearly a fundamental system of neighborhoods of zero for a locally m-convex topology $\mathfrak{J}$ on E. If $\mathfrak{W}$ is a fundamental system of idempotent, convex, equilibrated neighborhoods of zero for any other locally m-convex topology $\mathfrak{J}'$ on E for which all g_α are continuous, then each $W \in \mathfrak{W}$ is absorbing and $g_\alpha^{-1}(W)$ is a neighborhood of zero in E_α for all α; hence $\mathfrak{W} \subseteq \mathfrak{V}$ and $\mathfrak{J}' \subseteq \mathfrak{J}$, so therefore $\mathfrak{J}$ is the strongest such topology, i.e., $\mathfrak{J}$ is the algebraic inductive limit topology. If f is a continuous homomorphism from E with this topology into F, clearly all $f \circ g_\alpha$ are continuous since all g_α are continuous. Conversely, suppose f is a homomorphism such that $f \circ g_\alpha$ is continuous for all α. Let U be any convex, equilibrated, idempotent neighborhood of zero in F. Then $f^{-1}(U)$ is convex, equilibrated, idempotent, and absorbing. Finally, by hypothesis $g_\alpha^{-1}(f^{-1}(U)) = (f \circ g_\alpha)^{-1}(U)$ is a neighborhood of zero in E_α for all α; hence by the first part of this proposition, $f^{-1}(U)$ is a neighborhood of zero in E. Thus f is continuous.

The algebraic inductive limit topology is locally convex and is therefore weaker than the strongest locally convex topology on E for which all g_α are continuous, i.e., the linear inductive limit topology. Hence the two topologies coincide if and only if the linear inductive limit topology is locally m-convex. We next show that the two topologies may be distinct, even when they are defined by an increasing sequence $\{E_n\}_{n=1}^\infty$ of subalgebras of E such that $\bigcup_{n=1}^\infty E_n = E$, each E_n equipped with the topology induced by a metrizable, locally m-convex topology on E.

EXAMPLE 6. Let $E = \boldsymbol{K}[X_i]_{i=1}^\infty$ be the algebra of all polynomials in indeterminants $\{X_i\}_{i=1}^\infty$, E_n the subalgebra $\boldsymbol{K}[X_i]_{i=1}^n$ of E. For any sequence $\mathbf{x} = \{x_i\}_{i=1}^\infty$ of scalars, let $\mathbf{x}^*$ be the multiplicative linear form defined on E by $\mathbf{x}^*(h) = h(\mathbf{x})$. Let $H_n = [\{x_i\}_{i=1}^\infty \mid x_i$ is a non-negative rational for all i, and $x_i = 0$ if $i > m]$, and let $H = [\mathbf{x}^* \mid \mathbf{x} \in \bigcup_{n=1}^\infty H_n]$. It follows at once from a standard theorem of algebra (e.g., Proposition 8 of [3, p. 27]) that if $\mathbf{x}^*(h) = 0$ for all $\mathbf{x}^* \in H$, then $h = 0$. Hence if E' is the subspace of the algebraic dual E^* generated by H, E' is total. The weak topology $\sigma(E, E')$ is then clearly locally m-convex, and as E' has a denumerable base, is metrizable. We shall show that if each E_n is furnished with the locally m-convex topology induced by $\sigma(E, E')$, then the linear inductive and the algebraic inductive limit topologies defined on E by $\{E_n\}_{n=1}^\infty$ are distinct.

Let $B_n = [\mathbf{x}^* \mid \mathbf{x} = \{x_i\}_{i=1}^\infty \in H_n$ and each x_i is an integer $\leqq n]$. Since B_n is finite, $V_n = [h \in E_n \mid |\mathbf{x}^*(h)| < 2^{-n}$ for all $\mathbf{x}^* \in B_n]$ is a convex, equilibrated, idempotent neighborhood of zero in E_n. Let V be the convex envelope of $\bigcup_{n=1}^\infty V_n$. V is then clearly a neighborhood of zero for the linear inductive limit topology on E. We need two facts about these sets: (1) Let $\mathbf{x}^* \in B_j$, $g \in V_j$, and let $g = \sum_{n=1}^j g_n$ where g_n is the sum of those terms of g in E_n but not in

E_{n-1}. Then $|\mathbf{x}^*(g_n)| < 2^{n-1-j} < 1$ for $1 \leqq n \leqq j$. To prove the assertion we proceed by induction: Suppose $|\mathbf{x}^*(g_k)| < 2^{k-1-j}$ for all positive integers $k < n$. Let $\mathbf{x} = \{x_i\}_{i=1}^{\infty}$, and let $\mathbf{y} = \{y_i\}_{i=1}^{\infty}$, where $y_i = x_i$ for $i \leqq n$, $y_i = 0$ for $i > n$. Then $\mathbf{y}^*(g) = \sum_{k=1}^{n} \mathbf{y}^*(g_k)$, $\mathbf{y}^*(g_k) = \mathbf{x}^*(g_k)$ for $1 \leqq k \leqq n$, and $\mathbf{y}^* \in B_j$, whence $|\mathbf{y}^*(g)| < 2^{-j}$. By the inductive hypothesis, $|\mathbf{y}^*(g_k)| = |\mathbf{x}^*(g_k)| < 2^{k-1-j}$ for $k < n$. Hence $|\mathbf{x}^*(g_n)| = |\mathbf{y}^*(g_n)| \leqq |\mathbf{y}^*(g)| + \sum_{k=1}^{n-1} |\mathbf{y}^*(g_k)| < 2^{-j} + \sum_{k=1}^{n-1} 2^{k-1-j} = 2^{n-1-j}$. (2) If $h \in V$ is such that each term of h is in E_n but not in E_{n-1}, then for any $\mathbf{x}^* \in B_n$, $|\mathbf{x}^*(h)| < 1$. For let $h = \sum_{j=1}^{r} \lambda_j g_j$ where $r \geqq n$, $\lambda_j \geqq 0$ for all j, $\sum_{j=1}^{r} \lambda_j = 1$, and $g_j \in V_j$. Let $g_j = \sum_{i=1}^{j} g_{ji}$ where g_{ji} is the sum of those terms of g_j in E_i but not in E_{i-1}. Then by hypothesis $h = \sum_{j=n}^{r} \lambda_j g_{jn}$. For all $j \geqq n$, $\mathbf{x}^* \in B_n \subseteq B_j$, so by (1) $|\mathbf{x}^*(g_{jn})| < 1$; hence $|\mathbf{x}^*(h)| \leqq \sum_{j=n}^{r} \lambda_j |\mathbf{x}^*(g_{jn})| < 1$. Suppose now that V is a neighborhood of zero for the algebraic inductive limit topology. Then V contains an idempotent equilibrated neighborhood W of zero for the algebraic inductive limit topology. For each positive integer n, let $\mathbf{n} = \{n_j\}_{j=1}^{\infty}$ where $n_j = n$ for $j \leqq n$ and $n_j = 0$ for $j > n$. As W is absorbing and equilibrated, for each positive integer n we may choose $\alpha_n > 0$ such that $\alpha_n X_n \in W$. Then for any positive integer m, $(\alpha_1 X_1)^m (\alpha_n X_n) = \alpha_1^m \alpha_n X_1^m X_n \in W \subseteq V$. Hence by (2), as $\mathbf{n}^* \in B_n$, $|\mathbf{n}^*(\alpha_1^m \alpha_n X_1^m X_n)| < 1$, i.e., $(\alpha_1 n)^m (\alpha_n n) < 1$, and thus $\alpha_1 n < (\alpha_n n)^{-1/m}$. As this holds for all m, $\alpha_1 n \leqq \lim_{m \to \infty} (\alpha_n n)^{-1/m} = 1$, so $\alpha_1 \leqq 1/n$. As this is true for all n, $\alpha_1 = 0$, a contradiction. Hence V is not a neighborhood of zero for the algebraic inductive limit topology, and thus the two inductive limit topologies are distinct.

We seek now conditions insuring that the algebraic and the linear inductive limit topologies coincide. Michael [16, pp. 60–65] has given two such conditions. We shall give two others which are sufficient to cover most applications.

PROPOSITION 2. *Let E be a normed algebra, $\{E_\alpha\}$ subalgebras, each furnished with the topology induced from E, such that $\bigcup_\alpha E_\alpha = E$. Under either of the following two conditions, the algebraic inductive limit topology and the linear inductive limit topology on E with respect to $\{E_\alpha\}$ coincide*: (1) *Each E_α is an ideal*; (2) *$\{E_\alpha\}$, ordered by inclusion, is totally ordered.*

Proof. Let V be a convex, equilibrated neighborhood of zero for the linear inductive limit topology. For each α let ϵ_α be such that $0 < \epsilon_\alpha \leqq 1$ and such that $W_\alpha = [x \in E_\alpha \mid \|x\| \leqq \epsilon_\alpha] \subseteq V \cap E_\alpha$. Let $W = \bigcup_\alpha W_\alpha$. Then $W \subseteq V$, W is absorbing, and in both cases W is idempotent: Let $x, y \in W$, and let α and β be such that $x \in W_\alpha$, $y \in W_\beta$. (1) As E_α is an ideal, $xy \in E_\alpha$ and $\|y\| \leqq \epsilon_\beta \leqq 1$; hence $\|xy\| \leqq \|x\| \|y\| \leqq \|x\| \leqq \epsilon_\alpha$, so $xy \in W_\alpha \subseteq W$. (2) We may assume $E_\beta \subseteq E_\alpha$; then $xy \in E_\alpha$ and as in (1) $\|xy\| \leqq \epsilon_\alpha$, so $xy \in W_\alpha \subseteq W$. Now let U be the convex equilibrated envelope of W. Since V is convex and equilibrated and since $W \subseteq V$, $U \subseteq V$. Since W is idempotent and absorbing, so also is U. Finally $U \cap E_\alpha \supseteq W \cap E_\alpha \supseteq W_\alpha$, a neighborhood of zero in E_α. Thus by Proposition 1, U is a neighborhood of zero for the algebraic inductive limit topology on E. Hence the two inductive limit topologies coincide.

EXAMPLE 7. Let T be a locally compact space, $\mathcal{K}(T)$ the algebra of all continuous functions from T into $\boldsymbol{R}$ which vanish outside compact subsets of T. For a given compact subset L of T, we let $\mathcal{K}(T, L)$ be the ideal of all such functions vanishing outside L. $\mathcal{K}(T)$ has a natural norm topology, $\|x\| = \sup [|x(t)| \mid t \in T]$. The linear inductive limit topology on $\mathcal{K}(T)$ defined by the family $[\mathcal{K}(T, L) \mid L \text{ compact}]$ of subspaces equipped with the norm topology induced by the norm of $\mathcal{K}(T)$ is called the *measure topology* of $\mathcal{K}(T)$, for the Radon measures on T are precisely the members of the topological dual of $\mathcal{K}(T)$ equipped with this topology (see Exercise 1 of [9, p. 64]). Since each $\mathcal{K}(T, L)$ is an ideal and since $\mathcal{K}(T)$ is the union of such ideals, the measure topology is locally m-convex by (1) of Proposition 2.

EXAMPLE 8. Let T be a well-ordered set with least element a, equipped with the usual interval topology, E the algebra of all $\boldsymbol{R}$-valued, bound, continuous functions on T which are constant outside closed intervals. For $t \in T$ let $E_t = [x \in E \mid x \text{ is constant outside } [a, t]]$, equipped with the uniform norm. Then the linear inductive limit topology on E with respect to $\{E_t \mid t \in T\}$ is locally m-convex by (2) of Proposition 2.

PROPOSITION 3. *Let $\mathfrak{J}$ be a locally m-convex topology on E, $\{E_n\}_{n=1}^{\infty}$ an increasing sequence of ideals, each furnished with the topology induced by $\mathfrak{J}$, such that $\bigcup_{n=1}^{\infty} E_n = E$. Then the linear and the algebraic inductive limit topologies on E with respect to $\{E_n\}_{n=1}^{\infty}$ coincide.*

Proof. Let $\{p_\alpha\}_{\alpha \in \Gamma}$ be a family of pseudo-norms defining the topology $\mathfrak{J}$ such that $p_\alpha(xy) \leqq p_\alpha(x)p_\alpha(y)$ for all $x, y \in E$ and all $\alpha \in \Gamma$. Let V be a convex, equilibrated neighborhood of zero for the linear inductive limit topology. For all positive integers k choose ϵ_k and a finite subset Γ_k of Γ such that $0 < \epsilon_k \leqq 1$ and $V_k = [x \in E_k \mid p_\alpha(x) \leqq \epsilon_k \text{ for all } \alpha \in \Gamma_k]$ is contained in $V \cap E_k$. For all positive integers n, let $W_n = [x \in E_n \mid p_\alpha(x) \leqq \min_{1 \leqq k \leqq n} \epsilon_k \text{ for all } \alpha \in \bigcup_{k=1}^{n} \Gamma_k]$. W_n is then a neighborhood of zero in E_n. Let $W = \bigcup_{n=1}^{\infty} W_n$. W is clearly absorbing, and since $W_n \subseteq V_n$ for all n, $W \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq V$. W is idempotent: Let $x, y \in W$. Let n, m be such that $x \in W_n$, $y \in W_m$, and we may suppose $m \geqq n$. Then $xy \in E_n$ as E_n is an ideal. For any $\alpha \in \bigcup_{k=1}^{n} \Gamma_k$, α is also a member of $\bigcup_{k=1}^{m} \Gamma_k$, and hence $p_\alpha(y) \leqq \min_{1 \leqq k \leqq m} \epsilon_k \leqq 1$; hence $p_\alpha(xy) \leqq p_\alpha(x)p_\alpha(y) \leqq p_\alpha(x) \leqq \min_{1 \leqq k \leqq n} \epsilon_k$, so $xy \in W_n \subseteq W$. Now let U be the convex equilibrated envelope of W. As V is convex and equilibrated and as $W \subseteq V$, so also $U \subseteq V$. Since W is idempotent and absorbing, so also is U. Finally $U \cap E_n \supseteq W \cap E_n \supseteq W_n$, a neighborhood of zero in E_n. Hence by Proposition 1, U is a neighborhood of zero for the algebraic inductive limit topology on E. Hence the two inductive limit topologies coincide.

COROLLARY. *Let E be an algebra, the union of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of ideals. Let $\mathfrak{J}_n$ be a locally m-convex topology on E_n such that for all n, the topology induced on E_n by $\mathfrak{J}_{n+1}$ is $\mathfrak{J}_n$. Then the linear and the algebraic induc-*

tive limit topologies on E with respect to $\{E_n\}_{n=1}^{\infty}$ coincide if and only if for all n, the topology induced on E_n by the algebraic inductive limit topology is $\mathfrak{I}_n$.

Proof. The condition is sufficient by Proposition 3. It is necessary by Proposition 3 of [7, p. 64].

Example 6 shows that "metrizable" cannot replace "normed" in Proposition 2, nor can "subalgebra" replace "ideal" in Proposition 3.

EXAMPLE 9. Let E be the direct sum of algebras $\{E_i\}_{i=1}^{\infty}$, each E_i equipped with a locally m-convex topology $\mathfrak{I}_i$. Then the linear and algebraic inductive limit topologies on E with respect to $\{E_i\}_{i=1}^{\infty}$ coincide: For if F_n is the direct sum of $\{E_i\}_{i=1}^{n}$ with the product topology $\mathfrak{I}_n'$ of $\prod_{i=1}^{n} E_i$, the linear [respectively, algebraic] inductive limit topology on E with respect to $\{E_i\}_{i=1}^{\infty}$ is identical with the linear [respectively, algebraic] inductive limit topology with respect to $\{F_n\}_{n=1}^{\infty}$. $\{F_n\}_{n=1}^{\infty}$ is an increasing sequence of ideals whose union is E. If $\mathfrak{I}$ is the restriction to E of the topology of $\prod_{i=1}^{\infty} E_i$, $\mathfrak{I}$ induces $\mathfrak{I}_n'$ on F_n, and hence Proposition 3 is applicable.

More generally, if E is the direct sum of algebras $\{E_\alpha\}$, each E_α furnished with a locally m-convex topology, then E, furnished with the algebraic inductive limit topology defined by $\{E_\alpha\}$, is called the *topological direct sum* of $\{E_\alpha\}$.

EXAMPLE 10. Let P be an open subset of $\boldsymbol{R}^m$, $\mathfrak{D}_P$ the $\boldsymbol{K}$-algebra of all infinitely differentiable $\boldsymbol{K}$-valued functions on $\boldsymbol{R}^m$ vanishing outside compact subsets of P. If $\mathfrak{K}_P$ is the family of all closures of open, relatively compact subsets of P, $\mathfrak{D}_P = \bigcup [\mathcal{E}_K | K \in \mathfrak{K}_P]$, and the topology of Schwartz on $\mathfrak{D}_P$ is the linear inductive limit topology with respect to the family $[\mathcal{E}_K | K \in \mathfrak{K}_P]$ [19, pp. 67–68]. This topology is the same as the linear inductive limit topology with respect to $\{\mathcal{E}_{K_n}\}_{n=1}^{\infty}$, where each $K_n \in \mathfrak{K}_P$, $\bigcup_{n=1}^{\infty} K_n = P$, and K_n is contained in the interior of K_{n+1}. $\{\mathcal{E}_{K_n}\}_{n=1}^{\infty}$ is then an increasing sequence of ideals whose union is $\mathfrak{D}_P$, each ideal furnished with the locally m-convex topology induced from $\mathcal{E}_P$. Hence by Proposition 3, $\mathfrak{D}_P$ is locally m-convex. (This is shown by a different method in §15 of [16], which also contains special cases of Examples 7 and 9.)

3. ***i*-bornological algebras.** Henceforth, all locally m-convex algebras considered are Hausdorff unless the contrary is indicated.

If A is a subset of an algebra E, there exists a smallest subset containing A which is idempotent, namely, the intersection of all idempotent subsets of E containing A. This set is also clearly the same as $\bigcup_{n=1}^{\infty} A^n$.

DEFINITION 2. If A is any subset of an algebra E, the *idempotent envelope* of A is the smallest idempotent subset containing A.

DEFINITION 3. Let E be a locally m-convex algebra, $A \subseteq E$. A is *idempotently bound* (hereafter abbreviated to *i-bound*) if for some $\lambda > 0$, λA is contained in a bound, idempotent set, or equivalently, if for some $\lambda > 0$, the idempotent envelope of λA is bound.

DEFINITION 4. Let f be a function from locally m-convex algebra E into another locally m-convex algebra F. f is *idempotently bound* (hereafter abbreviated to *i-bound*) if for all i-bound subsets A of E, $f(A)$ is bound.

DEFINITION 5. A locally m-convex algebra E is *i-bornological* if every i-bound homomorphism from E into any locally m-convex algebra F is continuous.

We shall see that the i-bound sets form the algebraic counterpart to the bound sets in locally convex spaces. Not every one-point subset of a locally m-convex algebra need be i-bound; for example, if $x \in \mathcal{C}(\boldsymbol{R}, \boldsymbol{R})$ is an unbound function, $\{x\}$ is not i-bound for the compact-open topology on $\mathcal{C}(\boldsymbol{R}, \boldsymbol{R})$.

DEFINITION 6. A locally m-convex algebra E is *pointwise idempotently bound* (hereafter abbreviated to "p.i.b.") if for all $x \in E$, $\{x\}$ is i-bound.

Any normed algebra is clearly p.i.b.; hence by (5) of Proposition 4 below, a locally m-convex algebra E cannot be the algebraic inductive limit of normed algebras $\{E_\alpha\}$ with respect to homomorphisms $g_\alpha: E_\alpha \rightarrow E$ such that $\bigcup_\alpha g_\alpha(E_\alpha) = E$ unless E is p.i.b. Thus in proving the algebraic analogues of properties (B 1) and (B 4), defined at the beginning of the paper, equivalent (Theorem 4), we must drop the condition $\bigcup_\alpha g_\alpha(E_\alpha) = E$ from (B 4).

PROPOSITION 4. (1) *If A is i-bound, so is any scalar multiple of A, any subset, and the closed, convex, equilibrated envelope of A.* (2) *If E is a normed algebra, A is bound in E if and only if A is i-bound.* (3) *If E is commutative and if A and B are bound and idempotent, then AB and the closed, convex, equilibrated, idempotent envelope of $A \cup B$ are bound and idempotent; hence the product and the union of any finite family of i-bound subsets of E are i-bound.* (4) *If $\{x^n\}_{n=1}^{\infty}$ is bound, then for any $\alpha \in \boldsymbol{K}$ such that $|\alpha| < 1$, $(\alpha x)^n \rightarrow 0$; hence E is p.i.b. if and only if, for all $x \in E$, there exists $\lambda > 0$ such that $(\lambda x)^n \rightarrow 0$.* (5) *If E is the algebraic inductive limit of p.i.b. algebras $\{E_\alpha\}$ with respect to homomorphisms $g_\alpha: E_\alpha \rightarrow E$ such that $E = \bigcup_\alpha g_\alpha(E_\alpha)$, then E is p.i.b.*

Proof. (1) is obvious. (2) follows from the fact that the closed unit ball of a normed algebra is an i-bound neighborhood of zero. (3) As $(x, y) \rightarrow xy$ is bilinear and continuous, AB is bound; further, $(AB)^2 = ABAB = A^2B^2 \subseteq AB$; hence AB is bound and idempotent. But then $A \cup B \cup AB$ is also bound. And $(A \cup B \cup AB)^2 \subseteq A^2 \cup AB \cup A^2B \cup B^2 \cup BAB \cup ABAB \subseteq A \cup B \cup AB$. Thus the idempotent envelope of $A \cup B$ is $A \cup B \cup AB$, a bound set, so the result follows. (4) A subset A of a locally convex space is bound if and only if for every sequence $\{a_n\}_{n=1}^{\infty} \subseteq A$ and every sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n \rightarrow 0$, $\alpha_n a_n \rightarrow 0$. In particular, if $\{x^n\}_{n=1}^{\infty}$ is bound and $|\alpha| < 1$, $(\alpha x)^n = \alpha^n x^n \rightarrow 0$. Thus for any element y of a p.i.b. algebra, there exists $\lambda > 0$ such that $(\lambda y)^n \rightarrow 0$. Conversely, if $(\lambda x)^n \rightarrow 0$, then $\{(\lambda x)^n\}_{n=1}^{\infty}$ is a bound, idempotent set, so $\{x\}$ is i-bound. (5) Given $x \in E$, let α and $y \in E_\alpha$ be such that $g_\alpha(y) = x$. If $\lambda > 0$ is such that $(\lambda y)^n \rightarrow 0$, then $(\lambda x)^n = (\lambda g_\alpha(y))^n = g_\alpha(\lambda y)^n = g_\alpha((\lambda y)^n) \rightarrow 0$. Hence by (4) E is p.i.b.

EXAMPLE 11. Any normed algebra E is i-bornological: Every bound subset of a normed algebra is i-bound; hence every i-bound homomorphism on E is bound and hence continuous as E is a bornological space. More generally, if E is a bornological locally m-convex algebra such that every bound set is i-bound, then E is i-bornological.

We shall now associate with a given (Hausdorff) locally m-convex topology $\mathfrak{J}$ on E a stronger i-bornological topology $\mathfrak{J}^\star$, coinciding with $\mathfrak{J}$ if and only if $\mathfrak{J}$ is i-bornological. The characterization of $\mathfrak{J}^\star$ yields the proof that an i-bornological algebra is the algebraic inductive limit of normed algebras in a way very similar to the corresponding proof for bornological spaces. One feature of our treatment is the preservation of certain completeness properties from E to the inducing normed algebras. This enables certain problems about i-bornological algebras to be solved by reducing them to the corresponding normed algebra case (Theorem 2). A similar technique, exploited in Theorem 3, is passing to and from $\mathfrak{J}^\star$ to derive information about $\mathfrak{J}$.

DEFINITION 7. Let $\mathfrak{J}$ be a locally m-convex topology on E. A subset A of E is *i-bornivore* if A is convex, equilibrated, absorbing, idempotent, and if A absorbs every i-bound subset of $E;\mathfrak{J}$.

PROPOSITION 5. *Let $\mathfrak{J}$ be a locally m-convex topology on E.* (1) *The family of all i-bornivore sets is a fundamental system of neighborhoods of zero for a stronger locally m-convex topology $\mathfrak{J}^\star$ on E.* (2) *$\mathfrak{J}$ and $\mathfrak{J}^\star$ have the same i-bound sets.* (3) *E; $\mathfrak{J}^\star$ is i-bornological; moreover, if f is a homomorphism from E into any locally m-convex algebra F, then f is i-bound on E;$\mathfrak{J}$ if and only if f is continuous on E;$\mathfrak{J}^\star$.* (4) *E;$\mathfrak{J}$ is i-bornological if and only if $\mathfrak{J}=\mathfrak{J}^\star$, i.e., if and only if every i-bornivore set is a neighborhood of zero.*

Proof. Let $\mathfrak{A}$ be the collection of all i-bornivore sets of $E;\mathfrak{J}$. (1) Clearly $\mathfrak{A}$ is a filter base, and $0<|\lambda|\leqq 1$, $A\in\mathfrak{A}$ imply $\lambda A\in\mathfrak{A}$; hence by a remark of §1, $\mathfrak{A}$ is a fundamental system of neighborhoods of zero for a locally m-convex topology $\mathfrak{J}^\star$. $\mathfrak{J}^\star$ is stronger than $\mathfrak{J}$ since every convex, equilibrated, idempotent neighborhood of zero for $\mathfrak{J}$ is absorbing and absorbs every bound set. (2) is immediate by the definition of $\mathfrak{J}^\star$. (3) Let f be an i-bound homomorphism on $E;\mathfrak{J}^\star$ and let V be an idempotent, convex, equilibrated neighborhood of zero in the image space F. Then $f^{-1}(V)$ is clearly convex, equilibrated, idempotent, and absorbing. $f^{-1}(V)$ absorbs all i-bound sets: If B is i-bound, $f(B)$ is bound by hypothesis, so there exists $\lambda>0$ such that $f(B)\subseteq\lambda V$, and hence $B\subseteq\lambda f^{-1}(V)$. Hence $f^{-1}(V)\in\mathfrak{A}$, so f is continuous on $E;\mathfrak{J}^\star$. Thus $\mathfrak{J}^\star$ is an i-bornological topology on E. The remainder of (3) then follows from (2). (4) The condition is sufficient by (3). Necessity: If f is the identity map from $E;\mathfrak{J}$ onto $E;\mathfrak{J}^\star$, f is i-bound by (2). Hence by hypothesis f is continuous and so, as $\mathfrak{J}^\star$ is stronger than $\mathfrak{J}$, a homeomorphism. Thus $\mathfrak{J}=\mathfrak{J}^\star$.

COROLLARY. *If the topology of a locally m-convex algebra E is the strongest possible locally m-convex topology on E, then E is i-bornological. Conversely, if*

E is an i-bornological algebra such that the set H of i-bound elements is a finite-dimensional subalgebra of E, then the topology of E is the strongest possible locally m-convex topology.

Proof. Let $\mathfrak{J}$ be the topology of E. The hypothesis of the first assertion implies $\mathfrak{J}=\mathfrak{J}^{\star}$, so E is i-bornological. Suppose E is i-bornological and that H is a finite-dimensional subalgebra. Let $\mathfrak{J}'$ be the strongest locally m-convex topology on E, f the identity map from $E;\mathfrak{J}$ onto $E;\mathfrak{J}'$. If B is any i-bound subset of $E;\mathfrak{J}$, $B\subseteq H$. But as H is finite-dimensional, the restriction of f to H is continuous (Corollary 2 of Theorem 2 of [7, p. 28]), and hence $f(B)$ is bound. f is therefore an i-bound homomorphism, hence continuous, and therefore $\mathfrak{J}=\mathfrak{J}'$.

Note: We shall see later (Corollary of Proposition 11) that if E is commutative, H is necessarily a subalgebra.

We recall [7, p. 94] that if B is a closed, convex, equilibrated subset of a Hausdorff locally convex space, then on the subspace E_B generated by B, $p_B(x)=\inf\,[\lambda>0 \mid x\in\lambda B]$ is a norm whose closed unit ball is B; further if E is an algebra and B idempotent, p_B satisfies the multiplicative inequality $p_B(xy)\leqq p_B(x)p_B(y)$ (Lemma 1.2 of [16]).

THEOREM 1. *Let $E;\mathfrak{J}$ be an i-bornological algebra, $\mathfrak{B}$ the class of all closed, bound, convex, equilibrated, idempotent subsets of E. For each $B\in\mathfrak{B}$, the linear space E_B is a subalgebra, and E_B furnished with norm p_B is thus a normed algebra. Let $\mathfrak{J}_B$ be the norm topology on E_B, S_B the open unit ball in $E_B;\mathfrak{J}_B$. Then $E;\mathfrak{J}$ is the algebraic inductive limit of $\{E_B;\mathfrak{J}_B\}_{B\in\mathfrak{B}}$ with respect to the inclusion maps. Further:* (1) *E is p.i.b. if and only if $E=\bigcup_{B\in\mathfrak{B}} E_B$.* (2) *If E is commutative and p.i.b., then $\bigcup_{B\in\mathfrak{B}} S_B$ is a convex, equilibrated, idempotent neighborhood of zero.* (3) *If E is sequentially complete* (*i.e., if all Cauchy sequences converge*), *then so is each E_B, $B\in\mathfrak{B}$.* (4) *If E is advertibly complete, then so is each E_B, $B\in\mathfrak{B}$.*

Proof. Since each $B\in\mathfrak{B}$ is idempotent, the linear space generated by B is clearly a subalgebra. If V is an equilibrated neighborhood of zero for $\mathfrak{J}$, let $\lambda>0$ be such that $\lambda B\subseteq V$ for a given $B\in\mathfrak{B}$. Then $\lambda B\subseteq V\cap E_B$, and hence as λB is a neighborhood of zero in E_B, the topology induced on E_B by $\mathfrak{J}$ is weaker than $\mathfrak{J}_B$. So to prove $\mathfrak{J}$ is actually the algebraic inductive limit topology, it suffices, by Proposition 1, to show that if W is a convex, equilibrated, absorbing, idempotent subset of E such that $W\cap E_B$ is a neighborhood of zero in E_B for all $B\in\mathfrak{B}$, then W is a neighborhood of zero for $\mathfrak{J}$. But if $W\cap E_B$ is a neighborhood of zero in normed algebra E_B, then clearly $W\cap E_B$ absorbs B and hence also W absorbs B. Thus W absorbs all $B\in\mathfrak{B}$; but every i-bound subset of E is contained in a scalar multiple of some $B\in\mathfrak{B}$, so W absorbs all i-bound subsets of E. Hence W is i-bornivore and thus, by (4) of Proposition 5, W is a neighborhood of zero for $\mathfrak{J}$. Therefore E is the algebraic inductive limit of the family $\{E_B;\mathfrak{J}_B\}$ of normed algebras.

We turn now to the supplementary statements. (1) Suppose E is p.i.b. Then if $x\in E$, for some $\lambda>0$, the closed, convex, equilibrated, idempotent envelope B of $\{\lambda x\}$ is bound; hence $B\in\mathfrak{B}$ and $x\in E_B$. Conversely, suppose $E=\bigcup_{B\in\mathfrak{B}} E_B$. Given $x\in E$, choose $B\in\mathfrak{B}$ such that $x\in E_B$. Then for some $\lambda>0$, $\lambda x\in B$, a bound, idempotent set; hence $\{x\}$ is i-bound. (2) Let $S=\bigcup_{B\in\mathfrak{B}} S_B$. Each S_B is convex, equilibrated, and idempotent. Hence to show S has these properties, it clearly suffices to show that $\{S_B \mid B\in\mathfrak{B}\}$ is cofinal for the partial ordering defined by inclusion. Given A, $B\in\mathfrak{B}$, let C be the closed, convex, equilibrated, idempotent envelope of $A\cup B$. By (3) of Proposition 4, C is bound and hence $C\in\mathfrak{B}$. Also for $x\in E_A$, $p_A(x)=\inf\,[\lambda>0 \mid x\in\lambda A]\geqq\inf\,[\lambda>0\mid x\in\lambda C]=p_C(x)$, and similarly for $x\in E_B$, $p_B(x)\geqq p_C(x)$. Hence if $x\in S_A\cup S_B$, then either $p_A(x)<1$ or $p_B(x)<1$, so

$$p_C(x) \leqq \min\,\{p_A(x),\ p_B(x)\} < 1.$$

Thus $S_A\cup S_B\subseteq S_C$, and the assertion is proved. Since $E=\bigcup_{B\in\mathfrak{B}} E_B$, S is clearly absorbing. S certainly absorbs all $B\in\mathfrak{B}$ and hence all i-bound sets. Thus S is i-bornivore and hence by (4) of Proposition 5 is a neighborhood of zero. (3) Suppose now that E is sequentially complete. Proposition 8 and its corollary of [7, p. 11] hold if "sequentially complete" replaces "complete"; hence as each $B\in\mathfrak{B}$ is closed in the given topology $\mathfrak{I}$, E_B is sequentially complete for all $B\in\mathfrak{B}$. (4) Suppose E is advertibly complete. Let $\mathfrak{F}$ be a Cauchy filter on $E_B;\mathfrak{I}_B$ such that for some $z\in E_B$, $z\circ\mathfrak{F}\to 0$ and $\mathfrak{F}\circ z\to 0$ in E_B. Then as $\mathfrak{I}_B$ is stronger than the topology induced on E_B by $\mathfrak{I}$, $\mathfrak{F}$ is a Cauchy filter base on E and $z\circ\mathfrak{F}\to 0$, $\mathfrak{F}\circ z\to 0$ in $E;\mathfrak{I}$. Hence by assumption there exists $z'\in E$ such that $\mathfrak{F}\to z'$ in $E;\mathfrak{I}$. As E is Hausdorff, z' is clearly the adverse of z. We assert $z'\in E_B$: There exists $F\in\mathfrak{F}$ such that F is B-small. Let $y\in F$. Then $F+(-y)\subseteq B$, so $F\subseteq y+B$. But since B is closed in $E;\mathfrak{I}$, so also $y+B$ is closed in $E;\mathfrak{I}$. Hence $z'\in\overline{F}\subseteq$ the closure of $y+B=y+B\subseteq E_B$. Finally, $\mathfrak{F}\to z'$ in $E_B;\mathfrak{I}_B$ since $\mathfrak{F}=0\circ\mathfrak{F}=(z'\circ z)\circ\mathfrak{F}=z'\circ(z\circ\mathfrak{F})\to z'\circ 0=z'$ in $E_B;\mathfrak{I}_B$ by assumption. Thus E_B is advertibly complete.

COROLLARY. *A sequentially complete, i-bornological algebra is the algebraic inductive limit of Banach algebras. An advertibly complete, i-bornological algebra is the algebraic inductive limit of normed Q-algebras.*

Proof. A sequentially complete, normed algebra is complete. An advertibly complete, normed algebra is a Q-algebra by Theorem 7 of [25].

THEOREM 2. *Let E be an i-bornological algebra which is either sequentially complete or advertibly complete. Then every multiplicative linear form on E is continuous. If, in addition, E is commutative and p.i.b., $[x \mid -\sum_{n=1}^{\infty} x^n$ exists and is the adverse of $x]$ is a neighborhood of zero; in particular E is a Q-algebra.*

Proof. E is the algebraic inductive limit of normed Q-algebras $\{E_B\}_{B\in\mathfrak{B}}$ by the corollary of Theorem 1 (for every Banach algebra is a Q-algebra).

If u is a multiplicative linear form on E, its restriction to each E_B is again a multiplicative linear form which is thus continuous on E_B (cf. Lemma E.4 of [16, p. 77]). Hence, by Proposition 1, u is continuous on E. Suppose now that E is commutative and p.i.b. If S_B is the open unit ball of E_B, by (2) of Theorem 1 $S=\bigcup_{B\in\mathfrak{B}} S_B$ is a neighborhood of zero. Given $x\in S_B$, the proof of Theorem 7 of [25] shows that the sequence $\{-\sum_{j=1}^{n} x^j\}_{n=1}^{\infty}$ converges to the adverse of x for $\mathfrak{I}_B$ and hence also for the weaker topology induced on E_B by $\mathfrak{I}$. Thus $S\subseteq[x|-\sum_{n=1}^{\infty} x^n$ exists and is the adverse of $x]$.

THEOREM 3. *If E is an advertibly complete, locally m-convex algebra, every multiplicative linear form on E is i-bound.*

Proof. If $\mathfrak{I}$ is the given advertibly complete topology of E, $\mathfrak{I}^\star$ is also advertibly complete: For if $\mathfrak{F}$ is a Cauchy filter on $E;\mathfrak{I}^\star$ such that for some x, $\mathfrak{F}\circ x\to 0$ and $x\circ\mathfrak{F}\to 0$ in $E;\mathfrak{I}^\star$, clearly $\mathfrak{F}$ is Cauchy, $\mathfrak{F}\circ x\to 0$ and $x\circ\mathfrak{F}\to 0$ in $E;\mathfrak{I}$ as $\mathfrak{I}$ is weaker than $\mathfrak{I}^\star$. Hence by hypothesis there exists $x'\in E$ such that $\mathfrak{F}\to x'$ in $E;\mathfrak{I}$. As $\mathfrak{I}$ is Hausdorff, x' is the adverse of x. Hence in $E;\mathfrak{I}^\star$, $\mathfrak{F}=0\circ\mathfrak{F}=(x'\circ x)\circ\mathfrak{F}=x'\circ(x\circ\mathfrak{F})\to x'\circ 0=x'$, so $E;\mathfrak{I}^\star$ is advertibly complete. Hence by Theorem 2, every multiplicative linear form on E is continuous for topology $\mathfrak{I}^\star$, and hence also is i-bound on $E;\mathfrak{I}$ by (2) of Proposition 5.

Theorem 3 is a step towards an answer to the following question raised by Michael (Question 2 of [16, p. 50]): Is every multiplicative linear form on a commutative, complete, locally m-convex algebra necessarily a bound linear form?

If E is an algebra, there need not, in general, be any Hausdorff locally m-convex topologies on E, even though there may be Hausdorff locally convex topologies on E compatible with the algebraic structure of E (i.e., for which $(x, y)\to xy$ is continuous on $E\times E$). For example, Williamson [27] has shown that there exist Hausdorff, locally convex topologies on $\mathbf{C}(X)$, the field of all rational fractions in one indeterminant, compatible with its algebraic structure. But by the Gelfand-Mazur Theorem for locally m-convex algebras (Proposition 2.9 of [16, p. 10]), there can exist no Hausdorff locally m-convex topologies on $\mathbf{C}(X)$. Suppose, however, there does exist a Hausdorff locally m-convex topology on E yielding E' as topological dual. Then the topology generated by all locally m-convex topologies on E yielding E' as topological dual is again locally m-convex. Since each such topology is weaker than $\tau(E, E')$, so is the topology thus generated; hence its topological dual is also E'. Thus, if E' is a total subspace of the algebraic dual of E such that there exist locally m-convex topologies on E for which E' is the topological dual, there exists a strongest such locally m-convex topology, which we denote by "$\chi(E, E')$." It follows from the Mackey-Arens theorem (Theorem 4 of [15, p. 523] and Theorem 2 of [1, p. 790]) that a fundamental system of neighborhoods of zero for $\chi(E, E')$ is the class of all polars in E of weakly compact, convex subsets C of E' such that C^0 is idempotent. The author does not know if $\chi(E, E')$, when defined, is necessarily identical with $\tau(E, E')$.

DEFINITION 8. Let f be a linear transformation from locally m-convex algebra E to locally m-convex algebra F. We shall say f is *i-bornological* if, for every neighborhood V of zero in $F, f^{-1}(V)$ contains an i-bornivore set.

The author does not know an example of a bound linear form which is not i-bornological, nor of an i-bornological algebra which is not a bornological space. Any continuous linear transformation is i-bornological, for if V is a neighborhood of zero in $F, f^{-1}(V)$ contains a convex, equilibrated, idempotent neighborhood of zero in E which is, clearly, an i-bornivore set. Any i-bornological transformation f is i-bound: For if B is i-bound, V a neighborhood of zero in F, there exists $\lambda>0$ such that $B\subseteq\lambda f^{-1}(V)$ since $f^{-1}(V)$ contains an i-bornivore set; hence $f(B)\subseteq\lambda V$. Thus $f(B)$ is bound and so f is i-bound. Also, if f is a homomorphism, then f is i-bound if and only if f is i-bornological: For if f is i-bound and if V is a convex, equilibrated, idempotent neighborhood of zero in F, then $f^{-1}(V)$ is convex, equilibrated, idempotent, and absorbing. Also if B is i-bound, $f(B)$ is bound; thus there exists λ such that $f(B)\subseteq\lambda V$ and hence $B\subseteq\lambda f^{-1}(V)$. Therefore $f^{-1}(V)$ is i-bornivore.

THEOREM 4. *Let $E;\mathfrak{J}$ be a (Hausdorff) locally m-convex algebra, E' the topological dual of E. The following are equivalent*: (IB 1) *E is i-bornological.* (IB 2) *No strictly stronger locally m-convex topology on E has the same i-bound sets.* (IB 3) *Every i-bornivore set is a neighborhood of zero.* (IB 4) *E is the algebraic inductive limit of normed algebras.* (IB 5) *$\mathfrak{J}=\chi(E, E')$, and every i-bornological linear form on E is continuous.* (IB 6) *Every i-bornological linear transformation from E into any locally m-convex algebra is continuous.*

Proof. By Proposition 5, (IB 1) and (IB 3) are equivalent, and (IB 2) implies (IB 1). (IB 3) implies (IB 2): Let $\mathfrak{J}'$ be a locally m-convex topology on E stronger than $\mathfrak{J}$ but have the same i-bound sets as $\mathfrak{J}$. Then every convex, equilibrated, idempotent neighborhood V of zero for $\mathfrak{J}'$ is absorbing and absorbs every i-bound subset of $E;\mathfrak{J}$ and hence is an i-bornivore subset of $E;\mathfrak{J}$. Thus by (IB 3) V is a neighborhood of zero for $\mathfrak{J}$, and so $\mathfrak{J}=\mathfrak{J}'$. (IB 6) implies (IB 1), for the identity map from $E;\mathfrak{J}$ onto $E;\mathfrak{J}^\star$ is i-bound by (2) of Proposition 5, hence is i-bornological and thus continuous. Therefore $\mathfrak{J}=\mathfrak{J}^\star$ and $\mathfrak{J}$ is i-bornological. (IB 3) implies (IB 6): If f is an i-bornological linear transformation from E into F, for any neighborhood V of zero in $F, f^{-1}(V)$ contains an i-bornivore set which is by hypothesis a neighborhood of zero. Hence f is continuous. (IB 2) implies (IB 5): For $\mathfrak{J}$ and $\chi(E, E')$ have the same bound sets by Theorem 7 of [15, p. 524] and hence also the same i-bound sets. Therefore by hypothesis, $\mathfrak{J}=\chi(E, E')$. On the other hand, the equivalence of (IB 2) with (IB 6) shows that every i-bornological linear form is continuous. (IB 5) implies (IB 1): By hypothesis, $E;\mathfrak{J}$ and $E;\mathfrak{J}^\star$ have the same topological dual. Hence $\chi(E, E')=\mathfrak{J}\subseteq\mathfrak{J}^\star\subseteq\chi(E, E')$, so $\mathfrak{J}=\mathfrak{J}^\star$ and thus $E;\mathfrak{J}$ is i-bornological. (IB 1) implies (IB 4) by Theorem 1. It remains to show

(IB 4) implies (IB 1). Since every normed algebra is i-bornological, the desired implication follows from the more general result that the algebraic inductive limit of i-bornological algebras is i-bornological; this result is proved in the next section (Proposition 6).

EXAMPLE. If T is a locally compact space, the algebra $\mathcal{K}(T)$ with the measure topology (Example 7) is i-bornological by (IB 4).

4. **Properties of permanence.** Here we shall investigate the extent to which the property of being i-bornological is preserved under certain operations of algebra.

PROPOSITION 6. *The algebraic inductive limit of i-bornological algebras is i-bornological.*

Proof. Let E be the algebraic inductive limit of i-bornological algebras $\{E_\alpha\}$ with respect to homomorphisms $\{g_\alpha\}$. Let f be an i-bound homomorphism from E into a locally m-convex algebra F. Then since every g_α is continuous and thus i-bound, $f \circ g_\alpha$ is clearly i-bound for all α, and hence $f \circ g_\alpha$ is continuous for all α by hypothesis. But then by Proposition 1, f is continuous. Hence E is i-bornological.

COROLLARY 1. *If E is an i-bornological algebra, H a closed ideal of E, then E/H is i-bornological.*

Proof. The topology of E/H is the strongest locally m-convex topology on E/H such that the canonical map $\phi: E \to E/H$ is continuous. Since ϕ is a homomorphism, E/H is the algebraic inductive limit of $\{E\}$ with respect to $\{\phi\}$, so the result follows from Proposition 6.

COROLLARY 2. *If E is i-bornological, f a continuous, open homomorphism from E onto F, then F is i-bornological.*

Proof. F is topologically isomorphic with E/H, where H is the kernel of f.

COROLLARY 3. *The topological direct sum of i-bornological algebras is i-bornological.*

PROPOSITION 7. *If $E_1, \cdots, E_n$ are locally m-convex algebras, $E = \prod_{i=1}^{n} E_i$ is i-bornological if and only if each E_i is i-bornological.*

Proof. Necessity: The projection map from E onto E_i is a continuous, open homomorphism. Hence by Corollary 2 of Proposition 6 E_i is i-bornological. Sufficiency: By induction it suffices to show that if E and F are i-bornological, so is $E \times F$. If f: $E \times F \to G$ is an i-bound homomorphism, let g: $E \to G$ and h: $F \to G$ be defined by $g(x) = f(x, 0)$ and $h(y) = f(0, y)$. Then g and h are clearly i-bound homomorphisms and hence continuous. But then, since $f(x, y) = g(x) + h(y)$, it follows that f is continuous and hence $E \times F$ i-bornological.

If E is a locally m-convex algebra, we let E^+ be the algebra E with identity adjoined. That is, $E^+ = E \times \boldsymbol{K}$ (with the Cartesian product topology) with addition and scalar multiplication defined componentwise, and multiplication defined as follows: $(x, \alpha)(y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$. E^+ is then a locally m-convex algebra by Proposition 2.4 of [16, p. 7].

PROPOSITION 8. *E is an i-bornological algebra if and only if E^+ is i-bornological.*

Proof. We denote by pr_1 the map $(x, \lambda) \to x$ and by pr_2 the map $(x, \lambda) \to \lambda$. Both are linear, continuous, and open, and pr_2 is a homomorphism. Necessity: Let f be any i-bound homomorphism from E^+ into locally m-convex algebra F. Let $g(x) = f(x, 0)$ be defined on E. If B is a bound, idempotent subset of E, $B \times \{0\}$ is clearly bound and idempotent in E^+, and hence $g(B) = f(B \times \{0\})$ is bound in F. Therefore g is i-bound and hence continuous. Let $e = f(0, 1)$. Then e is the identity for $f(E^+)$, and hence $f(x, \lambda) = g(x) + \lambda e$. As g and the map $\lambda \to \lambda e$ are continuous, f is continuous. Hence E^+ is i-bornological. Sufficiency: First, we prove the following: If B is a bound, idempotent, convex, equilibrated subset of E^+, then $3^{-1} pr_1(B)$ is bound and idempotent in E. Since pr_1 is linear and continuous, $3^{-1} pr_1(B)$ is bound. Let $x, y \in pr_1(B)$. Then for some $\alpha, \beta \in \boldsymbol{K}$, $(x, \alpha) \in B$ and $(y, \beta) \in B$. Then $\alpha, \beta \in pr_2(B)$; as pr_2 is a continuous homomorphism, $pr_2(B)$ is bound and idempotent in $\boldsymbol{K}$ and hence is clearly contained in the closed unit ball $[\lambda \mid |\lambda| \leqq 1]$. Thus $|\alpha| \leqq 1$ and $|\beta| \leqq 1$. Now $(x, \alpha)(y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$, so $z = xy + \beta x + \alpha y \in pr_1(B)$. Hence $xy = z - \alpha y - \beta x \in 3pr_1(B)$, since $pr_1(B)$ is convex and equilibrated and since $|\alpha| \leqq 1$ and $|\beta| \leqq 1$. But then $(3^{-1}x)(3^{-1}y) = 9^{-1}xy \in 3^{-1} pr_1(B)$, so $3^{-1} pr_1(B)$ is idempotent, and our assertion is proved. It follows from this assertion that for any i-bound subset B of E^+, $pr_1(B)$ is i-bound in E. Now let g be any i-bound homomorphism from E into a locally m-convex algebra F. Let $f: E^+ \to F^+$ be defined by $f(x, \lambda) = (g(x), \lambda)$. Clearly f is a homomorphism. If B is any i-bound subset of E^+, $f(B) \subseteq g(pr_1(B)) \times pr_2(B)$. By the above $pr_1(B)$ is i-bound, so $g(pr_1(B))$ is bound. Also $pr_2(B)$ is bound, and hence $g(pr_1(B)) \times pr_2(B)$ is bound in F^+; hence $f(B)$ is bound. Thus f is an i-bound homomorphism and so, by hypothesis, is continuous. But then so also is g since $g(x) = (pr_1 \circ f)(x, 0)$. Hence E is i-bornological.

If H is a closed ideal of an i-bornological algebra E, H need not be i-bornological, even if E is the topological direct sum (as a locally convex space) of H and a subalgebra J (though, of course, if E is the topological direct sum of H and an ideal J, H is i-bornological by Proposition 7).

EXAMPLE 12. Let E be the subalgebra of $\boldsymbol{K}[X, Y]$ consisting of all polynomials in two indeterminants without constant term, equipped with the topology of Example 5. The set of i-bound elements of E is the subalgebra $\{0\}$: If not, by (4) of Proposition 4 there exists $z \neq 0$ such that $z^n \to 0$. Let $z = \sum_{i,j} \alpha_{ij} X^i Y^j$, and let i be the smallest integer such that $\alpha_{ik} \neq 0$ for some k.

Let j be the smallest integer such that $\alpha_{ij}\neq 0$. Then the coefficient of $X^{ni}Y^{nj}$ in z^n is α_{ij}^n. Choose β such that $0<\beta^{i+j}<|\alpha_{ij}|$. If $z^n\in V(\beta,\beta)$, then $|\alpha_{ij}^n|<\beta^{ni+nj}=(\beta^{i+j})^n$, impossible. Hence the sequence $\{z^n\}_{n=1}^{\infty}$ does not converge to zero, so the assertion is proved. Let H be the subspace spanned by $[X^iY^j|i\geqq 0, j\geqq 1]$. H is clearly an ideal. By the corollary of Proposition 5, E is i-bornological, and also if H is i-bornological, the topology of H must be the strongest locally m-convex topology possible on H. But it is not: Let W be the convex, equilibrated envelope of $\{2^{-n^2}X^nY\}_{n=0}^{\infty}\cup\{X^nY^m\}_{n=0,m=2}^{\infty\ \ \infty}$. W is the convex, equilibrated envelope of an idempotent set, and hence is idempotent. W is an absorbing subset of H, and thus is a neighborhood of zero in H for the strongest possible locally m-convex topology. Suppose for some $\alpha>0$, $\beta>0$, $W\supseteq V(\alpha,\beta)\cap H$. Then $(\alpha X)^m(\beta Y)\in W$ for all positive integers m, i.e., $\alpha^m\beta X^mY\in W$. Therefore $\alpha^m\beta\leqq 2^{-m^2}$ and so $\alpha\beta^{1/m}\leqq 2^{-m}$ for all m. Hence $\alpha=\lim_{m\to\infty}\alpha\beta^{1/m}\leqq\lim_{m\to\infty}2^{-m}=0$, a contradiction. Thus W is not a neighborhood of zero in H, and so H is not i-bornological. If J is the subalgebra of $\boldsymbol{K}[X]$ of all polynomials in one indeterminant without constant term, E is clearly the direct sum of J and H. The projection map pr_1 from E onto J is continuous, for given any $\beta>0$, $V(\beta,1)\subseteq pr_1^{-1}(V(\beta))$. Hence also the projection map from E onto H is continuous, so E is the topological direct sum of J and H; in particular, H is closed.

Finally, we observe that if E is a locally m-convex algebra over $\mathbf{C}$, E_r the algebra over $\boldsymbol{R}$ obtained from E by restricting the field of scalars, then E is i-bornological if and only if E_r is. It suffices to show that every i-bornivore set of E [respectively, E_r] contains an i-bornivore set of E_r [respectively, E]. But if V is an i-bornivore set of E, it is also an i-bornivore set of E_r. Conversely, let V be an i-bornivore set of E_r. If W is the $\mathbf{C}$-equilibrated envelope of $2^{-1}(V\cap iV)$, it is elementary that $W\subseteq V$, W is idempotent, and hence that W is an i-bornivore subset of E.

5. **Algebras of continuous functions and infinite Cartesian products.** Let T be a completely regular space, $\mathfrak{C}(T)$ the algebra of all real-valued continuous functions on T, $\mathfrak{U}$ the weakest uniform structure on T for which all $x\in\mathfrak{C}(T)$ are uniformly continuous, and $\beta(T)$ the Stone-Čech compactification of T. For each $t\in T$ let $\hat{t}$: $x\to x(t)$. Then $\hat{t}$ is a multiplicative linear form on $\mathfrak{C}(T)$. A filter $\mathfrak{F}$ on T is an *H-filter* if: (H 1) For any sequence $\{F_n\}_{n=1}^{\infty}$ of members of $\mathfrak{F}$, $\bigcap_{n=1}^{\infty}F_n\in\mathfrak{F}$; (H 2) $\mathfrak{F}\cap[x^{-1}(0)|x\in\mathfrak{C}(T)]$ is a filter base of $\mathfrak{F}$. An H-filter properly contained in no H-filter is called a *maximal H-filter*.

An important class of completely regular spaces, first investigated by Hewitt [12] and called by him Q-spaces, are those which satisfy any one and hence all of the following equivalent properties: (Q 1) $T;\mathfrak{U}$ is a complete uniform space. (Q 2) Every maximal H-filter converges (maximal H-filters play the same role that Z-maximal sets do in [12]). (Q 3) If $t\in\beta(T)$ is such that for every sequence $\{V_n\}_{n=1}^{\infty}$ of neighborhoods of t, $\bigcap_{n=1}^{\infty}V_n\cap T\neq\varnothing$, then $t\in T$. (Q 4) The canonical map $t\to\hat{t}$ is onto the set of all nonzero multiplicative linear forms on $\mathcal{C}(T)$. (Q 5) If S is a completely regular space containing a dense subspace T' homeomorphic with T such that every continuous real-valued function on T' is the restriction of a continuous real-valued function on S, then $T'=S$. (Q 6) T is homeomorphic with a closed subset of a Cartesian product of reals. (Q 7) (Nachbin-Shirota) $\mathcal{C}(T)$, furnished with the compact-open topology, is a bornological locally convex space. (For the proofs, see [12; 17; 21; 22].) We give an eighth equivalent characterization of Q-spaces:

THEOREM 5. *Completely regular space T is a Q-space if and only if the algebra $\mathcal{C}(T)$ of all continuous real-valued functions on T, equipped with the compact-open topology, is i-bornological.*

Proof. The proof is essentially contained in Nachbin's proof of the equivalence of (Q 7) and (Q 1). If $x, y\in\mathcal{C}(T)$, $[x, y]$ is the set $[z\in\mathcal{C}(T) \mid x(t)\leqq z(t) \leqq y(t)$ for all $t\in T]$. Let e be the identity element of $\mathcal{C}(T)$. Necessity: In the proof of Theorem 2 of [17], Nachbin showed that if V is a convex, equilibrated subset of $\mathcal{C}(T)$ absorbing every set of the form $[x, y]$, then V is a neighborhood of zero in $\mathcal{C}(T)$. Let W be any i-bornivore set. Let $x, y\in\mathcal{C}(T)$, and let $z=\max\{|x|, |y|, e\}$. Clearly $[-e, e]$ is a bound, idempotent subset of $\mathcal{C}(T)$. Let $\lambda>0$ be such that $[-e, e]\subseteq\lambda W$ and $z\in\lambda W$. For any $u\in[x, y]$, $u=(u/z)\cdot z$, and $u/z\in[-e, e]$. Hence $u\in(\lambda W)(\lambda W)=\lambda^2W^2\subseteq\lambda^2W$; therefore $[x, y]\subseteq\lambda^2W$. Hence by Nachbin's result W is a neighborhood of zero, so by (IB 3) $\mathcal{C}(T)$ is i-bornological. Sufficiency: Nachbin showed that if T is not a Q-space, there exists a bound, discontinuous linear form on $\mathcal{C}(T)$ which is actually a multiplicative linear form. Hence if T is not a Q-space, $\mathcal{C}(T)$ is not i-bornological.

Several interesting assertions in different fields of mathematics have recently been proved equivalent to the following unproved assertion: (U 1) If A is a set and if μ is a (countably additive) nonzero measure defined on all subsets of A, taking on only the values 0 and 1, then there exists $\alpha\in A$ such that $\mu(X)=1$ if and only if $\alpha\in X$. To stress that the assertion is essentially a set-theoretic one, we give it an equivalent formulation. Let us call an ultrafilter $\mathfrak{U}$ on A an *Ulam ultrafilter* if, for every sequence $\{F_n\}_{n=1}^{\infty}$ of members of $\mathfrak{U}$, $\bigcap_{n=1}^{\infty}F_n\in\mathfrak{U}$. (U 1) is then equivalent to the following:

AXIOM U. If A is a set and if $\mathfrak{U}$ is an Ulam ultrafilter on A, then there exists $\alpha\in A$ such that $\mathfrak{U}=[X\subseteq A \mid \alpha\in X]$.

(If μ is a measure on A having the properties of the hypothesis of (U 1), the sets of measure 1 form an Ulam ultrafilter; conversely, if $\mathfrak{U}$ is an Ulam ultrafilter, the set function μ, defined by $\mu(X)=1$ if $X\in\mathfrak{U}$, $\mu(X)=0$ if $X\notin\mathfrak{U}$, has the desired properties.)

In [24] Ulam showed that if a set A does not have the property of (U 1), then the cardinality of A must be at least as great as the first inaccessible cardinal. On the other hand, Shepherdson has shown (Theorem 3.71 of [20]) that if the usual axioms of set theory are consistent (not including the Axiom of Choice or the Generalized Continuum Hypothesis), then the set of axioms obtained by adding the Axiom of Choice, the Generalized Continuum Hypothesis, and the axiom that there exist no inaccessible cardinals is again consistent. Axiom U, by Ulam's result, is weaker than the assertion there exist no inaccessible cardinals, and hence is also consistent with the other axioms of set theory. In view of the desirable theorems which may be proved using Axiom U (a partial list follows), it might be desirable for mathematicians to add Axiom U (or some stronger axiom, such as the nonexistence of inaccessible cardinals) to the currently used axioms of set theory.

If A is a set, $\boldsymbol{R}^A$ is the Cartesian product $\prod_{\alpha \in A} \boldsymbol{R}_\alpha$, where for all α, $\boldsymbol{R}_\alpha = \boldsymbol{R}$. The following are equivalent to Axiom U: (U 2) If A is a set, the space of bound linear forms on the locally convex space $\boldsymbol{R}^A$ is generated by the family $\{pr_\beta\}_{\beta \in A}$ of projection mappings, where $pr_\beta((x_\alpha)_{\alpha \in A}) = x_\beta$. (U 3) Every discrete space is a Q-space. (U 4) Every metrizable space is a Q-space. (U 5) Every paracompact space is a Q-space. (U 6) Every completely regular space admitting a compatible, complete uniform structure is a Q-space. (U 7) The Cartesian product of any family of bornological spaces is bornological.

The equivalence of (U 2) and (U 1) is shown in [14], that of (U 7) and (U 1) in [11]. The equivalence of (U 3) and Axiom U is most easily seen by observing that on a discrete space, the class of maximal H-filters is identical with the class of Ulam ultrafilters; Axiom U and (Q 2) both assert that the filters in this class converge. Shirota (Theorem 3 of [21]) proved (U 3) implies (U 6); (U 6) implies (U 5) since every paracompact space admits a compatible, complete uniform structure ([18]); (U 5) implies (U 4) since every metrizable space is paracompact ([23]); and (U 4) implies (U 3) since a discrete space is metrizable. (Katětov (Theorem 3 of [13]) also showed that (U 3) implies (U 5).)

Theorem 6. *Axiom* U *is equivalent to the following*: (U 8) *For any class* $\{E_\alpha\}_{\alpha \in A}$ *of i-bornological algebras, each containing an identity,* $\prod_{\alpha \in A} E_\alpha$ *is i-bornological.*

Proof. First, observe that Axiom U is equivalent to the assertion that any Cartesian product of reals is i-bornological. For $\boldsymbol{R}^A$ is identical with the algebra $\mathcal{C}(A, \boldsymbol{R})$ of all continuous real-valued functions on A, furnished with the compact-open topology, where A is given the discrete topology. Hence by Theorem 5, $\boldsymbol{R}^A$ is i-bornological if and only if A is a Q-space; the equivalence then follows from (U 3). In particular, (U 8) implies Axiom U. So let us assume Axiom U; we shall prove (U 8) by showing (IB 5) holds for any product $E = \prod_\alpha E_\alpha$ of i-bornological algebras, each E_α possessing an identity e_α. By a remark of the preceding section, it suffices to consider the case where the scalar field is $\boldsymbol{R}$. We proceed by a series of lemmas:

Lemma 1. *Let U be an i-bornivore subset of E, $(x_\alpha) \in E$. Then there exists an i-bornivore subset V of $\boldsymbol{R}^A$ such that if $(\lambda_\alpha) \in V$, then $(\lambda_\alpha x_\alpha) \in U$.*

Proof. Let $V=[(\lambda_\alpha) \mid (\lambda_\alpha x_\alpha)\in U$ and $(\lambda_\alpha e_\alpha)\in U]$. V is convex, equilibrated and absorbing as U is. V is idempotent, for (λ_α), $(\mu_\alpha)\in V$ implies $(\lambda_\alpha\mu_\alpha x_\alpha) =(\lambda_\alpha x_\alpha)(\mu_\alpha e_\alpha)\in U^2\subseteq U$ and also $(\lambda_\alpha\mu_\alpha e_\alpha)=(\lambda_\alpha e_\alpha)(\mu_\alpha e_\alpha)\in U^2\subseteq U$, whence $(\lambda_\alpha\mu_\alpha)\in V$. Let $B=[(\lambda_\alpha) \mid |\lambda_\alpha| \leqq 1$, all $\alpha]$, and let $B'=[(\lambda_\alpha e_\alpha) \mid |\lambda_\alpha| \leqq 1$, all $\alpha]$. B' is clearly bound and idempotent in E. Hence as U is i-bornivore, we may choose $\beta \geqq 1$ such that $B'\subseteq \beta U\subseteq \beta^2 U$ and such that $(x_\alpha)\in\beta U$. Then $B'\cdot(x_\alpha) \subseteq\beta^2 U^2\subseteq\beta^2 U$. But by the definition of V, this insures that $B\subseteq\beta^2 V$. As all bound, idempotent subsets of $\boldsymbol{R}^A$ are contained in B, V absorbs all i-bound subsets of $\boldsymbol{R}^A$, and hence V is i-bornivore.

Now let u be any i-bornological linear form on E. For each $\beta\in A$ let I_β: $E_\beta\to E$ be defined by $I_\beta(x)=(x_\alpha)$ where $x_\beta=x$, $x_\alpha=0$ for all $\alpha\neq\beta$. Clearly I_β is a topological isomorphism into.

LEMMA 2. *There exists a finite subset A_1 of A such that for any $\alpha\notin A_1$, $u(I_\alpha(E_\alpha))=\{0\}$.*

Proof. If not, we may select an infinite sequence $\{\alpha_i\}_{i=1}^\infty$ of distinct α's and elements $x_{\alpha_i}\in E_{\alpha_i}$ such that $u(I_{\alpha_i}(x_{\alpha_i}))>0$. Let U be an i-bornivore set contained in $[x \mid |u(x)| \leqq 1]$. For $\alpha\notin\{\alpha_i\}_{i=1}^\infty$, let $x_\alpha=0$. By Lemma 1, let V be an i-bornivore set of $\boldsymbol{R}^A$ such that $(\lambda_\alpha)\in V$ implies $(\lambda_\alpha x_\alpha)\in U$. As remarked above, our hypothesis insures that $\boldsymbol{R}^A$ is i-bornological. Hence V is a neighborhood of zero in $\boldsymbol{R}^A$, and thus there exist neighborhoods S_α of zero in $\boldsymbol{R}$ and a finite subset A_1 of A such that $\alpha\notin A_1$ implies $S_\alpha=\boldsymbol{R}$ and $V\supseteq\prod_\alpha S_\alpha$. As A_1 is finite, some α_j is not in A_1, and hence $S_{\alpha_j}=\boldsymbol{R}$. Then for any positive integer n, $(n_\alpha)\in V$ where $n_{\alpha_j}=n$, $n_\alpha=0$ for $\alpha\neq\alpha_j$. This implies $(n_\alpha x_\alpha) =I_{\alpha_j}(nx_{\alpha_j})\in U$ for all n, and so $|u(I_{\alpha_j}(nx_{\alpha_j}))| \leqq 1$, i.e., $|u(I_{\alpha_j}(x_{\alpha_j}))| \leqq 1/n$ for all n. This implies $u(I_{\alpha_j}(x_{\alpha_j}))=0$, a contradiction.

Let $H=[(x_\alpha)\in E \mid$ for all but a finite number of $\alpha\in A$, $x_\alpha=0]$.

LEMMA 3. *The restriction of u to H is continuous.*

Proof. Let $H_1=\prod_{\alpha\in A_1}E_\alpha$, $H_2=(\prod_{\alpha\in A-A_1}E_\alpha)\cap H$. H is clearly the topological direct sum of H_1 and H_2, and by Lemma 2, $u=0$ on H_2. Also, as u is i-bornological on E, the restriction of u to H_1 is i-bornological. Hence by Proposition 11, the restriction of u to H_1 is continuous, and therefore by the preceding u is continuous on H.

LEMMA 4. *u is continuous.*

Proof. By Lemma 3 the restriction of u to the dense ideal H of E is continuous. If u is not continuous, let u_1 be the unique continuous linear form on E coinciding with u on H, and let $v=u-u_1$. As u is i-bornological and u_1 continuous, v is clearly i-bornological. Also $v=0$ on H, but $v\neq 0$. Let $(z_\alpha)\in E$ be such that $v((z_\alpha))\neq 0$. Let w be the linear form on $\boldsymbol{R}^A$ defined by $w((\lambda_\alpha)) =v((\lambda_\alpha z_\alpha))$. Then $w=0$ on a dense ideal of $\boldsymbol{R}^A$, but $w\neq 0$. Hence w is not continuous. Since $\boldsymbol{R}^A$ is i-bornological, we shall arrive at a contradiction if we show w is i-bornological. Given $\beta>0$, let U be an i-bornivore set contained in $[(x_\alpha)\in E \mid |v((x_\alpha))| \leqq \beta]$. By Lemma 1, there exists an i-bornivore set V in $\boldsymbol{R}^A$ such that $(\lambda_\alpha)\in V$ implies $(\lambda_\alpha z_\alpha)\in U$. But then for any $(\lambda_\alpha)\in V$, $|w((\lambda_\alpha))| = |v((\lambda_\alpha z_\alpha))| \leqq \beta$. Hence w is i-bornological, and the lemma is proved.

It remains to show that if E' is the topological dual of E, the topology of E is $\chi(E, E')$. In view of the characterization of $\chi(E, E')$ given in §3, a slight modification of the proof of the corollary of Theorem 2 of [11, p. 333] to take care of idempotentness yields a proof of the following more general result:

PROPOSITION 9. *Let $\{E_\alpha\}_{\alpha\in A}$ be a family of locally m-convex algebras, E'_α the topological dual of E_α, and let $E=\prod_\alpha E_\alpha$ with topological dual E'. If the topology of E_α is $\chi(E_\alpha, E'_\alpha)$ for all $\alpha\in A$, then the topology of E is $\chi(E, E')$*

Note: If $\{E_\alpha\}_{\alpha\in A}$ is a family of bornological spaces, $H=[(x_\alpha)\in\prod_\alpha E_\alpha \mid x_\alpha =0$ for all but a finite number of $\alpha\in A]$, the proof of Theorem 5 of [11, p. 334] shows that H is bornological. However, if each $E_\alpha=\boldsymbol{R}$ and if A is infinite, H is not i-bornological. For in that case H is simply the algebra of Example 14 of the following section, where the topological space of that example is A equipped with the discrete topology.

6. **P-algebras and metrizable algebras.** An important property of normed algebras, shared by a larger class of locally m-convex algebras, is that those elements whose powers converge to zero form a neighborhood of zero. In this section we shall see that for commutative, metrizable, locally m-convex algebras, this property is closely related with that of being i-bornological.

DEFINITION 9. A locally m-convex algebra E is a *P-algebra* if $[x \mid x^n\to 0]$ is a neighborhood of zero.

EXAMPLE. If T is a locally compact space, the i-bornological algebra $\mathcal{K}(T)$ with the measure topology (Example 7) is both a P-algebra and a Q-algebra. For if $V=[x\in\mathcal{K}(T) \mid |x(t)|<1$ for all $t\in T]$, V is a neighborhood of zero in the topology of uniform convergence and hence also in the stronger measure topology. If $x\in V$, clearly $x^n\to 0$ in the measure topology, and also the function $x/1-x$ is defined, a member of $\mathcal{K}(T)$, and the adverse of x.

By (4) of Proposition 4, any P-algebra is p.i.b. The following proposition shows that the properties of being p.i.b. and of being a P-algebra are actually equivalent under fairly mild restrictions.

PROPOSITION 10. *Let E be a commutative, i-barrelled algebra. Then E is p.i.b. if and only if E is a P-algebra. Further, if E is p.i.b., and, in addition, advertibly complete, then $[x \mid -\sum_{n=1}^\infty x^n$ exists and is the adverse of $x]$ is a neighborhood of zero, and hence E is a Q-algebra. In particular, a commutative, p.i.b. $\mathfrak{F}$-algebra is both a P- and a Q-algebra.*

Proof. Let $V=[x \mid x^n\to 0]$. If E is advertibly complete, we shall show that for any $x \in 2^{-1}V$, $-\sum_{n=1}^{\infty} x^n$ exists and is the adverse of x; the second statement then follows from the first. Let W be any convex, equilibrated neighborhood of zero, $z \in V$. Choose $\lambda > 0$ so that $z^n \in \lambda W$ for all n. Let $s_n = -\sum_{j=1}^{n} (2^{-1}z)^j$. Then $\sum_{j=m+1}^{p} (2^{-1}z)^j \in \sum_{j=m+1}^{p} 2^{-j}\lambda W \subseteq 2^{-m}\lambda W$ for all integers $p > m$. This shows that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Also $(2^{-1}z) \circ s_n = s_n \circ (2^{-1}z) = (2^{-1}z)^{n+1} \to 0$. Hence as E is advertibly complete, the Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ converges to an element w which is clearly the adverse of $2^{-1}z$ by continuity of $\circ$. It remains to prove the first assertion. Suppose E is p.i.b. Let $A = [x \mid \{x^n\}_{n=1}^{\infty}$ is bound$]$. Then A and hence also $4^{-1}\overline{A}$ are absorbing. Since $V \supseteq 2^{-1}A$ by (4) of Proposition 4, it suffices to show that A is convex, equilibrated, and idempotent, and that $2^{-1}\overline{A} \subseteq A$; for then $4^{-1}\overline{A}$ is an i-barrel contained in V. The proof thus reduces to that of the following proposition:

PROPOSITION 11. *If E is commutative and if $A = [x \in E \mid \{x^n\}_{n=1}^{\infty}$ is bound$]$, then A is convex, equilibrated, idempotent, and $2^{-1}\overline{A} \subseteq A$.*

Proof. We let C_j^n be the binomial coefficient $n!/j!(n-j)!$. Let $y, z \in A$, $0 \leqq \alpha \leqq 1$. If V is any convex, equilibrated, idempotent neighborhood of zero, let $\lambda > 0$ be such that for all n, $z^n \in \lambda V$ and $y^n \in \lambda V$. Then $(\alpha y + (1-\alpha)z)^n = \sum_{j=0}^{n} \alpha^{n-j}(1-\alpha)^j C_j^n y^{n-j} z^j \in \sum_{j=0}^{n} \alpha^{n-j}(1-\alpha)^j C_j^n \lambda^2 V = \lambda^2 V$ because of the familiar identity $\sum_{j=0}^{n} \alpha^{n-j}(1-\alpha)^j C_j^n = 1$. Hence $\alpha y + (1-\alpha)z \in A$, so A is convex. Also $\{(yz)^n\}_{n=1}^{\infty} = \{y^n z^n\}_{n=1}^{\infty} \subseteq \{y^n\}_{n=1}^{\infty} \cdot \{z^n\}_{n=1}^{\infty}$, a bound set, so A is idempotent. A is clearly equilibrated. Let $y \in \overline{A}$. If V is a convex, equilibrated, idempotent neighborhood of zero, let $x \in A$ be such that $y - x \in V$, and let $\lambda > 0$ be such that $\{x^n\}_{n=1}^{\infty} \subseteq \lambda V$. Then for any n, $(2^{-1}y)^n = (2^{-1}(y-x) + 2^{-1}x)^n = \sum_{j=0}^{n} 2^{-n} C_j^n (y-x)^{n-j} x^j \in \sum_{j=0}^{n} 2^{-n} C_j^n V \cdot \lambda V = \lambda V^2 \subseteq \lambda V$. Hence $2^{-1}y \in A$.

COROLLARY. *If E is commutative, $H = [x \mid \{x\}$ is i-bound$]$ is a subalgebra.*

Proof. H is clearly closed under multiplication and scalar multiplication. If $\{x^n\}_{n=1}^{\infty}$ and $\{y^n\}_{n=1}^{\infty}$ are bound, then by Proposition 11 $\{(2^{-1}(x+y))^n\}_{n=1}^{\infty}$ is bound, so $\{x+y\}$ is i-bound. It follows at once that H is closed under addition and is thus a subalgebra.

Although every p.i.b. $\mathfrak{F}$-algebra is a P-algebra, the next two examples show that there exist p.i.b. complete algebras and p.i.b. metrizable algebras which are not P-algebras.

EXAMPLE 13. Let T be a locally compact, noncompact, pseudo-compact space (a space is *pseudo-compact* [12] if every continuous real-valued function on it is bound). Then the algebra $\mathfrak{C}(T)$ of all continuous, real-valued functions on T, equipped with the compact-open topology, is p.i.b. and complete but is not a P-algebra. (An example of such a space T is the space of Exercise 22 of [4, p. 113].) The local compactness, pseudo-compactness, and noncompactness of T readily imply that $\mathfrak{C}(T)$ is respectively complete, p.i.b., but not a P-algebra.

The next example also shows that a metrizable, locally m-convex algebra need not be i-bornological. This is in contrast with the fact that every metrizable, locally convex space is bornological (Theorem 10 and Corollary of [15, pp. 527–528]).

EXAMPLE 14. Let T be a completely regular, noncompact space, E the algebra of all continuous real-valued functions on T vanishing outside compact sets, equipped with the compact-open topology $\mathfrak{J}$. Let $\mathfrak{J}'$ be the topology defined by the uniform norm. Then $B=[x\in E \mid |x(t)| \leqq 1 \text{ for all } t\in T]$ is bound and idempotent for both topologies, and also every bound, idempotent set for either topology is contained in B. Hence the i-bound sets for the two topologies coincide. As T is completely regular but not compact, $\mathfrak{J}'$ is strictly stronger than $\mathfrak{J}$. Hence by (IB 2) $E;\mathfrak{J}$ is not i-bornological. $E;\mathfrak{J}$ is clearly p.i.b. but is not a P-algebra, for given any compact $K\subseteq T$, there exists $x\in E$ such that $x=0$ on K but $x(t)>1$ for some $t\in T$; hence the sequence $\{x^n\}_{n=1}^{\infty}$ does not converge to zero. In particular, if T is the countable union of compact sets $\{K_n\}_{n=1}^{\infty}$ such that every compact subset of T is contained in some K_n, E is a metrizable, p.i.b. algebra which is neither a P-algebra nor i-bornological.

PROPOSITION 12. *A commutative, p.i.b., i-bornological algebra E is a P-algebra.*

Proof. By Theorem 1, E is the algebraic inductive limit of normed algebras $\{E_B\}$ such that if S_B is the open unit ball of E_B, $S=\bigcup_B S_B$ is a neighborhood of zero. But if $x\in S_B$, $x^n\to 0$ in normed algebra E_B and hence also for the weaker topology induced on E_B by that of E. Hence $x^n\to 0$ in E, so E is a P-algebra.

We now give criteria for p.i.b. metrizable algebras to be i-bornological.

THEOREM 7. *Let E be a commutative, metrizable, locally m-convex algebra. Then E is p.i.b. and i-bornological if and only if E is a P-algebra.*

Proof. The condition is necessary by Proposition 12. Sufficiency: We shall show that E satisfies (IB 3). Let A be an i-bornivore set, and assume A is not a neighborhood of zero. Let $V=[x \mid x^n\to 0]$, and let $\{V_n\}_{n=1}^{\infty}$ be a fundamental system of idempotent, convex, equilibrated neighborhoods of zero such that for all n, $V\supseteq V_n\supseteq V_{n+1}$. Then $\{n^{-1}V_n\}_{n=1}^{\infty}$ is a fundamental system of neighborhoods of zero, so since A is not a neighborhood of zero, there exists for all n an element $x_n\in n^{-1}V_n$ such that $x_n\notin A$. Let B be the idempotent envelope of $\{nx_n\}_{n=1}^{\infty}$. Elements of B are all of the form $\prod_{m\in S}(mx_m)^{r_m}$ where S is any nonempty finite subset of the positive integers, and where $r_m>0$ for all $m\in S$. We assert that B is bound: Consider any V_p. For any n, $nx_n\in V_n\subseteq V$, so $\lim_{r\to\infty}(nx_n)^r=0$. Hence there exists $r_0\geqq 1$ such that if $r\geqq r_0$, then $(mx_m)^r \in V_p$ for $1\leqq m<p$. Since V_p is absorbing, we may choose $\lambda\geqq 1$ so that if $0\leqq s_m<r_0$ for $1\leqq m<p$ (but with at least one such $s_m>0$), then $\prod_{m=1}^{p-1}(mx_m)^{s_m}$

$\in\lambda V_p$. Now consider any $\prod_{m\in S}(mx_m)^{r_m}\in B$: Let $S_1=[m\in S|1\leqq m<p$ and $r_m<r_0]$, $S_2=[m\in S|1\leqq m<p$ and $r_m\geqq r_0]$, and $S_3=[m\in S|m\geqq p]$. If $S_1\neq\emptyset$, then $\prod_{m\in S_1}(mx_m)^{r_m}\in\lambda V_p$. If $S_2\neq\emptyset$, then $\prod_{m\in S_2}(mx_m)^{r_m}\in V_p$ since V_p is idempotent and since each $(mx_m)^{r_m}\in V_p$ for $m\in S_2$. If $m\in S_3$, $(mx_m)^{r_m}\in V_m^{r_m}\subseteq V_m\subseteq V_p$; hence if $S_3\neq\emptyset$, $\prod_{m\in S_3}(mx_m)^{r_m}\in V_p$. Therefore, since at least one of S_1, S_2, and S_3 is nonempty, since V_p is idempotent and equilibrated and since $\lambda\geqq 1$, we have $\prod_{m\in S}(mx_m)^{r_m}\in\lambda V_p$. Hence B is a bound, idempotent set. But then A must absorb B. But A does not absorb B since for all positive integers n, $nx_n\in B$ whereas $nx_n\notin nA$. Hence A must be a neighborhood of zero, and therefore E is i-bornological.

Commutative, nonmetrizable P-algebras are not necessarily i-bornological, as we shall see later (Example 15).

COROLLARY 1. *If E is a commutative, metric locally m-convex algebra whose metric d satisfies $d(x^n, 0)\leqq d(x, 0)^n$ for all $x\in E$ and all positive integers n, then E is i-bornological.*

Proof. The hypothesis implies that if $d(x, 0)<1$, then $x^n\to 0$.

COROLLARY 2. *Any subalgebra of a commutative, metrizable, p.i.b., i-bornological algebra is i-bornological.*

The algebra E of Example 2 is commutative and metrizable. By induction it is easy to see that for all n, m, $v_m(x^n)\leqq np(x)^{n-1}v_m(x)$; hence if $p(x)<1$, $p(x^n)\to 0$ and $v_m(x^n)\to 0$ for all m, i.e., $x^n\to 0$ in the topology. Hence by Theorem 7, E is p.i.b. and i-bornological.

THEOREM 8. *If E is a commutative, metrizable, p.i.b., i-barrelled algebra (in particular, if E is a commutative, p.i.b. $\mathfrak{F}$-algebra), then E is i-bornological.*

Proof. By Proposition 10, E is a P-algebra, whence by Theorem 7, E is i-bornological.

The author does not know if every $\mathfrak{F}$-algebra is i-bornological.

COROLLARY. *If E is a commutative $\mathfrak{F}$-algebra, then E is the algebraic inductive limit of Banach algebras $\{E_\alpha\}$ with respect to homomorphisms $\{g_\alpha\}$ such that $E=\bigcup_\alpha g_\alpha(E_\alpha)$ if and only if E is p.i.b.*

Proof. The assertion follows from Theorem 1, Theorem 8, and (5) of Proposition 4.

We apply our results to algebras of differentiable functions. The notation is that of Example 3 and [19, p. 15].

LEMMA. *Let $p=(p_1, \cdots, p_m)$ be an m-tuple of non-negative integers, not all zero, let x be a **K**-valued function on $\mathbf{R}^m$ such that D^px exists and is continuous, and let K be a compact subset of $\mathbf{R}^m$. If $r=\sup\ [|x(t)|\,|t\in K]<1$, then* $\lim_{n\to\infty} N_K^p(x^n)=0$.

Proof. By induction, one may show that

$$D^p x^n = \sum n!(n - |p| + |j|)!^{-1} x^{n-|p|+|j|} h_{p,p-j}((D^q x)_{0<q\leqq p}),$$

the sum being over all m-tuples j such that $j \leqq p$ and $0 \leqq |j| \leqq |p| - 1$, where $h_{p,p-j}$ is a polynomial in $(p_1+1)(p_2+1) \cdots (p_m+1)-1$ indeterminants. The proof is straightforward but extremely tedious, and is therefore left to the interested reader. Now let $k_j = \sup_{t\in K} |h_{p,p-j}((D^q x)_{0<q\leqq p})(t)|$, let $k = \max [k_j | 0 \leqq |j| \leqq |p| - 1, j \leqq p]$, and let $c = (p_1+1)(p_2+1) \cdots (p_m+1)$. Then for $n > |p|$, $N_K^p(x^n) \leqq cn(n-1) \cdots (n-|p|+1)r^{n-|p|}k \leqq kcn^{|p|}r^{n-|p|}$. As $0 \leqq r < 1$, $\lim_{n\to\infty} kcn^{|p|}r^{n-|p|} = 0$, so $\lim_{n\to\infty} N_K^p(x^n) = 0$.

PROPOSITION 13. *If K is the closure of an open, relatively compact subset of $\boldsymbol{R}^m$, $\mathcal{E}_K$* (Example 3) *is a commutative, metrizable P-algebra and hence is i-bornological.*

Proof. By the lemma, if $N_K^0(x) < 1$, $N_K^p(x^n) \to 0$ for all nonzero m-tuples p. Clearly also $N_K^0(x^n) = N_K^0(x)^n \to 0$. Hence if $N_K^0(x) < 1$, $x^n \to 0$ in $\mathcal{E}_K$, so $\mathcal{E}_K$ is a P-algebra.

PROPOSITION 14. *If P is an open subset of R^m, $\mathfrak{D}_P$ (Example 10) is a p.i.b. i-bornological algebra.*

Proof. As seen in Example 10, $\mathfrak{D}_P$ is the algebraic inductive limit of p.i.b., i-bornological algebras whose union is $\mathfrak{D}_P$. Hence $\mathfrak{D}_P$ is p.i.b. and i-bornological by (5) of Proposition 4 and Proposition 6.

For a commutative, locally m-convex algebra to be p.i.b. and i-bornological it is necessary that it be a P-algebra by Proposition 12, and for metrizable algebras this condition is also sufficient. It is not, in general, sufficient for nonmetrizable commutative algebras, as the following example shows.

EXAMPLE 15. Let E be the algebra of all bound real-valued functions on $\boldsymbol{R}$ having continuous first derivatives. Let $N(x) = \sup [|x(t)| \,|\, t \in \boldsymbol{R}]$, and for every compact set K let $N_K(x) = \sup [|(Dx)(t)| \,|\, t \in K]$. Then E, equipped with the topology $\mathfrak{J}$ defined by the pseudo-norms N and $\{N_K | K$ compact and countable$\}$ is a commutative P-algebra but is not i-bornological. Proof: Let $\mathfrak{J}'$ be the topology defined by pseudo-norms N and $\{N_K | K$ compact$\}$. Both $\mathfrak{J}$ and $\mathfrak{J}'$ are locally m-convex topologies by an argument similar to that of Example 3. If $N(x) < 1$, by the lemma $N_K(x^n) \to 0$ for all compact sets, and also $N(x^n) \to 0$, and therefore $x^n \to 0$ in both topologies. Hence $E;\mathfrak{J}$ is a P-algebra. $\mathfrak{J}'$ is clearly strictly stronger than $\mathfrak{J}$, so a bound, idempotent set of $E;\mathfrak{J}'$ is bound and idempotent in $E;\mathfrak{J}$. Conversely, let B be bound and idempotent in $E;\mathfrak{J}$. If B is not bound in $E;\mathfrak{J}'$, for some compact $K N_K(B)$ is not bound, i.e., there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq B$ such that $N_K(x_n) > n$. Choose $t_n \in K$ so that $|(Dx_n)(t_n)| > n$. As $\{t_n\}_{n=1}^\infty \subseteq K$, we may extract a convergent subsequence $\{t_{n_j}\}_{j=1}^\infty$ converging to $s \in K$. Let $L = \{t_{n_j}\}_{j=1}^\infty \cup \{s\}$. L is then a countable, compact set, so $N_L(B)$ is a bound set of $\boldsymbol{R}$. But

$$N_L(x_{n_j}) \geqq |(Dx_{n_j})(t_{n_j})| > n_j \geqq j,$$

so $N_L(B)$ is not bound, a contradiction. Hence $E;\mathfrak{J}$ and $E;\mathfrak{J}'$ have the same i-bound sets. Thus by (IB 2) $E;\mathfrak{J}$ is not i-bornological.

7. **Some unresolved questions.**

1. We have seen that every commutative, p.i.b. $\mathfrak{F}$-algebra is i-bornological. Is every commutative $\mathfrak{F}$-algebra i-bornological? A special case of this question is the following:

2. Let E be the algebra of analytic functions on a domain of $\boldsymbol{C}$, equipped with the compact-open topology. E is then a commutative $\mathfrak{F}$-algebra. Is E i-bornological? If E is the algebra of entire functions, the subalgebra of i-bound elements is the one-dimensional subalgebra of constant functions. Hence by the corollary of Proposition 5, the question for this algebra becomes: Is the compact-open topology the strongest possible locally m-convex topology on the algebra of entire functions? Equivalently, if p is a pseudonorm on this algebra satisfying the multiplicative inequality $p(fg) \leqq p(f)p(g)$ for all entire functions f, g, is p necessarily continuous for the compact-open topology?

3. If E is a locally m-convex algebra with topological dual E', do $\tau(E, E')$ and $\chi(E, E')$ necessarily coincide?

4. Is a commutative, metrizable, p.i.b. Q-algebra necessarily a P-algebra?

5. Can the hypothesis concerning identities be removed from Theorem 6?

6. Is an i-bornological locally m-convex algebra necessarily bornological? Is a bound linear form on a locally m-convex algebra necessarily i-bornological?

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Harvard University,
Cambridge, Mass.
Duke University,
Durham, N. C.