ON THE CONFORMAL MAPPING OF MULTIPLY CONNECTED REGIONS(1)

BY

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It is the object of this paper to set forth a proof of Theorem 1 below, to the effect that an arbitrary plane region D bounded by a finite number of mutually disjoint Jordan curves can be mapped one to one and conformally onto a region Δ bounded by two level loci of a function which is the product of linear factors with exponents (positive and negative) not necessarily rational; the boundary curves of D can be divided arbitrarily into two classes, which correspond respectively to the two level loci bounding Δ. We establish also a limiting case (Theorem 3) of Theorem 1, in which such an arbitrary region D can be mapped one to one and conformally onto a region Δ bounded by a single level locus of a function of the kind already mentioned, where arbitrary points interior to D can be made to correspond to the zeros of that function. Theorem 1 includes the classical theorem on the mapping of a region D bounded by two disjoint Jordan curves onto a circular annulus, and Theorem 3 includes the Riemann mapping theorem on the mapping of a Jordan region. In both Theorem 1 and Theorem 3 the maps are essentially unique (Theorems 2 and 4).

Both Theorems 1 and 3 present new canonical maps for multiply connected regions, maps that are especially useful in the study of level loci of various functions, namely Green's functions, linear combinations of Green's functions, and harmonic measures—and in the study of the orthogonal trajectories of these loci. It is desirable to develop methods for the effective determination of these maps. The maps are useful certainly in the study of approximation [Walsh, 1955], although the applications have not yet been carried as far as possible, and the maps are presumably useful also in other parts of analysis.

De la Vallée Poussin [1930] and later Julia [1934] have studied the conformal mapping of an arbitrary multiply connected region onto a region bounded by the whole or (more commonly) parts of one or several lemniscates, namely level loci of a polynomial. Julia uses also the level loci of certain rational functions, especially those with fundamental circles, not considered here. De la Vallée Poussin states [1931] our Theorem 1 for the case $\nu=1$, namely the case that the boundary curves of D are divided into two classes

Received by the editors September 12, 1955.

(1) This research was done under the auspices of the U. S. Air Force, Office of Scientific Research of the Air Research and Development Command.

one of which contains but a single curve; however, his proof refers for details back to his preceding papers [1930, 1930a] which are not without flaw [de la Vallée Poussin, 1931a], so even for his case a new treatment would seem to be appropriate.

The present paper is the elaboration of two notes [Walsh, 1954] which contain the outlines of the proofs of Theorems 1 and 3, but without indication of Theorems 2 and 4.

I

THEOREM 1. *Let D be a region of the extended z-plane whose boundary consists of mutually disjoint Jordan curves $B_1, B_2, \cdots, B_\mu$; $C_1, C_2, \cdots, C_\nu$, $\mu\nu \neq 0$. There exists a conformal map of D onto a region Δ of the extended Z-plane, one to one and continuous in the closures of the two regions, where Δ is defined by*

$$(1) \quad 1 < |T(Z)| < e^{1/\tau}, \qquad T(Z) \equiv \frac{A(Z-a_1)^{M_1}(Z-a_2)^{M_2}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}(Z-b_2)^{N_2}\cdots(Z-b_\nu)^{N_\nu}},$$
$$\sum M_j = \sum N_j = 1, \ \tau > 0.$$

The exponents M_j and N_j are positive but need not be rational. The locus $|T(Z)| = 1$ consists of μ mutually disjoint Jordan curves, respective images of the B_j, which separate Δ from the a_j; the locus $|T(Z)| = e^{1/\tau}$ consists of ν mutually disjoint Jordan curves, respective images of the C_j, which separate Δ from the b_j.

We have written $T(Z)$ in the form where all the a_j and b_j are finite, but it is evident that an arbitrary linear transformation of the Z-plane can be made, so that in particular an a_j or a b_j may be infinite. In such a case, the form of $T(Z)$ is to be modified by simply omitting the factor corresponding to the point at infinity. In the sequel we use the form of $T(Z)$ and of similar functions as given in (1), with the tacit understanding that a zero or infinity at the infinite point is not excluded, even though the form in (1) then requires modification.

We postpone treatment of the case $\mu = \nu = 1$, for which the conclusion is classical, and suppose $\mu \geqq 2$, which involves if necessary interchanging the roles of the B_j and the C_j. We take also the Jordan curves B_j and C_j analytic, which is possible by a preliminary conformal transformation, and we suppose that D lies interior to C_1.

The function $T(Z)$ is transcendental, if the exponents are irrational. As a first part of the proof, we now show that D can be approximated by a region bounded by two level loci of a *rational* function.

There exists a unique function $u(z)$ harmonic in D, continuous in the closure $\overline{D}$ of D, equal to zero and unity on the B_j and C_j respectively. Then $u(z)$ is harmonic also slightly beyond the curves B_j and C_j, let us say in the closure of a finite region D' which contains $\overline{D}$ and whose boundary consists of $\mu+\nu$ analytic Jordan curves, respectively $B_1', B_2', \cdots, B_\mu', C_1', C_2', \cdots, C_\nu'$

near the B_j and C_j. We can choose this boundary as $B' = \sum B_j' : u(z) = -\delta_1 (<0)$ and $C' = \sum C_j' : u(z) = -\delta = 1 + \delta_1$. The conjugate $v(z)$ of $u(z)$ is not single valued in D'; nevertheless Green's formula valid for z in D' can be written [Walsh, 1935, §§8.7, 9.12; 1950, §§7.1, 8.1]

$$
(2) \qquad \begin{gathered} u(z) \equiv \int_0^\tau \log |z - t| \, d\sigma_t - \int_\tau^{2\tau} \log |z - t| \, d\sigma_t - \delta, \\ d\sigma = |dv| / 2\pi, \qquad \tau = \int_{B'} d\sigma = \int_{C'} d\sigma > 0. \end{gathered}
$$

The integrals in (2) are to be taken over B' and C' respectively. The number τ, a kind of modulus of D, and a conformal invariant, does not depend on δ_1. We can suppose that the derivative of the function $u(z) + iv(z)$ does not vanish in the closure of the point sets $-\delta_1 < u(z) < 0$, $1 < u(z) < -\delta$ in D'. Then on the boundary of D' we have $\partial u / \partial s = 0$, $\partial v / \partial s \neq 0$; thus $v(z)$ is monotonic on B' and on C'. On the boundary of D', if η denotes interior normal we have $(\partial u / \partial \eta) ds = -(\partial v / \partial s) ds = -dv$; on B' we have $\partial u / \partial \eta > 0$, and on C' we have $\partial u / \partial \eta < 0$.

We express the integrals in (2) as the limits of their Riemann sums; let the α_k and β_k, depending on n, divide B' and C' respectively in n equal parts—equal with respect to the parameter σ, not necessarily equal with respect to arc length. For z on any compact in D' we have uniformly

$$
u_n(z) \equiv \frac{\tau}{n} \sum_1^n \log |z - \alpha_k| - \frac{\tau}{n} \sum_1^n \log |z - \beta_k| - \delta \to u(z);
$$

the uniformity of convergence is a consequence of the uniform continuity of the functions involved, which in turn is a consequence of the boundedness of those harmonic functions.

It follows that *D is approached by D_n: $0 < u_n(z) < 1$, a region bounded by μ Jordan curves forming one level locus and by ν Jordan curves forming another level locus of a rational function*(²), approached in the sense that if $\mu + \nu$ arbitrary mutually disjoint annular regions (neighborhoods of the B_j and C_j) are given containing the B_j and C_j respectively, then for n sufficiently large the $\mu + \nu$ bounding curves of D_n lie in, and separate the boundary curves of, these annular regions. For instance let us choose as a neighborhood of B_1 an annular region α in D' bounded by one Jordan curve of each of the loci

(²) We have here a general result, that arbitrary finite sets of Jordan curves B_j, C_j bounding a region D can be approximated by respective level loci of a rational function whose zeros and poles lie exterior to D; the analyticity of the curves B_j and C_j is not essential. The limiting case where $\mu = \nu = 1$ and C_1 reduces to the point at infinity was established by Hilbert in 1897; the rational function is a polynomial. The present method is a generalization of that of Hilbert, involving rational functions more general than polynomials, and has been used a number of times by the present writer (loc. cit.).

$u(z)=\delta_2$ and $u(z)=-\delta_2$, $0<\delta_2<\delta_1<1/2$. For n sufficiently large we have in the closure of α the inequality $|u(z)-u_n(z)|<\delta_2/2$, hence on these respective curves $\delta_2/2<u_n(z)<3\delta_2/2$, $-3\delta_2/2<u_n(z)<-\delta_2/2$. The locus $u_n(z)=0$ obviously separates these two curves, that locus cuts neither curve, and (by the principle of maximum for harmonic functions) that locus consists in α of a single analytic Jordan curve. Application of a similar argument simultaneously for all curves B_j, C_j, shows that for n sufficiently large the locus $u_n(z)=0$ consists of μ curves respectively near the B_j, and the locus $u_n(z)=1$ consists of ν curves respectively near the C_j; these loci can contain no other points; for instance, in the interior of the curve $u_n(z)=0$ which lies near B_1 there lie certain points α_k but no points β_k, so we have $u_n(z)<0$ throughout that interior, again by the principle of maximum for harmonic functions. Each compact K in D lies in all the D_n for n sufficiently large (n depending on K), and each compact K exterior to D lies exterior to all the D_n for n sufficiently large (n depending on K).

The region D_n is defined by the inequalities

$$(3)\qquad 1<|R_n(z)|<e^{n/\tau},\qquad R_n(z)\equiv e^{-n\delta/\tau}\prod_1^n\frac{z-\alpha_k}{z-\beta_k}\equiv e^{n(u_n+iv_n)/\tau},$$

where $v_n(z)$ is conjugate to $u_n(z)$ in D'; thus D_n is bounded by two level loci of a *rational* function. As a second step in the proof, we proceed to show that D_n can be mapped onto a plane region bounded by two level loci of a simpler rational function likewise of degree n, having precisely μ distinct zeros and ν distinct poles.

The equation $w=R_n(z)$ defines the conformal map of the extended z-plane onto a Riemann surface σ_0 of n sheets over the extended w-plane, and the image of D_n is the totality of points w of σ_0 satisfying $1<|w|<e^{n/\tau}$, a connected set bounded wholly by points of the circles $|w|=1$ and $|w|=e^{n/\tau}$.

LEMMA 1. *If the function $f(z)$ is analytic except for precisely p poles and if $f(z)\neq 0$ in the closure of a bounded region E bounded by a finite number of Jordan curves, and if we have $|f(z)|=m$ on the entire boundary of E, then $f(z)$ takes every value w_0, $|w_0|>m$, precisely p times in E. The image of E under the transformation $w=f(z)$ is the portion $|w|>m$ of the extended w-plane covered precisely p times.*

When z traces the boundary of E, arg $[f(z)-w_0]$ for $|w_0|>m$ has zero net increase, but when z traces small circles about the respective poles of $f(z)$, on each of which we have $|f(z)|>|w_0|$, then arg $[f(z)-w_0]$ increases in totality $2\pi p$, so $f(z)$ takes on the value w_0 in E precisely p times. By the principle of maximum modulus, $f(z)$ takes on no value of modulus less than or equal to m in E, so the lemma is established. It is a corollary that *if $f(z)$ is analytic in such a region E and has there precisely p zeros, and if we have $|f(z)|=m$ on the entire boundary of E, then $f(z)$ takes every value w_0, $|w_0|<m$,*

precisely p times in E. The image of E under the transformation $w=f(z)$ is the portion $|w|<m$ of the w-plane covered precisely p times. For the proof, we merely apply the lemma to the function $1/f(z)$.

If E in Lemma 1 is bounded by a finite number of analytic Jordan curves, and if $f(z)$ is still analytic on the boundary, we can apply Lemma 1 to a region E_1 containing the closure $\overline{E}$ of E, E_1 being bounded by a finite number of Jordan curves on which we have $|f(z)|=m_1$. We must have $m_1<m$, for the boundary of E lies interior to E_1. The entire locus $|f(z)|=m$ in E_1 is the boundary of E, which therefore under the transformation $w=f(z)$ covers the circle $|w|=m$ precisely p times.

For n sufficiently large, the complement of D_n (with respect to the extended z-plane) consists of μ regions containing respectively, let us say, $m_1, m_2, \cdots, m_\mu$ points α_j, and ν regions containing respectively $n_1, n_2, \cdots, n_\nu$ points β_j; the m_k and n_k depend on n. It is easily proved that we have $(\sum m_j=\sum n_j=n)$.

$$(4)\quad \frac{m_j}{n}\to\frac{1}{\tau}\int_{B'_j}d\sigma=M_j,\qquad \frac{n_j}{n}\to\frac{1}{\tau}\int_{C'_j}d\sigma=N_j,\qquad \sum M_j=\sum N_j=1;$$

equations (4) define the M_j and N_j and the integrals may be taken over the B_j and C_j instead of the B'_j and C'_j.

The $\mu+\nu$ regions of the z-plane complementary to D_n are simply connected and are mapped by $w=R_n(z)$ onto μ simply connected subregions of σ_0 covering the disc $|w|<1$ precisely m_j times and ν subregions of σ_0 covering the region $|w|>e^{n/\tau}$ precisely n_j times, by Lemma 1. Since $u(z)+iv(z)$ has no critical points on or near the boundary of D, neither does $R_n(z)$, and the boundaries of these $\mu+\nu$ subregions of σ_0 consist of circles traced monotonically m_j and n_j times. Indeed, on these boundaries arg $w=nv_n/\tau$ varies monotonically.

Following a method used by Julia [1934, p. 82], we now form a new Riemann surface σ_1 by replacing the jth one of the μ subregions of σ_0 by a subregion containing m_j sheets with a single branch point $w=0$ of the Riemann surface for the inverse of $w=z^{m_j}$ covering the region $|w|<1$, and replacing the jth one of the ν subregions of σ_0 by a subregion containing n_j sheets with a single branch point $w=\infty$ of the Riemann surface for the inverse of $w=z^{n_j}$ covering $|w|>e^{n/\tau}$. This replacement can be made continuously along the boundaries $|w|=1$ and $|w|=e^{n/\tau}$ so that σ_1 is smooth in each of its n sheets above these boundaries, except of course for the branch lines. Since σ_0 is the image of the extended plane, and since each of the $\mu+\nu$ subregions of σ_0 is simply connected and replaced continuously by a simply connected region, also σ_1 is topologically the image of the extended plane, and (Schwarz) can be mapped conformally and one to one onto the extended Z-plane. Since σ_1 covers each point of the extended w-plane precisely n times, this mapping function is rational of degree n, necessarily of the form

$$w = S_n(Z) \equiv \frac{A_n(Z - a_1')^{m_1}(Z - a_2')^{m_2} \cdots (Z - a_\mu')^{m_\mu}}{(Z - b_1')^{n_1}(Z - b_2')^{n_2} \cdots (Z - b_\nu')^{n_\nu}}, \qquad \sum m_j = \sum n_j = n;$$

the a_j and b_j (depending on n) are distinct, exterior to the image $\Delta_n: 1 < |S_n(Z)| < e^{n/\tau}$ of D_n in the Z-plane.

The great advantage of replacing σ_0 by σ_1 is this relatively simple form of the function $S_n(Z)$ which defines the region Δ_n. We have now mapped the region D_n approximating D onto Δ_n. It remains to allow n to become infinite, and by studying the conformal maps of these variable regions D_n and Δ_n complete the proof of Theorem 1.

By a suitable linear transformation of the Z-plane we may choose $a_1' = 0$, $a_2' = 1$, $b_1' = \infty$ independent of n. As $n \to \infty$ there exists a partial sequence of n such that all the numbers $A_n^{1/n}$, a_j', b_j' approach limits A, a_j, b_j; only this partial sequence of the n is to be considered henceforth; by virtue of (4) the inequalities which define Δ_n approach the form (1), defining a region Δ. We proceed to show that D can be mapped onto Δ.

The transformation $R_n(z) = S_n(Z)$ of D_n onto Δ_n can be written $Z = Z_n(z)$. The functions $Z_n(z)$ admit in D_n the exceptional values 0, 1, ∞, hence even if not defined throughout D form a normal family in every closed subregion of D and also in D. Henceforth we consider only a subsequence of values n such that the $Z_n(z)$ uniformly approach a limit $Z_0(z)$ on every compact in D. This function $Z_0(z)$ cannot be identically constant; for instance if we have $Z_0(z) \equiv g \neq 0$, we choose a Jordan curve Γ in D near B_1 and surrounding B_1. The image of Γ under $Z = Z_n(z)$ surrounds $Z = 0$, whence

$$[\arg Z_n(z)]_\Gamma = 2\pi,$$

which contradicts $Z_n(z) \to g$ uniformly on Γ. Similarly (by use of an auxiliary linear transformation) $Z_n(z) \to \infty$ in D is seen to be impossible. Then the function $Z = Z_0(z)$ is univalent in D, and maps D onto some region Δ_0 of the Z-plane. The region Δ_0 like D is bounded by $\mu + \nu$ mutually disjoint continua, the respective images of the B_j and C_j. If Γ_j is an analytic Jordan curve in D near B_j and surrounding B_j, we have

$$(5) \qquad \frac{1}{2\pi i} \int_{\Gamma_j} \frac{Z_n'(z)dz}{Z_n(z) - a_k'} = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

and this integral approaches the value

$$\frac{1}{2\pi i} \int_{\Gamma_j} \frac{Z_0'(z)dz}{Z_0(z) - a_k} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

It is to be noted that $Z_0(z)$ does not assume the value a_k interior to D, as we shall shortly prove (Lemma 2), so the denominator of the integrand in (5) is bounded from zero. Consequently the image of Γ_j under $Z = Z_0(z)$ contains a_j in its interior but contains none of the points a_k with $k \neq j$. Similar reasoning regarding the points b_k shows that the points a_j and b_k are all distinct, that none lies in Δ_0, and that a Jordan curve in D separating B_j or C_j from all the other curves B_k and C_k has for its image in Δ_0 a Jordan curve which separates the image of B_j or C_j from the images of all the other curves B_k and C_k.

In our further detailed study of the mapping of variable regions, we make use of the ideas of Carathéodory [1932, §§120–123]; compare also Bieberbach [1931, p. 13].

LEMMA 2. *Let the functions $f_n(z)$ analytic and univalent in a region D^0 converge uniformly on every compact in D^0 to a function $f_0(z)$ which is not identically constant and therefore univalent. Let $w=f_0(z)$ map D^0 onto a region Δ_0 of the w-plane, and let Δ_0' be a closed subregion of Δ_0. Then for n sufficiently large, the image of D^0 by $w=f_n(z)$ covers Δ_0'.*

Hurwitz's theorem assures us that if w_0 is any point in Δ_0, then for n sufficiently large every $f_n(z)$ takes the value w_0 in D^0; Lemma 2 refers to *uniformity* with respect to all w_0 in Δ_0'.

Let Γ be a Jordan curve or a sum of a finite number of Jordan curves in D^0 whose image under the map $w=f_0(z)$ lies in $\Delta_0-\Delta_0'$ and separates Δ_0' from the boundary of Δ_0. For z on Γ and for all n sufficiently large we have uniformly for all w_0 in Δ_0'.

$$\left|\frac{f_n(z) - f_0(z)}{f_0(z) - w_0}\right| < 1,$$

so by Rouché's theorem $f_n(z)-w_0$ has the same number of zeros in the region bounded by Γ as does $f_0(z)-w_0$, namely one. Lemma 2 is established.

The functions $Z_n(z)$ are not necessarily defined throughout the whole region D of Theorem 1, but are defined (for n sufficiently large) on every closed subregion. The function $Z_0(z)$ cannot assume the value a_k interior to D, for if it did it would assume interior to D all values in a fixed neighborhood of a_k, hence by Lemma 2 also $Z_n(z)$ would assume for n sufficiently large in a closed subregion of D all values in a neighborhood of a_k, in particular $Z_n(z)$ would assume there the value a_k', contrary to our definition of Δ_n.

All the regions Δ_n for n sufficiently large contain the image of an arbitrary closed subregion of D under the transformation $Z=Z_0(z)$. The distinct points a_j' and b_j' used to define Δ_n approach distinct limits a_j and b_j, so the regions Δ_n approach their kernel, the region Δ defined by (1); of course the function $|S_n(Z)|^{1/n}$ for fixed Z is a continuous function of the variables $A_n^{1/n}$, a_j', b_j', m_j, n_j. The term *kernel* more precisely consists of the neighborhood of a specific point Z_1 plus fixed closed regions containing that neighborhood which lie in all the Δ_n for n sufficiently large. In the present case, any point Z_1 of Δ can be chosen, for two arbitrary points Z_1 and Z_2 of Δ lie in some closed region which is contained in all the Δ_n for n sufficiently large. The function $Z=Z_0(z)$ maps D onto some subregion Δ_0 of Δ.

We turn now to the consideration of the inverses $z=z_n(Z)$ of the functions $Z=Z_n(z)$. The $z_n(Z)$ are univalent in the respective regions Δ_n, hence for sufficiently large n are univalent in an arbitrary closed subregion of Δ. The functional values $z=z_n(Z)$ lie in D_n, hence are bounded, so in Δ the sequence forms a normal family; henceforth we consider only a subsequence which converges to some function $\zeta(Z)$ in Δ, uniformly in any closed subregion of Δ. The function $\zeta(Z)$ is either identically constant or univalent in Δ. If z_0 is any fixed point of D, we write

$$Z_n(z_0) = Z_n \to Z_0(z_0), \qquad z_n(Z_n) = z_0, \tag{6}$$

and the points Z_n lie in Δ_0, for sufficiently large index, because $Z_0(z_0)$ lies in Δ_0. Since the Z_n lie in Δ_0 for n sufficiently large, and since they approach $Z_0(z_0)$ in Δ_0, we have, by (6), $z_n(Z_n)=z_n[Z_n(z_0)]=z_0\to\zeta[Z_0(z_0)]$. That is to say, $\zeta(Z)$ is the inverse function of $Z_0(z)$ for z in D and Z in Δ_0. By a similar argument it follows that $Z_0(z)$ is the inverse function of $\zeta(Z)$, for Z in Δ and for z in a suitable subregion of D, namely the image of Δ under the transformation $z=\zeta(Z)$; compare Lemma 2.

The transformation $Z=Z_0(z)$ maps D onto a subregion Δ_0 of Δ, but every value Z_1 in Δ is taken on by $Z_0(z)$ at some point $z_1=\zeta(Z_1)$ in D. Since $Z_0(z)$ is univalent in D, it follows that $Z=Z_0(z)$ maps D conformally and one to one onto Δ.

The well known studies of the boundary behavior of the conformal mapping of Jordan regions, due to Carathéodory and Montel, establishing continuity and one-to-oneness in the closed regions, apply with no essential change in the present conditions, and show that the conformal map of D onto Δ can be extended so as to be one to one and continuous in the closed regions involved. Theorem 1 is established.

The proof of Theorem 1 as given does not include the classical case $\mu=\nu=1$, but the construction of the Riemann surface σ_1 is valid, and the transformation $R_n(z)=S_n(Z)$ maps D_n onto Δ_n, which may be taken as $1<|S_n(Z)|<e^{n/\tau}$,

$$S_n(Z) \equiv \frac{A_n(Z - a_1')^n}{(Z - b_1')^n},$$

which with the choice $a_1'=0$, $b_1'=\infty$ becomes $S_n(Z)\equiv A_nZ^n$. Then Δ_n can be expressed as

$$1 < |A_nZ^n| < e^{n/\tau};$$

two such regions for different values of A_n can be mapped onto each other by a dilatation with the origin fixed, so finally we choose Δ_n as $1<|Z|<e^{1/\tau}$. This region is independent of n, and the remainder of the proof of Theorem 1, continued study of the transformation $R_n(z)=S_n(Z)\equiv Z^n$, can be carried through even more simply than in the general case.

Indeed, the case $\mu=\nu=1$ is included in a larger category which, as we now show, can be treated still more simply. Here is included for instance also the case $\mu=2$, $\nu=1$, $M_1=M_2$, for which it is sufficient that B_1 and B_2 be mutually symmetric in a line L in which C_1 is symmetric.

If the numbers M_j and N_j of Theorem 1 defined by (4) are all rational, the region Δ is bounded by two level loci of a *rational* function. For if the integer N is suitably chosen, the numbers NM_j and NN_j are all integers, $[T(Z)]^N$ is a rational function of Z of degree N, and (1) can be written

$$1 < \left| [T(Z)]^N \right| < e^{N/\tau}. \tag{1'}$$

For this case Theorem 1 can be established without use of the two infinite sequences of regions D_n and Δ_n. Since the numbers

$$NM_j = \frac{N}{\tau}\int_{B_j} d\sigma, \qquad NN_j = \frac{N}{\tau}\int_{C_j} d\sigma,$$

are integers, the increase in $Nv(z)/2\pi\tau$ as z traces any one of the curves B_j, C_j is integral. Then the function $w=e^{N(u+iv)/\tau}$ is single valued in D and maps D conformally onto part of a Riemann surface σ_0; the image D_0 of D is the annulus $1<|w|<e^{N/\tau}$ covered precisely N times. We use here an extension of Lemma 1, which concerns an arbitrary function $f(z)$ analytic in a closed multiply connected region E bounded by a finite number of mutually disjoint Jordan curves, and we suppose $|f(z)|=m_0$ on several components P_1 of the boundary of E, $|f(z)|=m_1(>m_0)$ on the remaining part P_2 of the boundary, $f(z)\neq 0$ in E; it follows that each value w, $m_0<|w|<m_1$, is assumed by $f(z)$ in E the same number of times, namely the quotient by 2π of the increase in $\arg f(z)$ over P_1. We return to the region D; as z traces $\sum C_j$, $\arg e^{N(u+iv)/\tau}$ is increased by precisely $2\pi N$. Moreover, as z traces each of the curves B_j or C_j, $\arg e^{N(u+iv)/\tau}$ changes monotonically and is increased by precisely $-2\pi NM_j$ or $2\pi NN_j$, so w traces each of the $\mu+\nu$ boundary curves $|w|=1$ or $|w|=e^{N/\tau}$ monotonically $-NM_j$ or NN_j times. The Riemann surface σ_0 contains the image not merely of $\overline{D}$ but even of D', but is not known to cover the entire extended w-plane. Nevertheless continuous adjunction to D_0 of $\mu+\nu$ closed simply connected subregions $|w|\leqq 1$ and $|w|\geqq e^{N/\tau}$ of the Riemann surfaces for the inverses of $w=z^{NM_j}$ and $w=z^{NN_j}$ constructs a new Riemann surface σ_1 over the entire extended w-plane which is the topological image of the extended z-plane, namely D plus the closures of $\mu+\nu$ Jordan regions exterior to D whose boundaries are respectively the B_j and C_j. Schwarz's theorem now asserts that σ_1 can be mapped one to one and conformally onto the extended Z-plane. The mapping function is of the form $w=[T(Z)]^N$, and the one to one conformal image of D is the region (1′).

The proof just given is especially simple in the case $\mu=\nu=1$, for then $N=1$ and $T(Z)$ is of degree one; the extended w-plane and extended Z-plane may be taken as identical. The proof is also simple in the case $\mu=2$, $\nu=1$, $M_1=M_2$, for then $N=2$ and we may choose for instance $[T(Z)]^2\equiv A_0(Z^2-1)$.

II

It is evident that the region Δ of Theorem 1 is not uniquely determined, for one can still transform the Z-plane by an arbitrary linear transformation. Indeed, any possible region Δ, an image of D, may be found from a single such region Δ by a linear transformation:

THEOREM 2. *Let D be a region of the z-plane defined by the inequality*

$$1 < |R(z)| < e^{1/\tau}, \qquad R(z) \equiv \frac{A(z-a_1)^{m_1}\cdots(z-a_\mu)^{m_\mu}}{(z-b_1)^{n_1}\cdots(z-b_\nu)^{n_\nu}},$$

$$\sum m_j = \sum n_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $B_1, B_2, \cdots, B_\mu, C_1, C_2, \cdots, C_\nu$, where B_j separates a_j from D and C_j separates b_j from D. Let Δ be a region of the Z-plane defined by

$$1 < |R_1(Z)| < e^{1/\tau'}, \qquad R_1(Z) \equiv \frac{A'(Z-a_1')^{m_1'}\cdots(Z-a_\mu')^{m_\mu'}}{(Z-b_1')^{n_1'}\cdots(Z-b_\nu')^{n_\nu'}},$$

$\sum m_j' = \sum n_j' = 1$, *and whose boundary consists of mutually disjoint Jordan curves $B_1', B_2', \cdots, B_\mu', C_1', C_2', \cdots, C_\nu'$, where B_j' separates a_j' from Δ and C_j' separates b_j' from Δ. If there exists a one to one conformal transformation of D onto Δ so that each B_j corresponds to B_j' and each C_j corresponds to C_j', then the transformation is a linear transformation of the complex variable z, defined throughout the extended planes z and Z. We have $\tau=\tau'$, $m_j=m_j'$, $n_j=n_j'$. If* arg A' *is suitably chosen, we have $R(z)\equiv R_1(Z)$.*

If D is an arbitrary region whose boundary consists of $\mu+\nu$ Jordan curves, $\mu+\nu>2$, those curves can be separated into two classes containing respectively μ and ν curves in a wide variety of ways, and so also with Δ; compare Theorem 1. Theorem 2 is not concerned with all possible maps of D onto Δ, but with merely a certain map where we suppose that two classes of curves bounding D correspond respectively to two classes of curves bounding Δ.

If there exists a one-to-one conformal transformation of D onto Δ, that transformation can be extended so as to be one-to-one and continuous in the closed regions, so it is possible to consider the correspondence of boundary curves. Indeed, the Jordan curves bounding both D and Δ are analytic, so the transformation is even one to one and analytic in larger regions D' and Δ' containing respectively the closures $\overline{D}$ and $\overline{\Delta}$ of D and Δ.

For the present we suppose that D and Δ contain respectively the points $z=\infty$ and $Z=\infty$. Let $U(z)$ be the unique function harmonic in D, continuous in $\overline{D}$, equal to zero and unity on $B=\sum B_j$ and $C=\sum C_j$ respectively. Similarly let $U'(Z)$ be the unique function harmonic in Δ, continuous in $\overline{\Delta}$, equal to zero and unity on $B'=\sum B_j'$ and $C'=\sum C_j'$ respectively. Let $V(z)$ and $V'(Z)$ be conjugate to $U(z)$ and $U'(Z)$. If the given transformation is $Z=Z(z)$, we have $U'[Z(z)] \equiv U(z)$, $V'[Z(z)] \equiv V(z)$, if the determinations of the multiform functions $V(z)$ and $V'(Z)$ are suitably chosen. The function $\log |R(z)|$ is harmonic in D and takes the values zero and $1/\tau$ on B and C, so for z in D we have $|R(z)| \equiv e^{U(z)/\tau}$, $R(z) \equiv e^{[U(z)+iV(z)]/\tau}$, and similarly for Z in Δ we have $|R_1(Z)| \equiv e^{U'(Z)/\tau'}$, $R_1(Z) \equiv e^{[U'(Z)+iV'(Z)]/\tau'}$, again for suitable determination of $V(z)$ and $V'(Z)$.

The numbers a_j, b_j, a_j', b_j' are all finite; for the positive directions on the boundary curves with respect to D and Δ we have

$$[\arg R(z)]_{B_j} = -2\pi m_j, \qquad [\arg R(z)]_{C_j} = 2\pi n_j,$$
$$[\arg R_1(Z)]_{B_j'} = -2\pi m_j', \qquad [\arg R_1(Z)]_{C_j'} = 2\pi n_j'.$$

On the other hand we have

$$[\arg R(z)]_{B_j} = [V(z)/\tau]_{B_j}, \qquad [\arg R(z)]_{C_j} = [V(z)/\tau]_{C_j},$$

with similar equations for functions of Z, whence

$$[V(z)/\tau]_B = -2\pi \sum m_j = -2\pi,$$
$$[V'(Z)/\tau']_{B'} = -2\pi \sum m_j' = -2\pi,$$
$$\tau = \tau',\ m_j = m_j',\ n_j = n_j'.$$

If now D and Δ are bounded, say interior to C_1 and C_1' respectively, the functions $R(z)$ and $R_1[Z(z)]$ are analytic in $\overline{D}$, different from zero there, with

$$\left| \frac{R(z)}{R_1[Z(z)]} \right| = 1$$

on the boundary $B+C$ of D. Then this quotient is identically constant in D, and by changing $\arg A'$ if necessary we may write

$$R(z) \equiv R_1[Z(z)]; \tag{7}$$

this identity is valid not merely in $\overline{D}$ but also throughout D', and indeed wherever the functions involved may be continued analytically from D'. We now keep A and A' fixed.

The functions $U(z)$ and $U'(Z)$ have each precisely [Walsh, 1950, p. 274] $\mu+\nu-2$ critical points in D and Δ; this is the total number of critical points of $R(z)$ and $R_1(Z)$, so the latter functions have all their critical points in D and Δ respectively.

Let D_1 be the region whose boundary is B_1 which belongs to the complement of $\overline{D}$, and Δ_1 the region whose boundary is B_1' which belongs to the complement of $\overline{\Delta}$. When z moves from D across B_1 into D_1, the point $Z(z)$ moves from Δ across B_1' into Δ_1; in D_1 we have $|R(z)| < 1$ and in Δ_1 we have $|R_1(Z)| < 1$, so by (7) the point z cannot leave D_1 if Z remains in Δ_1, nor can Z leave Δ_1 if z remains in D_1.

The correspondence between z and Z defined by (7) involves the equation $|R(z)| = |R_1(Z)|$, and the level curves in D and Δ of these two functions $R(z)$ and $R_1(Z)$ correspond to each other. These level curves pass one through each point of D_1 and Δ_1 (except of course $z = a_1$ and $Z = a_1'$), have no multiple points in D_1 and Δ_1 (because $R(z)$ and $R_1(Z)$ have no critical points there), and by the principle of maximum modulus each level curve is an analytic Jordan curve containing a_1 or a_1' in its interior. When z or Z traces such a level curve in the positive sense, arg $[R(z)]$ or arg $[R_1(Z)]$ increases by $2\pi m_1 = 2\pi m_1'$. The orthogonal trajectories to these level curves are the loci arg $[R(z)] = \text{const}$ and arg $[R_1(Z)] = \text{const}$, which can be considered as Jordan arcs in D_1 and Δ_1 from a_1 and a_1' to B_1 and B_1'; any two such loci have no point of intersection in D_1 and Δ_1 except a_1 or a_1' unless they coincide throughout. One such locus extends from a_1 to each point of B_1 and one from a_1' to each point of B_1', and cuts each level curve in D_1 and Δ_1 once and only once.

Equation (7) *sets up a one-to-one analytic correspondence between the points of* D_1 *and those of* Δ_1. For fixed θ, let α and α' be respective loci arg $R(z) = \theta$ in D_1 and arg $R_1(Z) = \theta$ in Δ_1; on the subarc of α in D' and its image under the conformal map we have points z and Z corresponding to each other by (7). All along α and α' we now set up a correspondence by requiring $|R(z)| = |R_1(Z)|$; this correspondence is one to one in D_1 and Δ_1. We extend this correspondence continuously from α and α' in the same sense along each level locus in D_1 and the corresponding level locus in Δ_1, allowing arg $R(z)$ and arg $R_1(Z)$ to vary continuously and setting up the correspondence by means of the equation arg $R(z) = \arg R_1(Z)$; the total change of both arg $R(z)$ and arg $R_1(Z)$ along each level curve is precisely $2\pi m_1$. We thus have defined a one-to-one correspondence between the points z of $D_1 - a_1$ and the points Z of $\Delta_1 - a_1'$ which coincides in $D_1 \cdot D'$ and $\Delta_1 \cdot \Delta'$ with the analytic extension of the given conformal map. This correspondence in $D_1 - a_1$ and $\Delta_1 - a_1'$ is defined locally by equation (7), and is analytic, for if we set $R(z) = w = R_1(Z)$, then w is an analytic function of z and Z is an analytic function of w, since $R_1'(Z) \neq 0$ in Δ_1; hence Z is an analytic function of z. This function is continuous even at $z = a_1$ if we define $Z(a_1)$ as a_1', hence (Riemann) is analytic also for $z = a_1$.

The reasoning just given applies to each of the $\mu + \nu$ regions into which the z-plane is separated by $\overline{D}$, and to the analogous $\mu + \nu$ regions of the Z-plane, and shows that equation (7) defines a one to one conformal map not merely of D onto Δ but of the extended z-plane onto the extended Z-plane. Such a map is necessarily defined by a linear transformation, so Theorem 2 is established.

If one chooses $a_1 = a_1' = 0$, $b_1 = b_1' = \infty$, this linear transformation must be of the form $Z = \lambda z$, $\lambda \neq 0$, so D and Δ are similar figures—this condition of similarity is both necessary and sufficient that D and Δ be conformally representable on each other, with B_1 and C_1 corresponding respectively to B_1' and C_1'.

Theorem 2 naturally applies to the conformal transformations of a region D into itself, where $\sum B_j$ is invariant. If we again suppose D in the canonical form of Theorem 2 with $a_1=a_1'=0$, $b_1=b_1'=\infty$ (the conditions $a_1=a_1'$, $b_1=b_1'$ are equivalent to requiring that B_1 and C_1 shall be invariant), a transformation which carries D into itself must be of the form $Z=\lambda z$ with $|\lambda|=1$; otherwise a point z_0 of D could be carried into the points $\lambda z_0, \lambda^2 z_0, \lambda^3 z_0, \cdots$, which approach 0 or ∞, which is impossible. Then a necessary and sufficient condition that D admit a conformal transformation into itself (with $a_1=a_1'=0$, $b_1=b_1'=\infty$) other than the identity is that D admit a rotation about 0 into itself. A necessary condition is that the sets of numbers $m_2, m_3, \cdots, m_\mu$ and $n_2, n_3, \cdots, n_\nu$ each fall into ρ identical groups, where ρ is an integer greater than unity; the rotation about 0 will then be through the angle $2\pi/\rho$; but this condition is not sufficient.

It is clear that in the case $\mu=1$, $\nu=2$ with $n_1=n_2$ (or $\mu=2$, $\nu=1$ with $m_1=m_2$) the region D *always admits* a conformal transformation into itself, for one can fix a_1 while interchanging b_1 and b_2 (or fix b_1 while interchanging a_1 and a_2) by a linear transformation.

III

We proceed to show the validity of the limiting case of Theorem 1 where in the hypothesis the numbers M_j are kept fixed but the curves B_j are allowed to shrink to points:

THEOREM 3. *Let D be a region of the z-plane whose boundary consists of mutually disjoint Jordan curves $C_1, C_2, \cdots, C_\nu$, let $\alpha_1, \alpha_2, \cdots, \alpha_\mu$ be arbitrary distinct points of D, and let $M_1, M_2, \cdots, M_\mu$ be arbitrary positive numbers with $\sum M_j=1$. Then there exists a conformal map of D onto a region Δ of the Z-plane, one-to-one and continuous in the closed regions, where Δ is defined by*

$$(8)\qquad |T(Z)| < 1, \qquad T(Z) \equiv \frac{A(Z-a_1)^{M_1}(Z-a_2)^{M_2}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}(Z-b_2)^{N_2}\cdots(Z-b_\nu)^{N_\nu}},$$
$$N_j > 0, \quad \sum N_j = 1.$$

The a_j are the respective images of the α_j; the locus $|T(Z)|=1$ consists of ν analytic Jordan curves, which are respective images of the C_j, and which separate Δ from the b_j.

Theorem 3 is of particular interest in the case $\mu=1$, for if then we choose $a_1=\infty$, Δ is defined by

$$|(Z-b_1)^{N_1}(Z-b_2)^{N_2}\cdots(Z-b_\nu)^{N_\nu}| > |A|.$$

The exponents N_j are not necessarily rational, but if they are rational, with

$\mu=1$ and $a_1=\infty$, the boundary of Δ is a lemniscate. Theorem 3 includes the Riemann mapping theorem for a region bounded by a single Jordan curve.

The proof of Theorem 3 is so similar to that of Theorem 1 that we omit most of the details. We omit for the present the classical case $\mu=\nu=1$, and thanks to a preliminary transformation we suppose the Jordan curves C_j to be analytic, with D interior to C_1.

If $g_j(z)$ is Green's function for D with pole in α_j, we set

$$u(z) \equiv M_1 g_1(z) + M_2 g_2(z) + \cdots + M_\mu g_\mu(z),$$

a function harmonic in D except in the points α_j, and which can be extended harmonically across each C_j so as to be harnonic in a closed region D' which contains the closure $\overline{D}$ of D and whose boundary $C': u(z) = -\delta(<0)$ consists of ν analytic Jordan curves near the respective C_j. If $v(z)$ denotes the conjugate of $u(z)$, we assume the derivative of $u(z)+iv(z)$ not to vanish in the closure of the annular regions $D'-\overline{D}$. The formula of Green, valid for z in D', can be written [Walsh, 1935, p. 215]

$$u(z) = \int_0^1 \log |z-t|\, d\sigma - \sum_1^\mu M_j \log |z-\alpha_j| - \delta, \tag{9}$$

$$d\sigma = dv/2\pi, \qquad \int_{C'} d\sigma = 1.$$

The integral in (9) is to be taken over C'.

Let the β_k (depending on n) divide C' into n parts equal with respect to the parameter σ, and let integers m_j (depending on n) be chosen so that $\sum m_j = n$, $m_j/n \to M_j$. On any closed set in the region $D' - \sum \alpha_j$ we have uniformly as $n \to \infty$

$$u_n(z) \equiv \frac{1}{n}\sum_1^n \log |z-\beta_k| - \frac{1}{n}\sum_1^\mu m_j \log |z-\alpha_j| - \delta \to u(z).$$

As $n\to\infty$, the region D is approximated by the region $D_n: u_n(z)>0$, a region bounded by ν Jordan curves near (for n sufficiently large as close as desired to) the C_j. Thus D_n is defined by

$$|R_n(z)| > 1, \qquad R_n(z) \equiv \frac{e^{-n\delta}(z-\beta_1)(z-\beta_2)\cdots(z-\beta_n)}{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_\mu)^{m_\mu}}$$
$$\equiv e^{n(u_n+iv_n)},$$

where $v_n(z)$ is conjugate to $u_n(z)$ in D'. Each compact K of D lies in all the D_n for n sufficiently large, and each compact K exterior to D lies exterior to all the D_n sufficiently large.

The equation $w=R_n(z)$ defines the conformal map of the extended z-plane onto an n-sheeted Riemann surface σ_0 over the extended w-plane, and the image $|w|>1$ of D_n is connected, having μ branch points at infinity of respective orders m_j-1, and having for boundary ν circumferences $|w|=1$ of respective multiplicities n_j, namely the numbers of the β_k on the respective components of C', with $n_j/n\to\int_{C_j}d\sigma=N_j$ (definition of N_j). The ν regions of the z-plane complementary to D_n have as images ν simply connected regions of σ_0, covering $|w|<1$ respectively n_j times; the boundaries of these regions consist of the circle $|w|=1$ traced monotonically n_j times.

We construct a new Riemann surface σ_1 over the w-plane by replacing continuously each of these ν regions of σ_0 by the portion $|w|<1$ of the n_j-sheeted Riemann surface for the inverse of $w=z^{n_j}$, having then but a single branch point, namely $w=0$. Thus σ_1 like σ_0 is the topological image of the extended plane and covers each point of the w-plane precisely n times, hence can be mapped conformally onto the extended Z-plane, by the transformation $w=S_n(Z)$,

$$S_n(Z)\equiv\frac{A_n(Z-b_1')^{n_1}(Z-b_2')^{n_2}\cdots(Z-b_\nu')^{n_\nu}}{(Z-a_1')^{m_1}(Z-a_2')^{m_2}\cdots(Z-a_\mu')^{m_\mu}},$$

$\sum m_j=\sum n_j=n$; the a_j' and b_j' (depending on n) are all distinct, and the b_j' are exterior to the image Δ_n: $|S_n(Z)|>1$ of D_n, while a_j' lies interior to Δ_n as the image of α_j. By a linear transformation of the Z-plane let us choose $a_1'=0$, $b_1'=1$, $b_2'=\infty$ independent of n (if $\nu=1$ we choose $a_2'=\infty$ instead of $b_2'=\infty$). We write the transformation $R_n(z)=S_n(Z)$ as $Z=Z_n(z)$.

If we denote by D_0 the region $D_n-\alpha_1$ (if $\nu=1$ we choose $D_n-\alpha_1-\alpha_2$), the functions $Z_n(z)$ admit the exceptional values 0, 1, ∞ in D_0, hence form a normal family in $D-\alpha_1-\alpha_2$. We have also

$$[\arg Z_n(z)]_\Gamma=2\pi,$$

where Γ is a curve in D surrounding α_1 but not surrounding any other α_j, so no limit function of the sequence $Z_n(z)$ can be identically constant in D_0; every such limit function must be univalent in $D-\alpha_1$ and indeed in D.

As $n\to\infty$, the inequality $|S_n(Z)|>1$ defining Δ_n approaches the form (8), if we choose a subsequence of the n such that all the numbers $A_n^{1/n}$, a_j', b_j' approach limits. The remainder of the proof of Theorem 3 now follows closely that of Theorem 1, and is left to the reader.

The proof of Theorem 3 as indicated requires but minor modifications to apply to the case $\mu=\nu=1$, for then $S_n(Z)$ is of the form

$$\frac{A_n(Z-b_1')^n}{(Z-a_1')^n}.$$

The choice $a_1'=0$, $b_1'=\infty$ yields a family of regions Δ_n depending on A_n which are all trivially conformally equivalent, so we may choose $A_n=1$ and complete the proof using a region Δ_n which does not depend on n. However, in this proof of the Riemann mapping theorem for a region bounded by a Jordan curve, we have used the fact that such a curve may be taken as analytic, and have also used Schwarz's theorem on the conformal map of an arbitrary Riemann surface topologically equivalent to the sphere.

In Theorem 3 (compare Theorem 1) the numbers M_j and N_j may all be rational, in which case the region Δ is bounded by a level locus of the *rational* function $[T(Z)]^N$, where the integer N is so chosen that the numbers NM_j and NN_j are all integers. It follows from this choice of N that the function $w=e^{N(u+iv)}$ is single valued in D and maps D onto the subregion $|w|>1$ of an N-sheeted Riemann surface σ_0 over a portion of the extended w-plane. The Riemann surface σ_0 can be replaced by a simpler N-sheeted Riemann surface σ_1 over the extended w-plane without modifying the image of D, and the proof of Theorem 3 (like that of Theorem 1) can be completed without use of the infinite sequences of regions D_n and Δ_n.

IV

The map of Theorem 3, like that of Theorem 1, is uniquely determined except for a possible linear transformation:

THEOREM 4. *Let D be a region of the z-plane defined by the inequality*

$$|R(z)| < 1, \qquad R(z) \equiv \frac{A(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_\mu)^{m_\mu}}{(z-b_1)^{n_1}(z-b_2)^{n_2}\cdots(z-b_\nu)^{n_\nu}}, \qquad m_j > 0,$$

$$\sum m_j \equiv \sum n_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $C_1, C_2, \cdots, C_\nu$, which separate the b_j respectively from D. Let Δ be a region of the Z-plane defined by the inequality

$$|R_1(Z)| < 1, \qquad R_1(Z) \equiv \frac{A'(Z-a_1')^{m_1}(Z-a_2')^{m_2}\cdots(Z-a_\mu')^{m_\mu}}{(Z-b_1')^{n_1'}(Z-b_2')^{n_2'}\cdots(Z-b_\nu')^{n_\nu'}},$$

$$\sum n_j' = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $C_1', C_2', \cdots, C_\nu'$ which separate the b_j' respectively from Δ. If there exists a one to one conformal transformation of D onto Δ so that the a_j correspond respectively to the a_j', then this correspondence can be continued beyond D and Δ as a linear transformation of the extended z-plane onto the extended Z-plane. We have $n_j = n_j'$, and if arg A' *is suitably chosen, $R(z) \equiv R_1(Z)$.*

An element of contrast between Theorem 1 and Theorem 3 is that in the former the numbers m_j are determined initially by the properties of D, whereas in the latter theorem the (positive) m_j are entirely arbitrary, subject merely to the restriction $\sum m_j = 1$. Theorem 4 is false if the respective exponents in the numerator of $R_1(Z)$ are not supposed equal to those in the numerator of $R(z)$, as we shall prove. We choose an arbitrary region D_0 bounded by ν mutually disjoint Jordan curves, and points $\alpha_1, \alpha_2, \cdots, \alpha_\mu$ in D_0; then we can choose two different sets of exponents $m_1, m_2, \cdots, m_\mu$ and $m_1', m_2', \cdots, m_\mu'$ with $\sum m_j = \sum m_j' = 1$, and there exist by Theorem 3 two distinct maps of D_0 onto the regions D and Δ of Theorem 4 but with the exponents m_j' instead of the m_j in the definition of $R_1(Z)$. That is to say, this region D can be mapped one to one and conformally onto Δ, with coincidence of the two images of each α_j, but clearly the map cannot be extended so as to be a linear transformation of the extended z-plane onto the extended Z-plane.

If $g_j(z)$ denotes Green's function for D with pole in a_j, in D we have $\log |R(z)| \equiv \sum m_j g_j(z)$, and if $g_j'(z)$ denotes Green's function for Δ with pole in a_j', in Δ we have $\log |R_1(Z)| \equiv \sum m_j g_j'(Z)$. However, Green's function is invariant under a one-to-one conformal transformation, so if the given transformation of Theorem 4 is $Z = Z(z)$, throughout D we have $g_j'[Z(z)] \equiv g_j(z)$, whence $|R_1[Z(z)]| \equiv |R(z)|$. The quotient $R(z)/R_1[Z(z)]$ is analytic throughout D and does not vanish there, so by a change in the argument of A' if necessary we may write throughout D

$$R(z) \equiv R_1[Z(z)]. \tag{10}$$

If ϵ (>0) is sufficiently small, the loci $|R(z)| = \epsilon$ and $|R_1(Z)| = \epsilon$ lie in D and Δ respectively and consist each of μ small curves about the points a_j and the points a_j'. It follows from (10) that the given conformal map $Z = Z(z)$ transforms the region $1 < |1/R(z)| < 1/\epsilon$ onto the region $1 < |1/R_1(Z)| < 1/\epsilon$, so Theorem 4 follows from Theorem 2.

We add a few further remarks relative to the conformal transformations of a multiply connected region D into itself, where $\sum B_j$ is invariant, a topic to which Theorem 2 applies, and we naturally use the canonical form of Theorem 2 for D. A linear transformation which carries D into itself can be neither parabolic, hyperbolic, nor loxodromic, for none of these transformations is periodic; each such transformation carries an analytic Jordan curve B_j or C_j which is not a circle through the fixed points (or for a parabolic transformation through the fixed point) into an infinity of distinct images, hence cannot transform D into itself.

We consider then an elliptic transformation T which transforms D into D, where now $\mu + \nu > 2$, and we choose the fixed points of T as zero and infinity. If ρ (>1) is the smallest integer for which T^ρ is the identity, an integer which must exist by virtue of $\mu + \nu > 2$, each point a_j or b_j other than 0 and ∞ belongs to a set of ρ points a_j or b_j, a set which is transformed into itself by T (a rotation about 0), and for which m_j or n_j does not depend on j. Each curve B_j or C_j which is not a Jordan curve with ρ-fold rotational symmetry about 0 containing 0 in its interior thus belongs to a set of ρ curves B_j or C_j, a set invariant under T. No curve B_j or C_j can be a circle with center 0, for if it were the corresponding a_j or b_j would be 0 or ∞, and Schwarz reflection of the harmonic function log $|R(z)|$ in this circle with center 0 shows that $\mu+\nu\leqq 2$. If 0 is not a point a_j or b_j, it is a zero of the derivative $R'(z)$, for in the field of force [Walsh, 1950, p. 89] which determines the zeros of $R'(z)$ the point 0 is a point of zero force for each set of ρ points a_j and b_j. One sees by interchanging the roles of 0 and ∞ that ∞ also is either a point a_j or b_j or a zero of $R'(z)$.

Each zero of $R'(z)$ lies in D, for D contains $\mu+\nu-2$ such zeros [Walsh, 1950, p. 274], and $R'(z)$ has precisely $\mu+\nu-2$ zeros. The conditions on Γ that we have established as necessary are easily seen to be sufficient, so in conclusion we have: *If $\mu+\nu>2$, the region D in canonical form admits a conformal transformation into itself with $\sum B_j$ invariant if and only if D and $\sum B_j$ are invariant with respect to an elliptic transformation T whose pth power is the identity; each fixed point of T is either an a_j, a b_j, or a zero of $R'(z)$, the latter being necessarily in D.*

When D is given, there exist at most $\mu+\nu-2$ distinct zeros of $R'(z)$. The number ρ is a divisor of μ, $\mu-1$, or $\mu-2$ according as 0, 1, or 2 fixed points of T are points a_j, and ρ is likewise a divisor of ν, $\nu-1$, or $\nu-2$ according as 0, 1, or 2 fixed points of T are points b_j. Then for any given D, only relatively few transformations T are possible, especially in view of the equality necessary among the m_j and n_j. It should not be difficult to examine all the possible transformations to identify those which carry D into itself.

Of course this study of the transformations of D into itself is essentially the study of the automorphisms of the function $|R(z)|$; each such automorphism corresponds to an elliptic transformation T, essentially involving a multiple symmetry with each fixed point of T either an a_j, a b_j, or a critical point of $R(z)$.

In the study of interpolation or approximation to a given analytic function $f(z)$, it is often necessary to cover a given region by the level curves of a harmonic function, a function which depends on the particular procedure of interpolation or approximation. Both the degree of approximation to $f(z)$ and the regions of convergence of the approximating functions depend primarily on the regions of analyticity of $f(z)$ with respect to this system of curves. Theorem 3 is especially useful for this purpose of interpolation and approximation because if a given function $f(Z)$ is analytic in Δ, the a_j can be chosen as points of interpolation for a series of rational functions whose poles lie in the b_j. Theorem 1 is likewise useful, because a function $f(Z)$ analytic in Δ is the sum of two component functions analytic in the respective regions $|T(Z)|>1$ and $|T(Z)|<e^{1/\tau}$; interpolation series of rational functions for these two components using first the b_j as points of interpolation and the a_j as poles of the rational functions, then interchanging these roles of the a_j and the b_j,—such a series is analogous to and even a generalization of Laurent's series, and gives a useful representation of the given $f(Z)$ [compare Walsh, 1955].

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