SINGULAR INTEGRAL OPERATORS AND SPHERICAL HARMONICS

BY

J. HORVÁTH

1. **Introduction.** A singular integral operator $\mathcal{K}$, as applied to an integrable function $f(x)$ of $n>1$ real variables, is defined by the Cauchy principal value

$$\mathcal{K}[f] = \lim_{\epsilon\to 0} \int_{|x-t|>\epsilon} k\left(\frac{x-t}{|x-t|}\right)\frac{f(t)}{|x-t|^n}\,dt, \tag{1}$$

where the "characteristic function" $k(\sigma)$ is, say, continuous on the unit sphere and its integral vanishes there. In this paper I shall consider the operation (1) as the convolution of f with a distribution K in the sense of Laurent Schwartz, as it has been done already for $n=1$ by Schwartz himself [17, p. 115]. This point of view permits one to disregard the delicate question of the existence of the limit in (1) and to apply $\mathcal{K}$ more generally to a distribution T rather than to a function f.

We shall see that K belongs to one of the spaces where the Fourier transform $\mathfrak{F}(K)$ is defined. This gives an immediate solution, at least theoretically, to the problem of composition of singular operators: If $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined by the distributions K_1 and K_2 respectively, then the distribution K which corresponds to $\mathcal{K}_1 \circ \mathcal{K}_2$ will have $\mathfrak{F}(K)=\mathfrak{F}(K_1)\mathfrak{F}(K_2)$ as its Fourier transform. The expression $\mathfrak{F}(K)$ (which is in reality a function) has been introduced by Mihlin [15] and Giraud [9] under the name of "symbol" of the operator $\mathcal{K}$.

As an application I shall calculate $\mathfrak{F}(K)$ in the special case when $k(\sigma)$ is a homogeneous harmonic polynomial. The corresponding result has been announced first by Giraud [9] and a proof has been given more recently by Bochner [1]. The present method uses an apparently new process of generating a complete system of spherical harmonics with the aid of Grassmann's "algebra with complex multiplication" which is described in §3 and can be read independently of the rest of the paper and of the theory of distributions.

On the other hand in §§2 and 4 I make constant use of the theory of distributions and conserve the notations and terminology of Schwartz's book [16; 17], except that I note an integral extended over R^n by only one integral sign.

In a paper, now under preparation, I shall reconsider from the present point of view the known results concerning the composition and inversion of singular operators [9; 10; 15; 4, footnote p. 261].

Received by the editors July 19, 1955.

My sincerest thanks are due to Laurent Schwartz for his constant encouragement and help, and to Alexandre Grothendieck for some useful conversation. Parts of the results which follow have been announced earlier in three short notes [12; 13; 14].

2. **The singular operators as distributions.** We shall denote by $x=(x_1, \cdots, x_n)$, $t=(t_1, \cdots, t_n)$, and $u=(u_1, \cdots, u_n)$ points of the n-dimensional euclidean space R^n $(n \geqq 2)$. $r=|x|$ will be the norm of x defined as the positive square root of $x_1^2+ \cdots +x_n^2$. The unit sphere S_{n-1} is the locus of the points x with $|x|=1$. If $x \in R^n$, we shall denote its radial projection onto S_{n-1} by σ_x, i.e. $\sigma_x = x/|x|$. Then every $x \in R^n$ can be written in the form $x=r\sigma$, where $r=|x|$ is a positive real number and $\sigma=\sigma_x \in S_{n-1}$.

We shall consider a function $k(\sigma)$ defined and integrable on S_{n-1}, which verifies

$$\int_{S_{n-1}} k(\sigma) d\sigma = 0, \tag{2}$$

where $d\sigma$ is the surface element on S_{n-1}. Later on we shall make a stronger assumption concerning the integrability of $k(\sigma)$.

Let now $\phi \in (\mathfrak{D})$ and define the distribution

$$K = \text{v.p.} \frac{k(\sigma)}{r^n}, \tag{3}$$

where, of course, $r=|x|$, $\sigma=\sigma_x$, by

$$K(\phi) = \lim_{\epsilon \to 0} \int_{r>\epsilon} \frac{k(\sigma)}{r^n} \phi(x) dx. \tag{4}$$

PROPOSITION 1. *The limit in* (4) *exists and defines K as a distribution.*

Proof. According to the mean value theorem we have

$$\phi(x) = \phi(0) + \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} \phi(\theta x) \cdot x_\nu,$$

where $0<\theta=\theta(x)<1$. Hence, using (2),

$$\int_{r>\epsilon} \frac{k(\sigma)}{r^n} \phi(r\sigma) dx = \int_{\epsilon}^{\infty} dr \int_{S_{n-1}} \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} \phi(\theta x) \cdot \frac{x_\nu}{r} k(\sigma) d\sigma. \tag{5}$$

Now

$$\left| \int_{S_{n-1}} \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} \phi(\theta x) \frac{x_\nu}{r} k(\sigma) d\sigma \right| \leqq \left\{ \sum_{\nu=1}^{n} \max \left| \frac{\partial \phi}{\partial x_\nu} \right| \right\} \int_{S_{n-1}} |k(\sigma)| d\sigma,$$

and thus the expression in (5) tends to a finite limit as $\epsilon \to 0$.

Define now for $\epsilon > 0$ the function

$$K_\epsilon(x) = \begin{cases} \dfrac{k(\sigma)}{r^n}, & \text{if } r > \epsilon, \\ 0, & \text{if } r \leqq \epsilon. \end{cases}$$

$K_\epsilon(x)$ is a locally integrable function, since

$$\int_{\epsilon<|x|<R} |K_\epsilon(x)| \, dx = \int_{S_{n-1}} |k(\sigma)| \, d\sigma \int_\epsilon^R \frac{dr}{r} = \int_{S_{n-1}} |k(\sigma)| \, d\sigma \cdot \log \frac{R}{r},$$

and defines thus a distribution K_ϵ. We have just proved that $K_\epsilon(\phi)$ tends to $K(\phi)$ for every $\phi \in (\mathfrak{D})$, hence [16, Chap. III, Théorème XIII, p. 75] K is a distribution.

Let now s be a real number verifying $1 < s < \infty$ and suppose that

$$\int_{S_{n-1}} |k(\sigma)|^s d\sigma < \infty; \tag{6}$$

then we can prove

PROPOSITION 2(1). $K \in (\mathfrak{D}'_{L^s})$.

Proof. Let $\alpha(x) \in (\mathfrak{D})$ be such that $0 \leqq \alpha(x) \leqq 1$ and $\alpha(x) = 1$ for $|x| \leqq 1$. Then we can write

$$K = \alpha K + (1 - \alpha)K.$$

Now αK has compact support [16, p. 116] and so $\alpha K \in (\mathcal{E}') \subset (\mathfrak{D}'_{L^s})$. On the other hand, since

$$\int_1^\infty \int_{S_{n-1}} \frac{|k(\sigma)|^s}{r^{ns}} dx = \int_1^\infty \frac{dr}{r^{1+n(s-1)}} \cdot \int_{S_{n-1}} |k(\sigma)|^s d\sigma,$$

$(1-\alpha)K$ is a *function* vanishing for $|x| \leqq 1$ and belonging to L^s. Since $L^s \subset (\mathfrak{D}'_{L^s})$, the proposition is proved.

THEOREM 1. *If* (6) *is verified for some* $s > 1$, *then* $K \in (\mathfrak{D}'_{L^p})$ *for all* $1 < p < \infty$.

For if $1 < p < s$, then (6) implies

$$\int_{S_{n-1}} |k(\sigma)|^p d\sigma < \infty,$$

hence, by Proposition 2, $K \in (\mathfrak{D}'_{L^p})$. On the other hand if $s < p < \infty$, then $(\mathfrak{D}'_{L^s}) \subset (\mathfrak{D}'_{L^p})$.

For $T \in (\mathfrak{D}'_{L^p})$ we define the (generalized) *Hilbert transform* of T by

$$\mathfrak{K}[T] = K * T.$$

(1) For the definition and properties of the spaces $(\mathfrak{D}_{L^p})$ and $(\mathfrak{D}'_{L^p})$ see [17, Chapter VI, pp. 55–61].

It follows from Theorem 1 that for any $q>p$ we have $K*T\in(\mathfrak{D}'_{L^q})$ and that $T\rightarrow K*T$ is a continuous linear mapping of $(\mathfrak{D}'_{L^p})$ into $(\mathfrak{D}'_{L^q})$ [17, Chap. VI, Théorème XXVI.2°, p. 59]. But in reality more can be proved:

THEOREM 2. *If* (6) *is verified for some* $s>1$, *then* $T\rightarrow K*T$ *is a continuous mapping of* $(\mathfrak{D}'_{L^p})$ *into* $(\mathfrak{D}'_{L^p})$ *for all* $1<p<\infty$.

We shall denote by p' the conjugate exponent of p, i.e. $1/p+1/p'=1$. Let us recall that if $p<q$ and if we identify the distributions of $(\mathfrak{D}'_{L^p})$, which are linear forms on $(\mathfrak{D}_{L^{p'}})$, with their restrictions to $(\mathfrak{D}_{L^{q'}})$, then $(\mathfrak{D}'_{L^p})$ can be considered as a subspace of $(\mathfrak{D}'_{L^q})$ and the topology induced by $(\mathfrak{D}'_{L^q})$ on $(\mathfrak{D}'_{L^p})$ is weaker than that of $(\mathfrak{D}'_{L^p})$. We shall make use of the following corollary to the closed graph theorem [11, p. 32]: Let E and F be two metrizable and complete topological vector spaces. Let u be a linear mapping of E into F. If u is continuous for a weaker separated topology on F, then u is continuous.

As we know that $T\rightarrow K*T$ is a continuous mapping of $(\mathfrak{D}'_{L^p})$ into $(\mathfrak{D}'_{L^q})$, all we have to prove is that $K*T$ belongs to $(\mathfrak{D}'_{L^p})$, *i. e.* that $K*T$ is the restriction to $(\mathfrak{D}_{L^{q'}})$ of a continuous linear form on $(\mathfrak{D}_{L^{p'}})$. First we shall state two lemmas.

LEMMA 1. *Let* $\phi\in(\mathfrak{D})$. *Then*

$$K*\phi = \lim_{\epsilon\rightarrow 0}\int_{|x-t|>\epsilon} k\left(\frac{x-t}{|x-t|}\right)\frac{\phi(t)}{|x-t|^n}\,dt.$$

$K*\phi$ *is an infinitely differentiable function and*(2) $D^m(K*\phi)=K*D^m\phi$. [17, Chap. VI, Théorème XI, p. 22].

LEMMA 2. *For* $\phi\in(\mathfrak{D})$ *we have*

$$\|K*\phi\|_{L^{p'}} \leqq \alpha_{p'}\|\phi\|_{L^{p'}},$$

where $\alpha_{p}{}'$ *depends on* K *and* p', *but not on* ϕ.

This is a consequence of Lemma 1 and of a theorem of Calderón and Zygmund [3].

Proof of Theorem 2. For $\phi\in(\mathfrak{D})$ we have by Lemma 1 that $D^m(\check{K}*\phi)=\check{K}*D^m\phi$ and hence by Lemma 2

$$\|D^m(\check{K}*\phi)\|_{L^{p'}} \leqq \alpha_{p'}\|D^m\phi\|_{L^{p'}}.$$

This shows that $\check{K}*\phi\in(\mathfrak{D}_{L^{p'}})$ and that if we consider on $(\mathfrak{D})$ the topology induced by $(\mathfrak{D}_{L^{p'}})$, then the linear mapping $\phi\rightarrow\check{K}*\phi$ of $(\mathfrak{D})$ into $(\mathfrak{D}_{L^{p'}})$ is continuous. Let now $T\in(\mathfrak{D}'_{L^p})$, then the relation [17, formula (VI, 4; 11), p. 24]

$$\langle K*T,\phi\rangle = \langle T,\check{K}*\phi\rangle$$

(2) $D^m=\partial_1^{m_1\cdots m_n}/\partial x^{m_1}\cdots\partial x^{m_n}$.

shows that $K * T$ is a continuous linear form on $(\mathfrak{D})$ considered with the above topology. As $(\mathfrak{D})$ is dense in $(\mathfrak{D}_{L^{p'}})$ [17, p. 55], $K * T$ can be extended to a continuous linear form on the whole space $(\mathfrak{D}_{L^{p'}})$, which proves the theorem.

REMARK. If $T=f\in L^p$, then it can be shown, of course, that also $K * f$ is a *function* belonging to L^p. It then follows from the theorem used in the proof of Theorem 2 that $f\rightarrow K * f$ is a continuous mapping of L^p into L^p. $K * f$ is the limit of $K_\epsilon * f$ in L^p and also in the sense of pointwise convergence almost everywhere [2; 3].

3. **Grassmann's complex algebra and spherical harmonics.** The *Grassmann algebra with complex multiplication* [5, §15] is the commutative graded algebra [6, p. 7]

$$Q = \sum_{j=1}^{\infty} Q_j$$

over the field $\boldsymbol{R}$ of the real numbers, where the subspace Q_1 is isomorphic to $\boldsymbol{R}^n$ and its canonical basis $e_1, e_2, \cdots, e_n$ verifies the relation

$$e_1^2 + e_2^2 + \cdots + e_n^2 = 0. \tag{7}$$

A basis of Q_j is formed by the $C_{n+j-2,j}$ products of the form

$$e_1^{l_1} e_2^{l_2} \cdots e_{n-1}^{l_{n-1}} \quad \text{with} \quad l_1 + l_2 + \cdots + l_{n-1} = j, \tag{8}$$

and the $C_{n+j-3,j-1}$ products of the form

$$e_1^{l_1} e_2^{l_2} \cdots e_{n-1}^{l_{n-1}} e_n \quad \text{with} \quad l_1 + l_2 + \cdots + l_{n-1} = j - 1, \tag{9}$$

since, in virtue of (7), every product

$$e_1^{s_1} e_2^{s_2} \cdots e_n^{s_n} \quad \text{with} \quad s_1 + s_2 + \cdots + s_n = j$$

can be expressed as a linear combination of terms (8) and (9). The vector-space Q_j has dimension(3)

$$C_{n+j-2,j} + C_{n+j-3,j-1} = (n + 2j - 2)\, \frac{(n + j - 3)!}{j!(n - 2)!}. \tag{10}$$

We shall now show that *the coordinates of*

$$x^j = (x_1e_1 + x_2e_2 + \cdots + x_ne_n)^j \qquad (j = 1, 2, 3, \cdots)$$

in Q_j *with respect to the basis given in* (8) *and* (9) *are linearly independent*

(3) Clearly Q is the ideal formed by the elements of degree ≥ 1 of $S(R^n)/\mathfrak{a}$, where $S(\boldsymbol{R}^n)$ is the symmetrical algebra [6, p. 21] of $\boldsymbol{R}^n$ and $\mathfrak{a}$ is the ideal generated by the element $e_1^2 + \cdots e_n^2$.

homogeneous harmonic polynomials of degree j. As the maximum number of independent harmonic polynomials of degree j is given by (10) [8, p. 237], the coordinates of x^j ($j=1, 2, 3, \cdots$), together with the constant 1, form a complete orthogonal system on $\boldsymbol{S}_{n-1}$.

By the polynomial theorem we have

$$x^j = \sum_{s_1+\cdots+s_n=j} \frac{j!}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n} e_1^{s_1} \cdots e_n^{s_n}.$$

Let first s_n be even. Then (7) gives

$$\begin{aligned} e_n^{s_n} &= (-e_1^2 - e_2^2 - \cdots - e_{n-1}^2)^{s_n/2} \\ &= (-1)^{s_n/2} \sum \frac{\left(\frac{s_n}{2}\right)!}{\left(\frac{\mu_1}{2}\right)! \cdots \left(\frac{\mu_{n-1}}{2}\right)!} e_1^{\mu_1} \cdots e_{n-1}^{\mu_{n-1}}, \end{aligned}$$

where the summation is extended over all systems $\mu_1, \mu_2, \cdots, \mu_{n-1}$ with

$$\begin{aligned} &\mu_1 + \mu_2 + \cdots + \mu_{n-1} = s_n, \\ &\mu_1, \mu_2, \cdots, \mu_{n-1} \text{ even}, \\ &0 \leqq \mu_i \leqq s_n \ (i = 1, 2, \cdots, n-1). \end{aligned} \tag{11}$$

It follows that the coefficient of (8) in the expression of x^j will be

$$\begin{aligned} Y_j^{(l_1,\cdots,l_{n-1})}(x) &= \sum (-1)^{s_n/2} \frac{j!}{(l_1-\mu_1)! \cdots (l_{n-1}-\mu_{n-1})! s_n!} \\ &\quad \cdot \frac{(s_n/2)!}{(\mu_1/2)! \cdots (\mu_{n-1}/2)!} x_1^{l_1-\mu_1} \cdots x_{n-1}^{l_{n-1}-\mu_{n-1}} x_n^{s_n} \end{aligned} \tag{12}$$

where the summation is extended over $s_n=0, 2, 4, \cdots$ and all systems $\mu_1, \cdots, \mu_{n-1}$ which satisfy (11).

Next let s_n be odd. Then

$$e_n^{s_n-1} = (-1)^{(s_n-1)/2} \sum \frac{((s_n-1)/2)!}{(\mu_1/2)! \ldots (\mu_{n-1}/2)!} e_1^{\mu_1} \cdots e_{n-1}^{\mu_{n-1}},$$

where the summation is extended over all systems $\mu_1, \mu_2, \cdots, \mu_{n-1}$ with

$$\begin{aligned} &\mu_1 + \mu_2 + \cdots + \mu_{n-1} = s_n - 1, \\ &\mu_1, \mu_2, \cdots, \mu_{n-1} \text{ even}, \\ &0 \leqq \mu_i \leqq s_n - 1 \quad (i = 1, 2, \cdots, n-1). \end{aligned} \tag{13}$$

The coefficient of (9) in the development of x^j is

$$(14)\quad Y_j^{(l_1,\cdots,l_{n-1},1)}(x) = \sum (-1)^{(s_n-1)/2} \frac{j!}{(l_1-\mu_1)!\cdots(l_{n-1}-\mu_{n-1})!s_n!} \cdot \frac{((s_n-1)/2)!}{(\mu_1/2)!\cdots(\mu_{n-1}/2)!} x_1^{l_1-\mu_1}\cdots x_{n-1}^{l_{n-1}-\mu_{n-1}} x_n^{s_n},$$

where the summation is extended over

$$(13).\qquad s_n = 1, 3, 5, \cdots \quad \text{and all} \quad \mu_1, \cdots, \mu_{n-1}$$

We have to prove that the homogeneous polynomials given in (12) and (14) are harmonic. It is obvious that ΔY_j is a homogeneous polynomial of degree $j-2$, i.e. the linear combination of monomials of the form

$$(15)\qquad x_1^{l_1-\nu_1} x_2^{l_2-\nu_2}\cdots x_{n-1}^{l_{n-1}-\nu_{n-1}} x_n^{\rho_n},$$

where

$$(16)\qquad (l_1-\nu_1)+\cdots+(l_{n-1}-\nu_{n-1})+\rho_n = j-2.$$

This same term is obtained by differentiating twice the n monomials whose variable parts are

$$\begin{array}{l} x_1^{l_1-\nu_1+2}\cdots x_{n-1}^{l_{n-1}-\nu_{n-1}} x_n^{\rho_n}, \\ \cdots\cdots\cdots\cdots, \\ x_1^{l_1-\nu_1}\cdots x_{n-1}^{l_{n-1}-\nu_{n-1}+2} x_n^{\rho_n}, \\ x_1^{l_1-\nu_1}\cdots x_{n-1}^{l_{n-1}-\nu_{n-1}} x_n^{\rho_n+2}, \end{array}$$

with respect to $x_1, \cdots, x_{n-1}, x_n$ respectively. Thus in the case of a polynomial (12) the coefficient of (15) in ΔY_j is

$$\sum_{i=1}^{n-1} (-1)^{\rho_n/2}\binom{\nu_i}{2}\frac{j!}{(l_1-\nu_1)!\cdots(l_{n-1}-\nu_{n-1})!\rho_n!}\,\frac{(\rho_n/2)!}{(\nu_1/2)!\cdots(\nu_{n-1}/2)!} + (-1)^{(\rho_n+2)/2}\frac{j!}{(l_1-\nu_1)!\cdots(l_{n-1}-\nu_{n-1})!\rho_n!}\,\frac{((\rho_n+2)/2)!}{(\nu_1/2)!\cdots(\nu_{n-1}/2)!}.$$

This expression can be written as a product, where one factor is

$$(17)\qquad \sum_{i=1}^{n-1}\nu_i - \rho_n - 2.$$

Since we have

$$\sum_{i=1}^{n-1} l_i = j,$$

by subtraction from (16) we obtain that (17) equals zero, i.e. $\Delta Y_j = 0$ for the polynomials (12).

In the case of a Y_j of type (14) the coefficient of (15) in ΔY_j is

$$\sum_{i=1}^{n-1} (-1)^{(\rho_n-1)/2} \left(\frac{\nu_i}{2}\right) \frac{j!}{(l_1-\nu_1)!\cdots(l_{n-1}-\nu_{n-1})!\rho_n!} \frac{((\rho_n-1)/2)!}{(\nu_1/2)!\cdots(\nu_{n-1}/2)!}$$
$$+ (-1)^{(\rho_n+1)/2} \frac{j!}{(l_1-\nu_1)!\cdots(l_{n-1}-\nu_{n-1})!\rho_n!} \frac{((\rho_n+1)/2)!}{(\nu_1/2)!\cdots(\nu_{n-1}/2)!}.$$

This expression contains the factor

$$\sum_{i=1}^{n-1} \nu_i - \rho_n - 1. \tag{18}$$

Since we have now

$$\sum_{i=1}^{n-1} l_i + 1 = j,$$

by subtraction from (16) we get that (18) equals zero and the polynomials (14) are harmonic too.

On the other hand it is obvious that the polynomials Y_j are linearly independent, since every one of them contains exactly one term in which x_n enters with degree $l_n \leqq 1$, and all these terms have different systems of exponents $l_1, l_2, \cdots, l_n$.

Let us also observe that

$$\int_{S_{n-1}} Y_j(\sigma) d\sigma = 0 \tag{19}$$

for $j \geqq 1$, as this expresses the well known orthogonality of Y_j and the constant function on the sphere S_{n-1} [8, p. 241].

4. **The Fourier transforms of certain special operators.** We shall now consider the vectorial distribution

$$H_j = \frac{\Gamma((n+j)/2)}{\pi^{n/2}\Gamma(j/2)} \text{v.p.} \frac{\sigma^j}{r^n} = \frac{\Gamma((n+j)/2)}{\pi^{n/2}\Gamma(j/2)} \text{v.p.} \frac{x^j}{|x|^{n+j}}.$$

H_j is a continuous linear mapping of $(\mathfrak{D}_{L^{p'}})$ into Q_j [16, p. 30] whose components are

$$\frac{\Gamma((n+j)/2)}{\pi^{n/2}\Gamma(j/2)} \text{v.p.} \frac{Y_j(\sigma)}{r^n} \qquad (j = 1, 2, 3, \cdots).$$

From (19) we see that these components are indeed distributions of the type K considered in §2. Our main objective is to prove that *the Fourier transform of* $\boldsymbol{H}_j$ *is*

$$\mathfrak{F}(\boldsymbol{H}_j) = (-i)^j \frac{u^j}{|u|^j}. \tag{20}$$

Considering the components we obtain the result of Giraud referred to in the introduction, according to which

$$\mathfrak{F}\left(\text{v.p.} \frac{Y_j(\sigma_x)}{|x|^n}\right) = \frac{\pi^{n/2}\Gamma(j/2)}{i^j\Gamma((n+j)/2)} Y_j(\sigma\mu),$$

where, by linearity, Y_j can be *any* homogeneous harmonic polynomial of degree j.

To prove (20) let us remark first that

$$\text{v.p.} \frac{x^j}{r^{n+j}} = x^j\left(\text{Pf.} \frac{1}{r^{n+j}}\right) \tag{21}$$

[16, p. 45], since by (5) the infinite part of the integral

$$\int_{R^n} \frac{\sigma^j}{r^n} \phi(x)dx$$

is zero. Let us suppose first that j is odd. Then we have [17, formula (VII, 7; 13), p. 113]

$$\mathfrak{F}\left(\text{Pf.} \frac{1}{r^{n+j}}\right) = \pi^{n/2+j} \frac{\Gamma(-j/2)}{\Gamma((n+j)/2)} |u|^j.$$

A well known property of the gamma function [7, p. 3, formula (6)] gives

$$\Gamma\left(-\frac{j}{2}\right) = \Gamma\left(1 - \frac{j+2}{2}\right) = \frac{\pi}{(-1)^{(j+1)/2}\Gamma((j+2)/2)}$$

and so

$$\mathfrak{F}\left(\text{Pf.} \frac{1}{r^{n+j}}\right) = \frac{(-1)^{(j+1)/2}\pi^{n/2+j+1}}{\Gamma((j+2)/2)\Gamma((n+j)/2)} |u|^j. \tag{22}$$

We introduce now the operator

$$\boldsymbol{D} = \sum_{\nu=1}^{n} e_\nu \frac{\partial}{\partial u_\nu}$$

and apply the formula

$$\mathfrak{F}(x^jT) = (-2\pi i)^{-j}\boldsymbol{D}^j\mathfrak{F}(T) \tag{23}$$

[17, (VII, 7; 3), p. 109]. An elementary calculation shows that

$$D^j|u|^j = (-1)^{(j-1)/2}1^2\cdot 3^2 \cdots (j-2)^2 j \frac{u^j}{|u|^j}, \tag{24}$$

hence (21), (23), (22), and (24) give

$$\mathfrak{F}\left(\text{v.p. } \frac{x^j}{r^{n+j}}\right) = \frac{\pi^{n/2}}{\Gamma((n+j)/2)} \frac{\pi\cdot 1^2\cdot 3^2 \cdots (j-2)^2 j}{2^j i^j \Gamma((j+2)/2)} \frac{u^j}{|u|^j}.$$

Since

$$\frac{\pi\cdot 1^2\cdot 3^2 \cdots (j-2)^2 j}{2^j} = \Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{j+2}{2}\right)$$

we obtain finally

$$\mathfrak{F}\left(\text{v.p. } \frac{x^j}{r^{n+j}}\right) = \frac{\pi^{n/2}\Gamma(j/2)}{\Gamma((n+j)/2)} (-i)^j \frac{u^j}{|u|^j},$$

which proves (20) for odd j.

Let now j be even, then [17, (VII, 7; 14), p. 114]

$$\mathfrak{F}\left(\text{Pf. } \frac{1}{r^{n+j}}\right) = \frac{\pi^{n/2+j}}{\Gamma((n+j)/2)} 2 \frac{(-1)^{j/2}}{\Gamma((j+2)/2)} |u|^j \left[\log \frac{1}{\pi|u|} + A_{n,j}\right], \tag{25}$$

where $A_{n,j}$ is a constant. Let us prove that

$$D^j|u|^j\left[\log \frac{1}{\pi|u|} + A_{n,j}\right] = (-1)^{j/2}2^2\cdot 4^2 \cdots (j-2)^2 j \frac{u^j}{|u|^j}. \tag{26}$$

Observe first that

$$D \log \frac{1}{\pi|u|} = -\frac{u}{|u|^2} \tag{27}$$

and

$$Du = D(u_1e_1 + \cdots + u_ne_n) = e_1^2 + \cdots + e_n^2 = 0, \tag{28}$$

which follows from (7). Now for $1 \leqq l \leqq j/2$ we have

$$\begin{aligned} &D^l|u|^j\left[\log \frac{1}{\pi|u|} + A_{n,j}\right] \\ &= j(j-2)\cdots(j-2l+2)u^l|u|^{j-2l}\left[\log \frac{1}{\pi|u|} + A_{n,j}\right] + \alpha_l u^l|u|^{j-2l}, \end{aligned} \tag{29}$$

where α_l is a constant. This can be proved by mathematical induction. In fact let (29) be true, then the relations (27), (28), and (24) (for $j=1$) yield

$$D^{l+1}|u|^j\left[\log\frac{1}{\pi|u|}+A_{n,j}\right]$$

$$= j(j-2)\cdots(j-2l)u^{l+1}|u|^{j-2l-2}\left[\log\frac{1}{\pi|u|}+A_{n,j}\right]$$

$$+ j(j-2)\cdots(j-2l+2)u^{l}|u|^{j-2l}\left(-\frac{u}{|u|^2}\right)+\alpha_l(j-2l)u^{l+1}|u|^{j-2l+2}$$

$$= j(j-2)\cdots(j-2(l+1)+2)u^{l+1}|u|^{j-2(l+1)}\left[\log\frac{1}{\pi|u|}+A_{n,j}\right]$$

$$+ \alpha_{l+1}u^{l+1}|u|^{j-2(l+1)},$$

which proves (29). In particular we have for $l=j/2$:

$$D^{j/2}|u|^j\left[\log\frac{1}{\pi|u|}+A_{n,j}\right]$$

$$= j(j-2)\cdots 2u^{j/2}\left[\log\frac{1}{\pi|u|}+A_{n,j}\right]+\alpha_{j/2}u^{j/2},$$

from where, in virtue of (27) and (28),

$$D^{j/2+1}|u|^j\left[\log\frac{1}{\pi|u|}+A_{n,j}\right] = -j(j-2)\cdots 2u^{(j+2)/2}|u|^{-2}.$$

Since obviously

$$D^{j/2-1}\frac{u^{(j+2)/2}}{|u|^2} = (-2)(-4)\cdots(-j+2)\frac{u^j}{|u|^j},$$

we have proved (26).

It now follows from (21), (23), (25) and (26) that

$$\mathfrak{F}\left(\text{v.p. }\frac{x^j}{r^{n+j}}\right) = \frac{\pi^{n/2}}{\Gamma((n+j)/2)}\,\frac{2^2\cdot 4^2\cdots(j-1)^2 j}{2^{j-1}i^j\Gamma((j+2)/2)}\,\frac{u^j}{|u|^j}.$$

Since for even j

$$\frac{2^2\cdot 4^2\cdots(j-1)^2 j}{2^{j-1}} = \Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{j+2}{2}\right),$$

we obtain

$$\mathfrak{F}\left(\text{v.p. }\frac{x^j}{r^{n+j}}\right) = \frac{\pi^{n/2}\Gamma(j/2)}{\Gamma((n+j)/2)}(-i)^j\frac{u^j}{|u|^j}$$

which proves (20) also in the case of an even j.

In virtue of the result just obtained, the vectorial Hilbert transforms

$$\mathcal{H}_j[T] = H_j * T \qquad (j = 1, 2, 3, \cdots)$$

form a one-parameter semi-group of operators, since (20) implies the composition formula

$$\mathcal{H}_j[\mathcal{H}_l[T]] = H_j * H_l * T = H_{j+l} * T = \mathcal{H}_{j+l}[T].$$

If we put $H_0 = \delta$, then $\mathcal{H}_0$ will be the identity operator.

In the case $n=2$ the operators $\mathcal{H}_j$ can also be defined for negative j and then the $\mathcal{H}_j$ form a group. The composition formula in this case has been obtained by Tricomi and Mihlin [15; 12; 14](4).

References

1. S. Bochner, *Theta relations with spherical harmonics*, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 804–808.

2. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. vol. 88 (1952) pp. 85–139.

3. (a) ———, *On singular integrals*, Bull. Amer. Math. Soc. Abstract 59-5-535.

(b) A. P. Calderón, *Integrales singulares*, Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Villavicencio-Mendoza 1954, pp. 319–328.

4. A. P. Calderón and A. Zygmund, *Singular integrals and periodic functions*, Studia Mathematica vol. 14 (1954) pp. 249–271.

5. E. Cartan, *Nombres complexes*, Encyclop. Sc. Math., éd. franc., I 5., 1908 = Oeuvres complètes, Partie II. vol. 1 pp. 107–246.

6. C. C. Chevalley, *Théorie des groupes de Lie*, vol. 2 Paris, 1951.

7. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, vol. 1, New York, 1953.

8. ———, ibid. vol. 2.

9. G. Giraud, *Sur une classe générale d'équations à intégrales principales*, C.R. Acad. Sci. Paris vol. 202 (1936) pp. 2124–2127.

10. ———, *Complément à un résultat sur les équations à intégrale principales*, C.R. Acad. Sci. Paris vol. 203 (1936) pp. 292–294.

11. A. Grothendieck, *Espaces vectoriels topologiques*, Mimeographed lecture notes, Saõ Paulo, 1954.

12. J. Horváth, *Sur l'itération de la transformée de Hilbert d'une distribution complexe*, C.R. Acad. Sci. Paris vol. 237 (1953) pp. 1480–1482.

13. ———, *Hilbert transforms of distributions in R^n*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, pp. 122–123.

14. ———, *Transformadas de Hilbert de distribuciones*, Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino America, Villavicencio-Mendoza, 1954, pp. 61–70.

15. S. G. Mihlin, *Singular integral equations*, Uspehi Matematičeskih Nauk (N.S.) vol. 3 (1948) pp. 29–112. Amer. Math. Soc. Translations no. 24, 1950.

16. L. Schwartz, *Théorie des distributions*, vol. 1, Paris, 1950.

17. ———, ibid., vol. 2, Paris, 1951.

Universidad de los Andes,
Bogotá, Columbia

(4) In my note [12] the coefficient in H_k should be $|k|/2\pi$ rather than $k/2\pi$.