SOME REMARKS ABOUT ELEMENTARY DIVISOR RINGS(1)

BY

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In this and the following paper [2], we are concerned with obtaining conditions on a commutative ring S with identity element in order that every matrix over S can be reduced to an equivalent diagonal matrix(2). Following Kaplansky [4], we call such rings *elementary divisor* rings. A necessary condition is that S satisfy

F: *all finitely generated ideals are principal.*

It has been known for some time that if S satisfies the ascending chain condition on ideals, and has no zero-divisors, then **F** is also sufficient. Helmer [3] showed that the chain condition can be replaced by the less restrictive hypothesis that S be *adequate* (i.e., of any two elements, one has a "largest" divisor that is relatively prime to the other(2)). Kaplansky [4] generalized this further by permitting zero-divisors, provided that they are all in the (Perlis-Jacobson) radical.

By a slight modification of Kaplansky's argument, we find that the condition on zero-divisors can be replaced by the hypothesis that S be an *Hermite* ring (i.e., every matrix over S can be reduced to triangular form(2)). This is an improvement, since, in any case, it is necessary that S be an Hermite ring, while, on the other hand, it is not necessary that all zero-divisors be in the radical. In fact, we show that every *regular* commutative ring with identity is adequate. However, the condition that S be adequate is not necessary either.

We succeed in obtaining a necessary and sufficient condition that S be an elementary divisor ring. Along the way, we obtain a necessary and sufficient condition that S be an Hermite ring. In the paper that follows [2], we make constant use of these results. In particular, we construct examples of rings that satisfy **F** but are not Hermite rings, and examples of Hermite rings that are not elementary divisor rings. However, all these examples contain zero-divisors; therefore, the question as to whether there exist corresponding examples that are *integral domains* is left unsettled.

DEFINITION 1. An m by n matrix A over S *admits triangular reduction* if

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(2) The precise definition is given below.

there exist nonsingular[3] matrices U, V such that $AU=[b_{i,j}]$ is triangular (i.e., $b_{i,j}=0$ whenever $i<j$), and VA is triangular; A *admits diagonal reduction* if there exist nonsingular matrices P, Q such that $PAQ=[c_{i,j}]$ is diagonal (i.e., $c_{i,j}=0$ whenever $i\neq j$), and every $c_{i,i}$ is a divisor of $c_{i+1,i+1}$ [4][4].

THEOREM 2 (KAPLANSKY). *Let S be a commutative ring with identity. If all* 1 *by* 2 *and all* 2 *by* 1 *matrices over S admit diagonal reduction, then every matrix over S admits triangular reduction; in this case, S is called an Hermite ring. If, in addition, all* 2 *by* 2 *matrices over S admit diagonal reduction, then every matrix over S admits diagonal reduction; in this case, S is called an elementary divisor ring.*

For the proof, see [4, Theorems 3.5 and 5.1].

Obviously, every elementary divisor ring is an Hermite ring. Furthermore, every Hermite ring satisfies **F** [4, p. 465].

In order to prove that a given commutative ring is an Hermite ring, it suffices, by symmetry, to show only that every 1 by 2 matrix admits diagonal reduction.

THEOREM 3. *A commutative ring S with identity is an Hermite ring if and only if it satisfies the condition*

T: *for all $a, b\in S$, there exist $a_1, b_1, d\in S$ such that $a=a_1d$, $b=b_1d$, and* $(a_1, b_1)=(1)$.

Proof. Suppose that S satisfies **T**. In order to show that S is an Hermite ring, it suffices to show that every 1 by 2 matrix $[a\ b]$ admits diagonal reduction (Theorem 2 ff.). Let a_1, b_1, d, s, t satisfy $a=a_1d$, $b=b_1d$, and $sa_1+tb_1=1$. Let

$$Q=\begin{bmatrix} s & -b_1 \\ t & a_1 \end{bmatrix}. \tag{1}$$

Then Q is nonsingular, and $[a\ b]Q=[d\ 0]$.

Conversely, suppose that S is an Hermite ring. Let $a, b\in S$. By hypothesis, there exists a nonsingular matrix Q, which we denote as in (1), such that $[a\ b]Q=[d\ 0]$ for some $d\in S$. Then $ab_1=ba_1$, and $sa+tb=d$. Since Q is nonsingular, we may assume that $sa_1+tb_1=1$. Then $sa_1a+tb_1a=a$, whence $sa_1a+ta_1b=a$, i.e., $a_1d=a$. Similarly, $b_1d=b$.

The following lemma, due essentially to Kaplansky [4, §4], shows that in dealing with condition **T** relative to any specific pair a, b, it suffices to consider any particular generator of the ideal (a, b).

LEMMA 4. *Let $a, b\in S$. If a_1, b_1, d exist as in condition* **T** (*whence* $(a, b)=(d)$),

[3] By *nonsingular*, we mean that U (resp. V) has a two-sided inverse in the ring of all n by n (resp. m by m) matrices over S.

[4] Kaplansky [4] does not require commutativity of S.

then for all d' *with* $(a, b)=(d')$, *there exist* a_1', b_1' *such that* $a=a_1'd'$, $b=b_1'd'$, *and* $(a_1', b_1')=(1)$.

Proof. Write $d=kd'$, $d'=ld$, and choose s, t such that $sa_1+tb_1=1$. Define $a_1'=klt-t+a_1k$, and $b_1'=s-kls+b_1k$. Then $a_1'd'=a$, $b_1'd'=b$, and $(sl-b_1)a_1'+(tl+a_1)b_1'=1$.

As a straightforward consequence of Lemma 4, we have:

COROLLARY 5. *If* S *satisfies condition* **T**, *then given* a, b, c, d *with* $(a, b, c)=(d)$, *there exist* a_1, b_1, c_1 *such that* $a=a_1d$, $b=b_1d$, $c=c_1d$, *and* $(a_1, b_1, c_1)=(1)$.

THEOREM 6. *A commutative ring* S *with identity is an elementary divisor ring if and only if it is an Hermite ring that satisfies the condition*

D′: *for all* $a, b, c\in S$ *with* $(a, b, c)=(1)$, *there exist* $p, q\in S$ *such that* $(pa, pb+qc)=(1)$.

Thus, S *is an elementary divisor ring if and only if it satisfies* **T** *and* **D′**.

Proof. We have already remarked that every elementary divisor ring is an Hermite ring. The necessity of the condition **D′** is established in the proof of [4, Theorem 5.2].

The sufficiency of the two conditions is obtained by making the following two changes in the proof of [4, Theorem 5.2]. First, delete the reference to [4, Theorem 3.2]. Second, justify the fact that $xa_1+yb_1+zc_1$ is a unit by referring to our Corollary 5.

DEFINITION 7 (HELMER)(5). A commutative ring S with identity is said to be *adequate* if it satisfies the two conditions **F** and

A: *for every* $a, b\in S$, *with* $a\neq 0$, *there exist* $a_1, d\in S$ *such that* (i) $a=a_1d$, (ii) $(a_1, b)=(1)$, *and* (iii) *for every nonunit divisor* d' *of* d, *we have* $(d', b)\neq(1)$.

If in the proof of [4, Theorem 5.3], we replace the reference to [4, Theorem 5.2] by a reference to our Theorem 6, we obtain:

THEOREM 8. *An adequate ring is an elementary divisor ring if and only if it is an Hermite ring.*

DEFINITION 9 (VON NEUMANN)(6). A commutative ring S with identity is said to be *regular* if for every $a\in S$, there exists $x\in S$ such that $a^2x=a$.

von Neumann [5] shows that in any regular ring, every principal ideal is generated by an idempotent; in fact, if $a^2x=a$, then $e=ax$ is idempotent, and $(a)=(e)$. Furthermore, every finitely generated ideal is principal; for if $b^2y=b$, $f=by$, and $d=e+f-ef$, then $a=ad$, $b=bd$, and $d\in(e, f)=(a, b)$,

(5) Helmer's definition [3] was restricted to integral domains. More general commutative rings with identity were first investigated in this connection by Kaplansky [4].

(6) In von Neumann's definition [5], it is not assumed that S be commutative. The defining condition in the general case is $axa=a$.

whence $(a, b)=(d)$. Moreover, every element is a *unit* multiple of an idempotent(7):

LEMMA 10. *For any element a of a regular ring S (commutative, with identity), there exists a unit u such that* $a^2u=a$ *(whence* $e=au$ *is idempotent).*

Proof. Let x satisfy $a^2x=a$, and let z satisfy $x^2z=x$. Define $u=1+x-xz$. Since $axz=(a^2x)xz=a^2x=a$, we have $a^2u=a$. Now obviously, $(u, x)=(1)$. But $xu=x^2$, whence x belongs to every maximal ideal that contains u. It follows that u is a unit.

THEOREM 11. *Every regular ring S (commutative, with identity) is adequate.*

Proof. We have already remarked that S must satisfy **F**. In order to show that S satisfies **A**, consider any $a, b\in S$. By Lemma 10, we may work instead with the idempotents e, f of which a, b are unit multiples. Define $d=e+f-ef$; then, as noted above, $(d)=(e, f)$. Put $e_1=1-f+ef$. Then $e=e_1d$ and $(e_1, f)=(1)$. Since d divides f, no nonunit divisor d' of d can be relatively prime to f.

REMARK 12. Kaplansky points out [4, p. 474] that by using results developed in [1], one can show that every commutative regular ring S with identity is an elementary divisor ring. This can also be seen as follows. Working again with the idempotents e and f, let d and e_1 be as above, and define $f_1=f$. Then $e=e_1d$, $f=f_1d$, and $(e_1, f_1)=(1)$. It follows that S is an Hermite ring (Theorem 3). Therefore, by Theorem 8, S is an elementary divisor ring.

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(7) The arguments that follow are motivated by [1, Theorem 2.2].