RINGS OF ANALYTIC AND MEROMORPHIC FUNCTIONS(1)

BY

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1. **Introduction.** It has been known for some time that the conformal structure of a domain in the complex plane is determined by the algebraic structure of certain rings of analytic functions on it. Bers [1] proved that if D_1 and D_2 are two plane domains such that there is an algebraic isomorphism(2) between the rings $A(D_1)$ and $A(D_2)$ of all analytic functions on the two domains, then the domains are conformally equivalent. Rudin [9] has extended this theorem to the case in which D_1 and D_2 are arbitrary open Riemann surfaces.

In another direction Kakutani and Chevalley (unpublished, cf. Kakutani [4] and Rudin [8]) have shown that if D_1 and D_2 are two plane domains with no AB-removable(3) points, then they are conformally equivalent if the rings $B(D_1)$ and $B(D_2)$ of bounded analytic functions on them are algebraically isomorphic.

There are essentially two tools available for the proof of theorems of this type. One is the existence of a univalent function in the ring of analytic functions at hand, while the other is the knowledge of the ideal structure of various rings of analytic functions. The early proofs in this field all made use of the existence of a univalent function and (implicitly) the fact that the univalence of a function can be characterized in algebraic terms. These two circles of ideas can be contrasted by comparing the proof of the Chevalley-Kakutani theorem given by Rudin [8] with the outline given by Kakutani [4].

The existence of a univalent function is a very powerful tool, and its use enables one to extend some of these results to rings of functions of several complex variables and to some rings of meromorphic functions. Unfortunately, it is no longer available when we consider nonplanar domains, and we must fall back on ideal theory. The ideal theory of the ring of entire functions has been rather fully discussed by Henriksen [7], and the results of Florack [3] allow one to extend his discussion to the ring of all analytic functions on an open Riemann surface. In §3 of the present paper we use this ideal theory

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(2) Bers does not assume a priori that the complex constants are preserved under the given isomorphism. In this paper, however, we shall always use isomorphism and homomorphism to mean isomorphism and homomorphism as an algebra over the complex field. Chevalley and Kakutani have proved their theorem assuming isomorphism over the complex field, while Rudin has proved this theorem without a priori assumption on the complex constants.

(3) A boundary point of D is said to be AB-removable if every bounded analytic function in D is analytic at the point.

to give a characterization of the algebraic homomorphisms of the ring of analytic functions on one Riemann surface into the corresponding ring on another surface.

On a compact surface the only everywhere analytic functions are the constants, but it is well known (cf. [2]) that the conformal structure is determined by the algebraic structure of the field of meromorphic functions on the surface. This suggests the investigation of rings and fields of meromorphic functions on an arbitrary surface, and in §4 we give a generalization of the Chevalley-Kakutani theorem in terms of isomorphisms between two rings of meromorphic functions on plane domains. The natural tool for such an investigation is valuation theory. However, our knowledge of valuations on a ring of meromorphic functions is rudimentary. The argument used in §4 makes use of the fact that these rings contain the Banach algebras of bounded analytic functions on the two domains, and we give an example in §6 which indicates the impossibility of extending this technique to nonplanar domains.

In §5 we analyze the valuations on some fields of meromorphic functions on a plane domain of finite connectivity. This analysis makes strong use of the existence of a univalent function, and many questions remain open.

2. **The ring of analytic functions.** We consider an open Riemann surface W and the ring $A=A(W)$ of all analytic functions on W. In this section we discuss some of the divisibility properties and ideal theory of this ring. We first note that an element f of A is a unit if and only if it has no zeros. This suggests the introduction of the zero set $N(f)=\{p\in W:f(p)=0\}$ of the function f. We take $N(f)$ to be an algebraic set, i.e., its members have integral multiplicities which are greater than or equal to one and we define multiplicities for a union and intersection as the maxima and minima of the corresponding multiplicities. Every such zero set is discrete, and we shall call a discrete algebraic set an N-set. Florack's generalization [3] of the Weierstrass theorem then states that every N-set is the zero set of a function in A. It is apparent that a function f divides a function g if and only if $N(f)\subset N(g)$, and consequently f and g are associated if and only if they have the same zero-sets.

This last remark implies that any finite number of functions $\{f_1,f_2,\cdots,f_n\}$ in A have a greatest common divisor d, that is, a function which divides each f_i and which is divisible by any other function which does so. For we merely take the intersection of the sets $N(f_i)$ and Florack's result implies that this is the zero set of a function d which must be a greatest common divisor of the $\{f_i\}$. An important property of greatest common divisors is given by the following proposition which was given by Helmer [6] in the case of the ring of entire functions on the complex plane:

PROPOSITION 1. *Let $f_1,\cdots,f_n$ be functions in the ring A. Then they have a greatest common divisor d, and if d is any greatest common divisor there exist functions $e_1,\cdots,e_n$ in A such that*

$$d=e_1f_1+\cdots+e_nf_n.$$

Proof. By induction it suffices to prove the proposition for $n=2$. Since the greatest common divisor of f_1/d and f_2/d is 1, we may assume also that d is 1, that is, that f_1 and f_2 have no common zeros. By a theorem of Florack we can construct a function e_2 in A whose expansion at each zero p of f_1 agrees with the expansion of f_2^{-1} to the same number of terms as the order of p as a zero of f_1. Consequently $N(1-e_2f_2)\supset N(f_1)$ and so

$$e_1 = (1 - e_2f_2)/f_1$$

is in A. But this gives us $1=e_1f_1+e_2f_2$, proving the proposition.

We now prove several propositions about the ideals in the ring A. The term maximal ideal is used to mean a maximal proper ideal. We note that the ideal $\{f\}$ generated by a function f consists of all g such that $N(g)\supset N(f)$. With this in mind we prove the following proposition.

PROPOSITION 2. *The principal ideal generated by f is maximal if and only if f has exactly one zero. Thus a principal maximal ideal is the set of all functions which vanish at a fixed point p of the Riemann surface.*

Proof. If f has exactly one zero, say at p, then the homomorphism $g\rightarrow g(p)$ maps A onto the complex field and has kernel $\{f\}$. Thus $\{f\}$ is maximal.

Suppose on the other hand that $\{f\}$ is maximal. Since $\{f\}$ is proper, f must have at least one zero, say at p. By a theorem of Florack there is a function g in A which has a simple zero at p and is nonzero elsewhere. The ideal $\{g\}$ is proper and contains $\{f\}$. By the maximality of $\{f\}$, the ideals $\{f\}$ and $\{g\}$ must be identical, whence $g\in\{f\}$, and $g=hf$. But this implies f has at most one zero, completing the proof of the proposition.

This proposition tells us which principal ideals are maximal while the following one tells us which maximal ideals are principal.

PROPOSITION 3. *A maximal ideal I is principal if and only if it is the kernel of a homomorphism π of A onto the complex numbers such that $\pi(\lambda)=\lambda$ for all complex constants λ.*

Proof. If I is principal, then by Proposition 2 it consists of all functions which vanish at a given point p. The homomorphism $\pi g=g(p)$ satisfies the requirements of the proposition.

Suppose on the other hand that I is the kernel of such a homomorphism π. Let $f\in I$. Since I is proper, f has at least one zero. Let $\{z_n\}$ be the zeros of f, disregarding multiplicities. There is a function g in A such that $g(z_n)=n$ and $g'(z_n)$ is not zero. Then f and $g-\pi(g)$ can have at most one zero in common and so their greatest common divisor d has at most one zero. By Proposition 1 we may write d as $e_1f+e_2(g-\pi g)$, whence $d\in I$. Thus d has at least one zero and the ideal $\{d\}$ is maximal. Since I is a proper ideal containing $\{d\}$ we must have $I=\{d\}$, proving the proposition.

PROPOSITION 4. *An ideal I is an ideal consisting of all functions which vanish at some point $p \in W$ if and only if it is the kernel of a homomorphism π of A into the complex numbers such that $\pi(\lambda)=\lambda$ for all complex constants λ.*

This proposition is an immediate corollary of Propositions 2 and 3 and is the heart of proof of Theorem 1 in the next section. It is much more general than Propositions 2 and 3 separately since it holds not only for the ring of analytic functions on a Riemann surface but also for the ring of all continuous functions on a compact space and the ring of all functions of n complex variables which are analytic in and continuous on the boundary of a domain of existence in complex n-space.

3. **Homomorphisms of one ring of analytic functions into another.** Let W_1 and W_2 be two open Riemann surfaces and $A(W_1)$ and $A(W_2)$ the rings of all analytic functions on them. Then we have the following theorem:

THEOREM 1. *Let ϕ be a homomorphism of $A(W_2)$ into $A(W_1)$ such that $\phi(\lambda)=\lambda$ for all complex constants. Then there is a unique analytic mapping ψ of W_1 into W_2 such that $\phi(f)=f\circ\psi$. If in particular the rings $A(W_1)$ and $A(W_2)$ are isomorphic, then W_1 and W_2 are conformally equivalent.*

Proof. Let p be a point of W_1 and denote by I_p the ideal consisting of all functions in $A(W_1)$ which vanish at p. By Proposition 4 there is a homomorphism π of $A(W_1)$ onto the complex numbers which preserves constants and whose kernel is I_p.

Now $\pi\circ\phi$ is a homomorphism of A_2 into the complex numbers for which $\pi\circ\phi(\lambda)=\lambda$ for each constant λ. By Proposition 4 the kernel of $\pi\circ\phi$ must be an ideal $I_{\psi(p)}$ consisting of all functions which vanish at a point $\psi(p)$ on W_2. We have thus established a mapping ψ of W_1 into W_2.

Let f be any function in $A(W_2)$ and suppose that the value of $\phi(f)$ at p is λ. Then $\phi f-\lambda\in I_p$ and $(f-\lambda)\in I_{\psi(p)}$. Thus the value of f at $\psi(p)$ is also λ, and we have shown that $\phi(f)=f\circ\psi$.

We next show that ψ is continuous. Since W_1 and W_2 both satisfy the second axiom of countability, it suffices to show that if $p_n\to p$ on W_1, then $\psi(p_n)\to\psi(p)$ on W_2. Suppose not. Then either we may pick out a subsequence (again called $\{p_n\}$) such that $\psi(p_n)$ converges to a point $q\neq\psi(p)$ or else we may pick out a discrete subsequence. In the first case we let f be a function in $A(W_2)$ which is zero at $\psi(p)$ and nonzero at q. Then $\phi f(p_n)\to\phi f(p)=0$, while $f(\psi(p_n))\to f(q)\neq 0$, a contradiction since $\phi f=f\circ\psi$. In the second case we construct a function f which is one at each point $\psi(p_n)$ and zero at $\psi(p)$, and the same contradiction follows. Thus ψ must be continuous.

Let p be a point of W_1, and let $f\in A(W_2)$ be a function with a simple zero at $\psi(p)$. Set $g=\phi f$. Then there is a neighborhood U of $\psi(p)$ in which the function f is univalent. Take a neighborhood V of p such that $V\subset\psi^{-1}(U)$ and such that $g(V)\subset f(U)$. Then in V we have the representation $f^{-1}\circ g$ for the mapping ψ, and hence ψ is analytic.

To show that ψ is unique, suppose ψ_1 were another mapping such that $\phi f = f \circ \psi_1 = f \circ \psi$. If there were a point p where $\psi(p) \neq \psi_1(p)$ we could construct a function $f \in A(W_2)$ with different values at these points and arrive at a contradiction.

If ϕ is an isomorphism onto, then it has an inverse ϕ^{-1}. Let ψ and ψ' be the analytic mappings associated with them. Then $\psi \circ \psi'$ and $\psi' \circ \psi$ are analytic mappings of W_1 and W_2, respectively onto themselves which induce the identity homomorphisms of $A(W_1)$ and $A(W_2)$, respectively. By uniqueness we see that $\psi \circ \psi'$ and $\psi' \circ \psi$ must be the identity maps on W_1 and W_2. Thus $\psi' = \psi^{-1}$, and ψ is a one-to-one conformal correspondence between W_1 and W_2. This completes the proof.

From this theorem it follows that ϕ is one-to-one, unless its range is the complex numbers, for if ψ is not a point-mapping its range must be open. But $0 = \phi(f) = f \circ \psi$ means that f vanishes on the range of ψ, and hence everywhere. We summarize this and other properties in the following corollary.

COROLLARY. *The homomorphism ϕ of Theorem 1 is always one-to-one unless its range is one-dimensional. The mapping ψ is onto if and only if ϕ takes each nonunit of $A(W_2)$ into a nonunit of $A(W_1)$. The mapping ψ is one-to-one if and only if ϕ takes each irreducible element of $A(W_2)$ into an irreducible element or a unit of $A(W_1)$.*

In conclusion we remark that the problem of determining whether or not an open Riemann surface W_1 can be mapped into W_2 now becomes identical with the problem of imbedding $A(W_2)$ in $A(W_1)$.

4. **Characterization by rings of meromorphic functions.** For plane domains it is known that the ring of bounded analytic functions characterize the conformal structure of the domain to within AB-removable sets. More precisely, Chevalley and Kakutani (unpublished, see [4] and [8]) have shown that if D_1 and D_2 are two plane domains with the property that to each boundary point there is a bounded analytic in the domain with a singularity at the point, then the isomorphism of the rings $B(D_1)$ and $B(D_2)$ of bounded analytic functions on D_1 and D_2 implies the conformal equivalence of D_1 and D_2. We prove the following generalization:

THEOREM 2. *Let D_1 and D_2 be two plane domains with no AB-removable boundary points, and let R_1 and R_2 be any two rings of meromorphic functions on these domains such that $R_1 \supset B(D_1)$ and $R_2 \supset B(D_2)$. If R_1 and R_2 are algebraically isomorphic, then D_1 and D_2 are conformally equivalent.*

Proof. Let ϕ be an isomorphism from R_1 to R_2. We shall reduce this theorem to the Chevalley-Kakutani theorem by showing that ϕ is an isomorphism between $B(D_1)$ and $B(D_2)$. If $f \in B(D_1)$, say $|f| \leqq M$, then $(1 - f/\lambda)^{1/n}$ is in $B(D_1)$ for every n, provided $|\lambda| > M$. Hence, $(1 - \phi(f)/\lambda)^{1/n}$ is in R_2 for every n and every $\lambda \geqq M$. But this implies $1 - \phi(f)/\lambda$ does not vanish in D_2, whence we must have $|\phi(f)| \leqq M$ and $\phi(f) \in B(D_2)$. Thus ϕ takes $B(D_1)$ into $B(D_2)$

and similarly ϕ^{-1} takes $B(D_2)$ into $B(D_1)$. Consequently $B(D_1)$ and $B(D_2)$ are isomorphic and so D_1 and D_2 are conformally equivalent.

5. **Some remarks on valuations.** Let F be a field containing the complex numbers. For our purposes a valuation v on F is a homomorphism of the multiplicative group of F onto the additive group of integers such that

$$v(f_1 + f_2) \geqq \min\left[v(f_1), v(f_2)\right] \quad \text{and} \quad v(\lambda) = 0$$

for each nonzero complex number λ (cf. [2] and [11]). The set $O = \{f: v(f) \geqq 0\}$ is called the valuation ring of v, and the set $P = \{f: v(f) > 0\}$ is called the valuation ideal of v. The reasoning given in the proof of Theorem 2 is essentially that of the following remark.

PROPOSITION 5. *Let F be the field of quotients*(4) *of a commutative Banach algebra B. Then B is contained in the valuation ring of every valuation on F.*

Proof. Let $f \in B$, and let λ be a complex number such that $\|f\| < |\lambda|$. Then for each integer $n \neq 0$ the series

$$\sum_{\nu=0}^{\infty} \binom{1/n}{\nu} \frac{f^\nu}{\lambda^\nu}$$

converges in norm. If we set g equal to $\lambda^{1/n}$ times the sum of this series, we see that

$$g^n = \lambda + f.$$

Now let v be any valuation on F. Then

$$nv(g) = v(\lambda + f).$$

Thus $v(\lambda+f)$ is divisible by every integer and so must be zero. Thus $\lambda+f$, and consequently also f, belongs to the valuation ring of v, proving the proposition.

A mapping π of F into the complex sphere will be called an extended homomorphism if there is a valuation v on F such that

(i) The set $O = \{f: \pi(f) \neq \infty\}$ is the valuation ring of v;

(ii) The set $\{f: \pi(f) = 0\}$ is the valuation ideal of v; and

(iii) On O the mapping π is a homomorphism which preserves the complex numbers.

PROPOSITION 6. *Let F be the field of quotients of a Banach algebra B and let π be an extended homomorphism of F into the complex numbers. Then π is a homomorphism of B into the complex numbers and on B we have $\|\pi\| = 1$.*

(4) We thus tacitly assume B has no zero divisors. For convenience, we assume B has a unit.

Proof. Let $f \in B$, $\|f\| \leqq 1$. Then for each λ such that $\|f\| < \lambda$, we have $(\lambda - f)^{-1}$ in B, and hence

$$\pi[(\lambda - f)^{-1}] \neq \infty$$

by Proposition 5. Thus

$$0 \neq \pi(\lambda - f) = \lambda - \pi f,$$

and so πf cannot be a complex number larger than 1, whence $|\pi f| \leqq 1$, and so

$$\|\pi\| \leqq 1.$$

Since $\pi(1) = 1$, we have

$$\|\pi\| = 1,$$

proving the proposition.

If F is a field of meromorphic functions on a domain D, then for each $z_0 \in D$ we have an extended homomorphism π into the complex numbers defined by $\pi f = f(z_0)$. The natural question that arises is that of whether all extended homomorphisms of the field F arise in this manner. For the fields of all meromorphic functions and of all meromorphic functions of bounded characteristic, this is an open question. An affirmative answer in the former case would enable us to extend Theorem 1 by replacing rings of analytic functions by rings of meromorphic functions.

In the more restrictive case in which F is the field of quotients of the ring $C(D)$ of analytic functions on D which are continuous on $\overline{D}$, we can answer the question in the affirmative provided D is bounded by a finite number of Jordan curves. The question is also affirmatively settled for the field of all meromorphic functions on a compact Riemann surface [2]. The former case is given by the following proposition.

Proposition 7. *Let D be a plane domain bounded by a finite number of Jordan curves, and let F be the field of quotients of $C(D)$. Then for each extended homomorphism π of F into the complex sphere there is a point $z_0 \in D$ such that $\pi f = f(z_0)$.*

Proof. Without loss of generality we may assume that D is bounded and is bounded by a finite number of analytic boundary contours $\{C_\nu\}$. Let z be the identity function on D, and set $\pi z = z_0$. Then $z - z_0$ is in the valuation ideal P of π. We shall show that $z_0 \in D$.

If z_1 is any point not in D, we choose a point z_2 exterior to D and lying in the same component of the complement of D as z_1. Then the function $f = (z - z_1)/(z - z_2)$ has an nth root in F for every n. If v is the valuation associated with π, then as before we have $v(f) = 0$. Thus

$$0 \neq \pi(f) = \frac{\pi(z) - z_1}{\pi(z) - z_2} = \frac{z_0 - z_1}{z_0 - z_2},$$

whence $z_0 \neq z_1$. Thus $z_0 \in D$.

Since π is a homomorphism, we have $\pi r = r(z_0)$ for each rational function $r(z)$. But the rational functions are dense in $C(D)$ by Runge's theorem, and π is continuous on $C(D)$ by Proposition 5. Thus $\pi f = f(z_0)$ for $f \in C(D)$, and the extension to F is immediate.

6. **A counterexample.** It is not quite clear what the natural generalization of the Chevalley-Kakutani theorem to Riemann surfaces should be, but we give here an example of a Riemann surface W of infinite genus for which $B(W)$ is the ring of bounded analytic functions in the unit circle. In fact, let W be the two-sheeted covering surface over $0 < |z| < 1$ with branch points at $z = 1/n$, $n = 2, 3, \cdots$. Let f be a bounded analytic function on W, and let g be the difference of f on the two sheets. The function g is well-defined apart from sign, and so the function g^2 is a bounded analytic function in $0 < |z| < 1$. Hence it must be analytic in $|z| < 1$. But g^2 vanishes at the points $z = 1/n$ and so must vanish identically. From this it follows that f is a bounded analytic function of z, and so the ring $B(W)$ is just the ring of bounded analytic functions in the unit circle.

The behavior in this example is not an isolated phenomenon but must occur on any parabolic end of a Riemann surface, for Heins [5] has shown that on such an end there is a function with the property that every bounded analytic function is a function of this given one.

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