SUMS OF NORMAL ENDOMORPHISMS

BY

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1. **Introduction.** It is, of course, well-known that the set of endomorphisms of a commutative group G form a ring R with respect to the usual definition of sum and product of endomorphism, i.e., for η_1 and η_2 in R, $\eta_1\eta_2$ and $\eta_1+\eta_2$ are given by $\eta_1\eta_2(x)=\eta_1(\eta_2(x))$ and $\eta_1+\eta_2(x)=\eta_1(x)\eta_2(x)$.

If G is not commutative the situation is quite different. For one thing sums of endomorphisms need not be endomorphisms. For another the endomorphisms of G generate a near ring of mappings, a system which satisfies all the ring axioms save two. Addition is not necessarily commutative and one of the distributive laws need not hold.

Fitting investigated the structure of the algebraic system E consisting of the set of all normal endomorphisms of G and the above operation [1](1). E is like a ring save that sum is not defined for all pairs of elements.

This paper is concerned with closure properties in E and with the larger system of mappings K_G generated by the elements of E with respect to sum and product. K_G is shown to be a ring. Here, and henceforth, the word endomorphism designates a mapping which is either an endomorphism in the usual sense (here called a direct endomorphism) or a skew endomorphism. If η is a direct endomorphism then $-\eta$, given by $-\eta(x)=\eta(x)^{-1}$, is a skew endomorphism and conversely. Thus propositions about direct endomorphisms can be dualized.

The propositions on closure in E are given in Theorems 3 and 6. The related proofs bear heavily on the fact that a direct normal endomorphism η is idempotent on the commutator subgroup.

If G is noncommutative, has no nontrivial direct abelian factor, and satisfies the ascending and descending chain conditions, then η is nilpotent if and only if η maps G into its center.

It is well known that the sum of two direct normal automorphisms is an endomorphism only if G is commutative. Their difference, however, is an endomorphism into the center in all cases. It follows that, if G fulfills the above conditions and is indecomposable, K_G is generated by the identity map and the endomorphisms into the center (Corollary 6.2).

The following lemma is well-known.

LEMMA 1. *If η_1 and η_2 are normal endomorphisms on G then $\eta_1+\eta_2=\eta_2+\eta_1$.*

Presented to the Society, November 18, 1955; received by the editors February 4, 1956.

(1) Numbers in brackets refer to the bibliography at the end of the paper.

Let $\eta_1 = \sum_{i=1}^{k} \eta_{1i}$ and $\eta_2 = \sum_{j=1}^{h} \eta_{2j}$ be sums of normal endomorphisms. Then

$$\eta_1\eta_2(x) = \eta_1 \sum_{j=1}^{h} \eta_{2j}(x) = \eta_1 \prod_{j=1}^{h} [\eta_{2j}(x)] = \sum_{i=1}^{k} \eta_{1i}\left[\prod_{j=1}^{h} \eta_{2j}(x)\right]$$
$$= \prod_{i=1}^{k} \eta_{1i}\left[\prod_{j=1}^{h} \eta_{2j}(x)\right] = \prod_{i,j=1}^{k,h} [\eta_{1i}\eta_{2j}(x)].$$

Thus a product of sums of normal endomorphisms is again a sum of normal endomorphisms.

THEOREM 1. *K_G is a ring.*

Proof. The additive inverse of a normal endomorphism is again a normal endomorphism. Hence, in view of Lemma 1, K_G is a commutative group with respect to addition. The only ring axiom requiring verification is the left distributive law which can be established as follows.

Let η_1 and η_2 be arbitrary elements of K_G and let η denote a normal direct endomorphism. Then

$$\eta[\eta_1 + \eta_2](x) = \eta[\eta_1(x)\eta_2(x)] = \eta\eta_1(x)\eta\eta_2(x) = \eta\eta_1 + \eta\eta_2(x).$$

Thus, $\eta(\eta_1+\eta_2)=\eta\eta_1+\eta\eta_2$. The conclusion also follows if η is assumed to be skew by Lemma 1. A straightforward finite induction argument on the number of summands in η completes the proof. K_G will be referred to as the normal endomorphism ring of G. The symbols ϵ and θ will denote the unit element and zero of K_G respectively.

2. Closure properties.

THEOREM 2. *If η is a direct normal endomorphism on G then $\eta^2(z)=\eta(z)$ for all $z \in C_G$ the commutator subgroup of G.*

Proof. Let $z = xyx^{-1}y^{-1}$. By repeated application of the normality property we obtain $\eta(z)=\eta(xyx^{-1}y^{-1})=\eta(x)\eta(y)\eta(x)^{-1}\eta(y^{-1})=\eta[\eta(x)y\eta(x)^{-1}]\eta(y^{-1})$ $=\eta[\eta(x)y\eta(x)^{-1}y^{-1}]=\eta[\eta(x)\eta(yx^{-1}y^{-1})]=\eta^2(xyx^{-1}y^{-1})$. Thus $\eta(z)=\eta^2(z)$ for a set of generators of C_G from which Theorem 2 follows.

COROLLARY 2.1. *A normal direct automorphism on a group G leaves the commutator subgroup elementwise fixed.*

Proof. Applying η^{-1} to both sides of the expression $\eta^2(z)=\eta(z)$ yields the result.

The symbol $\eta(X)$ shall denote the set of elements $\eta(x)$ for $x \in X$.

COROLLARY 2.2. *The kernel K of a normal nilpotent direct endomorphism η on a group G contains the commutator subgroup of G and $\eta(G)$ is contained in the center Z_G of G.*

Proof. By Theorem 2, η induces an idempotent nilpotent endomorphism on C_G which then must be the zero map. The following equality is equivalent to the remaining assertion of the corollary:

$$y^{-1}\eta(x)y\eta(x)^{-1} = \eta(y^{-1}xy)\eta(x^{-1}) = \eta(y^{-1}xyx^{-1}) = e$$

for all x and y in G.

COROLLARY 2.3. *Let G satisfy the ascending and descending chain conditions for normal subgroups. Let H and K denote indecomposable noncommutative direct factors of G. There exists a normal automorphism of G mapping H onto K if and only if $C_H = C_K$.*

Proof. Let η denote a normal automorphism of G which maps H onto K. Then $C_H = \eta(C_H) \subset C_K$ and $C_K = \eta^{-1}(C_K) \subset C_H$. Hence $C_H = C_K$.

Conversely, let $H \times H_1 \times \cdots \times H_n$ and $K \times K_1 \times \cdots \times K_n$ be direct decompositions of G into indecomposable subgroups. By the Krull Schmidt Theorem [2, p. 156] there is a normal automorphism η which maps H onto K or some K_i. Since $C_H = C_K \neq e$ it follows that $\eta(H) = K$.

Corollary 2.3 implies that if H and K are indecomposable direct factors of G either $C_H = C_K$ or $C_H \cap C_K = e$.

Let N, A, and E represent the sets of normal nilpotent (necessarily direct) endomorphisms, normal direct automorphisms and normal direct endomorphisms of G respectively.

If η_1 and η_2 are in E then

$$(1) \quad \eta_1 - \eta_2(xy) = [\eta_1 - \eta_2(x)][\eta_1 - \eta_2(y)] \cdot \eta_2[\eta_1(xyx^{-1}y^{-1})(yxy^{-1}x^{-1})].$$

In order to verify relation (1) we observe that

$$\eta_1 - \eta_2(xy) = [\eta_1 - \eta_2(x)][\eta_1 - \eta_2(y)] \cdot [\eta_2(y)\eta_1(y)^{-1}\eta_2(x)\eta_1(y)\eta_2(y)^{-1}\eta_2(x)^{-1}].$$

The third expression in brackets can be rearranged to obtain relation (1) by making repeated use of the normality property as follows.

$$\begin{aligned}
\eta_2(y)\eta_1(y)^{-1}\eta_2(x)\eta_1(y)\eta_2(y)^{-1}\eta_2(x)^{-1} &= \eta_2(y)\eta_2[\eta_1(y)^{-1}x\eta_1(y)]\eta_2(y)^{-1}\eta_2(x)^{-1} \\
&= \eta_2[y\eta_1(y)^{-1}x\eta_1(y)y^{-1}x^{-1}] \\
&= \eta_2[\eta_1(y)^{-1}yx\eta_1(y)x^{-1}y^{-1}yxy^{-1}x^{-1}] \\
&= \eta_2[\eta_1(y)^{-1}\eta_1(yxyx^{-1}y^{-1})yxy^{-1}x^{-1}] \\
&= \eta_2[\eta_1(xyx^{-1}y^{-1}(yxy^{-1}x^{-1}].
\end{aligned}$$

Relation (1) shows that if η_1 leaves commutators fixed $\eta_1 - \eta_2$ is in E. Moreover, if η_2 also leaves commutators fixed $\eta_1 - \eta_2(C_G) = e$ and hence $\eta_1 - \eta_2(G) \subset Z_G$. We have established the first part of Theorem 3. The remainder is a direct consequence of Corollary 2.2.

THEOREM 3. *If $\eta_1 \in A$ and $\eta_2 \in E$ then $\eta_1 - \eta_2 \in E$. If also $\eta_2 \in A$ then*

$\eta_1-\eta_2(G)\subset Z_G$. If $\eta_1\in E$ and $\eta_2\in N$ then $\eta_1-\eta_2\in E$. If also $\eta_1\in N$ then $\eta_1-\eta_2(G)\subset Z_G$.

One would not expect to be able to tell as much about a normal endomorphism by its behavior on C_G if G contains a nontrivial direct abelian factor as otherwise. As a matter of fact if G does satisfy the ascending and descending chain conditions for normal subgroups and does not have a nontrivial direct abelian factor the above results (Theorem 3) can be sharpened. To this end the following results are useful, *where it is assumed for the remainder of this section that G is noncommutative and satisfies both chain conditions.*

LEMMA 2. *If η is a normal direct endomorphism and $\eta(C_G)=C_G$ then the kernel H of η is contained in the center of G.*

Proof. For $y\in G$ and $x\in H$ we have

$$\eta(xyx^{-1}y^{-1}) = \eta(x)\eta(y)\eta(x^{-1})\eta(y^{-1}) = \eta(y)\eta(y^{-1}) = e.$$

The condition $\eta(C_G)=C_G$ implies that η leaves commutators fixed. Thus $xyx^{-1}y^{-1}=e$, $xy=yx$ and the lemma follows.

THEOREM 4. *If $\eta(C_G)=C_G$, where η is a normal direct endomorphism, either G has a nontrivial direct abelian factor or η is an automorphism.*

Proof. By Fitting's Lemma [2, p. 155] $G=Y\otimes G_k$, for a suitable integer k, where Y is the radical of η and $G_k=\eta^k(G)$. If η is not an automorphism Y is not the trivial group. Now every x in Y is in the kernel of η^n for some n. Since $\eta^n(C_G)=C_G$ if $\eta(C_G)=C_G$ it follows by Lemma 2 that every x in Y is in the center of G, hence Y is a commutative group.

THEOREM 5. *If $\eta(C_G)=e$, where η is a normal direct endomorphism, either G has a nontrivial direct abelian factor or η is nilpotent.*

Proof. $G=Y\otimes G_k$ as above. Also, since C_G is in the kernel of η, $\eta(G)$, and hence G_k, is commutative. Thus, either $G_k=e$ and η is nilpotent or G has a nontrivial direct abelian factor.

THEOREM 6. *Given that G has no nontrivial direct abelian factor, then*

(2) *N is a group with respect to addition*;

(3) *$\eta_1-\eta_2\in A$ and $\eta_1+\eta_2\in A$ for η_1 in A and η_2 in N*;

(4) *$\eta_1-\eta_2\in N$ for η_1 in A and η_2 in A.*

Proof. By Theorem 3, $\eta_1-\eta_2$ and $\eta_1+\eta_2$ are normal direct endomorphisms in each of the above cases. That they are nilpotent or automorphisms in the different cases is shown by applying Theorems 4 or 5 after observing the effect of each on C_G.

It follows from Theorem 6 that the direct normal automorphisms of a group which has no direct abelian factor are entirely determined by the set of normal nilpotent direct endomorphisms as indicated by the following.

COROLLARY 6.1. *Given that G has no nontrivial direct abelian factor, G has a direct normal automorphism other than the identity map if and only if G has an endomorphism into its center other than* θ. *Every* $\eta \in A$ *is of the form* $\epsilon+\zeta$ *where* $\zeta \in N$ *and conversely every such mapping is in* A.

Proof. If G has distinct direct normal automorphisms η_1 and η_2 then $\eta_1-\eta_2$ is nilpotent by Theorem 6 and distinct from θ. By Corollary 2.2, $\eta_1-\eta_2(G)$ is in the center of G. Conversely, if η is an endomorphism such that $\eta(G)$ is in the center of G then η is normal direct nilpotent. If $\eta \neq \theta$, $\epsilon+\eta$ is a normal direct automorphism distinct from ϵ. The remainder of the corollary follows directly.

COROLLARY 6.2. *If G is indecomposable* K_G *is generated by* ϵ *and the set* N.

Proof. The result follows from Fitting's Lemma, which asserts that if $\eta \in E$ then $\eta \in N$ or $\eta \in A$, and Corollary 6.1.

THEOREM 7. *Given that G has no direct abelian factor. Then, if* C_G *has index* n *and* Z_G *has order* m, *G has a normal nilpotent endomorphism different from* θ *and hence a normal direct automorphism different from* ϵ *if and only if* n *and* m *have a common factor other than* 1.

Proof. Let $p \neq 1$ be a prime which divides both n and m. The order of G/C_G is then a multiple of p. It follows that G/C_G is homomorphic to the group G_p of order p. Moreover, since p divides the order of Z_G, Z_G contains a group of order p. It follows that G possesses an endomorphism ζ ($\neq \theta$) into Z_G. By the proof of Corollary 6.1 ζ is normal nilpotent. Also, by Corollary 6.1, G has a normal direct automorphism other than ϵ.

Conversely, if η is a normal direct automorphism of G then $\zeta=\epsilon-\eta$ is normal nilpotent and is an endomorphism into Z_G. The order k ($\neq 1$) of $\zeta(G)$ divides the order of Z_G. Let K denote the kernel of ζ. Then $K \supset C_G$ and the index of C_G equals k times the index of C_G in K.

These results are related to Fitting's Theorem 7 [1, p. 533] which states that if G is indecomposable N is a two-sided nilpotent ideal in E and E has the property that every η in E and not in N is a unit. Fitting's definition of a two-sided ideal in E is like the customary definition save that, as in E, sum is not defined for all pairs of elements. Theorem 6 and Corollary 6.1 imply that N is a ring in the usual sense and that N and $\epsilon+N$ are the only cosets of E (mod N).

3. Characterization of mappings in K_G.

THEOREM 8. *A normal mapping* η *on a noncommutative group G is a sum of n normal direct endomorphisms each of which leaves* C_G *elementwise fixed if and only if* η *has the following properties.*

$$\eta(x)^{-1}y\eta(x) = x^{-n}yx^{n}, \tag{5}$$

$$\eta(xy) = \eta(x)\eta(y)[y^{-n}x^{-n}(xy)^{n}]. \tag{6}$$

Proof. In proving Theorem 3 it was actually shown that if η_1 and $\eta_2 \in E$ and if $\eta_i(C_G) = C_G$ $(i=1, 2)$ then $\eta_1 - \eta_2$ is an endomorphism into the center. This implies that η is as in Theorem 8 if and only if $\eta = n\epsilon + \zeta$ where $\zeta(G) \subset Z_G$. To establish the necessity of the conditions we may assume then that $\eta = n\epsilon + \zeta$. Thus

$$\eta(x)^{-1}y\eta(x) = x^{-n}\zeta(x)^{-1}yx^{n}\zeta(x) = x^{-n}yx^{n}$$

and

$$\eta(xy) = (xy)^{n}\zeta(xy) = x^{n}\zeta(x)y^{n}\zeta(y)[y^{-n}x^{-n}(xy)^{n}] = \eta(x)\eta(y)y^{-n}x^{-n}(xy)^{n}.$$

To verify the sufficiency of conditions (5) and (6) we use an inductive argument. If $n=1$ relation (6) asserts that η is a direct endomorphism. Also by property (5), which then holds,

$$\eta(y^{-1}x^{-1}yx) = y^{-1}\eta(x)^{-1}y\eta(x) = y^{-1}x^{-1}yx$$

and, hence, η is the identity mapping on C_G.

Assuming the conditions sufficient for all $1 < n < k$ we assume $n = k$ and let $\eta_1 = \eta - \epsilon$. We have

$$\eta_1(x)^{-1}y\eta_1(x) = x\eta(x)^{-1}y\eta(x)x^{-1} = x^{-(n-1)}yx^{n-1}.$$

Also,

$$\eta_1(xy) = \eta(xy)(xy)^{-1} = \eta(x)\eta(y)y^{-n}x^{-n}(xy)^{n}(xy)^{-1}.$$

Thus, remembering that $\eta(z)^{-1}z = z\eta(z)^{-1}$, we have

$$\eta_1(xy) = \eta_1(x)\eta_1(y)[y\eta(y)^{-1}x\eta(x)^{-1}\eta(x)\eta(y)y^{-n}x^{-n}(xy)^{n-1}].$$

Using property (5) the expression in brackets reduces to $y^{-(n-1)}x^{-(n-1)}(xy)^{n-1}$. Thus by the inductive hypothesis η_1 is a sum of $k-1$ normal direct endomorphisms each of which is the identity on C_G and since $\eta = \eta_1 + \epsilon$ the theorem is proved.

COROLLARY 8.1. *A normal mapping ζ is a sum of k normal skew endomorphisms each of which is the negative of the identity mapping on C_G if and only if*

$$\zeta(x)y\zeta(x)^{-1} = x^{-n}yx^{n} \tag{7}$$

and

$$\zeta(xy) = (xy)^{-n}x^{n}y^{n}\zeta(y)\zeta(x). \tag{8}$$

Proof. The corollary follows directly from the fact that ζ is a sum of k normal skew endomorphisms each of which is the negative of the identity on

C_G if and only if $-\zeta$ is a sum of k normal direct endomorphisms each of which is the identity on C_G, the fact that $-\zeta(x)=\zeta(x)^{-1}$ and Theorem 8.

THEOREM 9. *If G satisfies both chain conditions and is indecomposable η belongs to K_G if and only if η satisfies relations* (5) *and* (6) *or relations* (7) *and* (8) *for some $n\geqq 0$.*

Proof. The conclusion is immediate if G is commutative. Fitting's Lemma implies that if G is indecomposable and η is a normal direct endomorphism then η is either an automorphism or nilpotent. Thus, in view of Theorem 6, a sum of normal endomorphisms is a sum of n normal direct automorphisms, of n normal skew automorphisms or is nilpotent. This, in view of Theorem 8 and its corollary, implies the theorem.

Theorem 9 can also be stated in the following manner.

If G satisfies both chain conditions and is indecomposable η belongs to K_G if and only if either η or $-\eta$ satisfies relations (5) and (6) for some $n\geqq 0$.

Let G_i for $i=1, \cdots, n$ represent arbitrary groups. With each element $\eta=\eta_1, \cdots, \eta_n$ in the direct sum $K=K_{G_1}\oplus \cdots \oplus K_{G_n}$ we associate the mapping $\eta'=\eta_1'+\cdots+\eta_n'$ where $\eta_i'=\eta_i e_i'$ and e_i' is the projection of $G'=G_1\otimes \cdots \otimes G_n$ onto G_i. This correspondence is an isomorphism between K and a subring K' of $K_{G'}$. Thus $\eta'\in K'$ if and only if $\eta'=\eta_1'+\cdots+\eta_n'$ where η_i' is induced by $\eta_i\in K_{G_i}$ for $i=1, \cdots, n$. K and K' are identified in the following theorem.

THEOREM 10. *Let $G'=G_1\otimes \cdots \otimes G_n$ where the G_i are arbitrary groups. Then $\eta'\in K_{G'}$ if and only if $\eta'=\eta_1'+\eta_2'$ where $\eta_1'\in K_{G_1}\oplus \cdots \oplus K_{G_n}$ and η_2' is an endomorphism into the center of G.*

Proof. The sufficiency of the condition follows from the above discussion and the fact that an endomorphism into the center is normal.

To establish the necessity of the condition we note that if $\eta'\in K_{G'}$ then

$$\eta' = (e_1'+\cdots+e_n')\eta'(e_1'+\cdots+e_n')$$
$$= e_1'\eta'e_1' + \cdots + e_n'\eta e_n' + \sum_{i\neq j} e_i'\eta' e_j'$$

where $e_{i'}\eta' e_i'$ is induced by an η_i in K_{G_i} and $\eta_2'=\sum_{i\neq j} e_i'\eta' e_j'$ is a sum of endomorphisms into the center of G and hence is an endomorphism into the center of G.

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