ON A FAMILY OF LIE ALGEBRAS OF CHARACTERISTIC \( p \)

BY

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Introduction. We study a family of Lie algebras of characteristic \( p \) which are defined as subalgebras of the derivation algebra of the group algebra of an elementary \( p \)-group. In particular we show that simple Lie algebras of dimensions \( m(p^n - 1), mp^n, p^n - 2 \), where \( m \) and \( n \) are arbitrary integers such that \( 1 \leq m < n \), and where \( p > 2 \) only for the dimensions \( p^n \) and \( p^n - 2 \), are associated with this family. The algebras studied by M. S. Frank [2] are included in our family, but those of dimension \( m(p^n - 1) \) in general appear to be new.

Since this paper was written, the paper of A. A. Albert and M. S. Frank [1] has been published. The relation between the algebras studied in [1] and those in this paper will be mentioned in §9, although it is not thoroughly clarified yet.

1. Definition of the family \( \mathfrak{F} \). Let \( \Phi \) be an algebraically closed field of characteristic \( p > 0 \), and \( \mathfrak{A} \) the group algebra over \( \Phi \) of an abelian group \( \mathfrak{G} \) of type \( (p, p, \ldots, p) \) and order \( p^n \). Let \( D_0, \ldots, D_m \) be derivations(2) of \( \mathfrak{A} \) such that \( D_i \circ D_j = 0 \) for all \( i, j \), and let \( a_0, \ldots, a_m \in \mathfrak{A} \) be such that

\[
D_i a_j = D_j a_i \quad (i, j = 0, 1, \ldots, m).
\]

Consider the set \( \mathfrak{L} = \mathfrak{L}(D_i, a_i) \) of all derivations of the form \( D = f_0 D_0 + \cdots + f_m D_m \), where \( f_i \in \mathfrak{A} \) satisfy \( \sum D_i f_i = \sum a_i f_i \). By an elementary computation, we see easily that \( \mathfrak{L} \) is a subalgebra of the derivation algebra(2) of \( \mathfrak{A} \). (The case when \( m + 1 = n \), \( a_0 = \cdots = a_m = 0 \), \( D_i = \partial / \partial g_i \), where \( g_0, \ldots, g_m \) is a set of independent generators of the group \( \mathfrak{G} \), was considered by M. S. Frank [2], and the case \( m + 1 = n \), \( a_i = 1 \), \( D_i = \partial / \partial g_i \), by A. A. Albert and M. S. Frank [1].)

In this paper, we study the family \( \mathfrak{L} \) of algebras \( \mathfrak{L}(D_i, a_i) \), where \( D_0, \ldots, D_m \) satisfy the following conditions:

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(2) By a derivation \( D \) of an algebra \( \mathfrak{A} \) over a field \( \Phi \) we mean a linear mapping of \( \mathfrak{A} \), regarded as a vector space over \( \Phi \), into itself such that \( D(fg) = (Df)g + f(Dg) \) for all \( f, g \) in \( \mathfrak{A} \). If \( D_1, D_2 \) are derivations of \( \mathfrak{A} \), then \( D_1 \circ D_2 = D_2 D_1 - D_1 D_2 \) is easily seen to be a derivation of \( \mathfrak{A} \). The totality of derivations of \( \mathfrak{A} \) forms a Lie algebra over \( \Phi \) with the ordinary addition and the multiplication \( \circ \). It is called the derivation algebra of \( \mathfrak{A} \).
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(1.0.2) \( D_i \circ D_j = 0 \) for all \( i, j \);

(1.0.3) \( \sum f_i D_i = 0 \), where \( f_i \in \mathfrak{A} \), implies \( f_i = 0 \) for all \( i \);

(1.0.4) \( D_j f = 0 \) for all \( i \) implies \( f \in \Phi \);

(1.0.5) If \( f \in \mathfrak{A} \) is such that \( D_if = \lambda_i f \), where \( \lambda_i \in \Phi \), for all \( i \), then \( f = 0 \) or \( f \) is a unit in \( \mathfrak{A} \).

The elements \( a_0, \ldots, a_m \) of \( \mathfrak{A} \) will be always assumed to be chosen such that (1.0.1) is satisfied. An ordered set \((D_0, \ldots, D_m)\) of derivations of \( \mathfrak{A} \) will be called a semi-system if (1.0.2)—(1.0.4) are satisfied, and a system if (1.0.2)—(1.0.5) are satisfied(1). Since we fix \( m > 0 \) throughout this paper, a semi-system or a system \((D_0, \ldots, D_m)\) will usually be denoted by the notation \((D_i)\).

It is shown in [4] that \( m < n \) must hold for a system. The following lemma is also shown in [4]:

**Lemma 1.1.** For a system \((D_i)\), if \( f \) and \( a_i \in \mathfrak{A} \) are such that \( D_if = a_if \) for all \( i \), then \( f = 0 \) or \( f \) is a unit in \( \mathfrak{A} \).

2. **Equivalent systems.** Two semi-systems \((D_i)\) and \((D'_i)\) are said to be equivalent if there exist \( c, y \in \mathfrak{A} \) such that

\[
(2.0.1) \quad D'_i = \sum_{s=0}^{m} c_{is} D_s
\]

for \( i = 0, \ldots, m \), and such that \( \det (c_{ij}) \) is a unit in \( \mathfrak{A} \). From the properties (1.0.2)—(1.0.3) for \((D'_i)\) it follows easily that

\[
(2.0.2) \quad D'_i c_{jk} = D'_j c_{ik}
\]

for all \( i, j, \) and \( k \).

**Lemma 2.1.** A semi-system equivalent to a system is a system.

**Proof.** Let \((D_i)\) be a semi-system equivalent to a system \((D'_i)\), and let the relation (2.0.1) hold. Suppose \( f \in \mathfrak{A} \) and \( \lambda_i \in \Phi \) are such that \( D_if = \lambda_i f \) for all \( i \). Then (2.0.1) yields \( D'_if = (\sum c_{is} \lambda_s)f \) for all \( i \). Then from Lemma 1.1 it follows that \( f = 0 \) or \( f \) is a unit in \( \mathfrak{A} \). Therefore \((D'_i)\) is a system.

Let \((D_i)\) and \((D'_i)\) be equivalent systems related by (2.0.1). Let \( (c_{ij})^{-1} = (c'_{ij}) \). Then \( D_i = \sum c'_{is} D'_s \), and \( \sum f_i D_i = \sum f'_i D'_i \), where \( f'_i = \sum f c'_{si} \). It may be readily verified that \( \sum D'_if_i = \sum a_if_i \) if and only if \( \sum D'_if_i = \sum a'_if'_i \), where

\[
(2.2.1) \quad a'_i = \sum_s (a_sc_{is} - D_sc_{is}), \quad i = 0, 1, \ldots, n.
\]

Thus we may state

(1) Semi-system and system in this paper may be called in the language of [4] "orthogonal system satisfying (1.0.4)" and "orthogonal system satisfying (1.0.4)—(1.0.5)," respectively.
Theorem 2.2. If the system \((D_i')\) is given by (2.0.1), then \(\mathfrak{g}(D_i, a_i) = \mathfrak{g}(D_i', a_i')\), where \(a_i'\) are given by (2.2.1).

The following lemma is useful in changing the formula (2.2.1).

Lemma 2.3. Let \((D_i)\) be a system, and let \(a_{ij} \in \mathfrak{g}\) be such that \(D_i a_{jk} = D_j a_{ik}\) for all \(i, j, k = 0, 1, \ldots, m\). Let \(\bar{a}_{ij}\) be the cofactor of \(a_{ij}\) in the determinant of the \((m+1) \times (m+1)\) matrix \((a_{ij})\). Then \(\sum_{i=0}^m D_s \bar{a}_{is} = 0\) for all \(s\).

Proof. For simplicity we assume that \(s = 0\). The other cases may be proved similarly. Since

\[
\det (a_{ij}) = \sum \epsilon(s_0 s_1 \cdots s_m) a_{s_0 s_1} \cdots a_{s_m m},
\]

where \(\epsilon(s_0 s_1 \cdots s_m)\) denotes +1 if the permutation 

\[
p = \begin{pmatrix} 0 & 1 & \cdots & m \\ s_0 & s_1 & \cdots & s_m \end{pmatrix}
\]

is even, -1 if \(p\) is odd, therefore

\[
\bar{a}_{0s} = \sum' \epsilon(s_1 \cdots s_m) a_{s_1} \cdots a_{s_m m},
\]

where the summation \(\sum'\) runs over all permutations \(p\) such that \(s_0 = s\). Since \(D_s\) is a derivation, we have

\[
\sum D_s \bar{a}_{0s} = \sum \epsilon(s_1 \cdots s_m) \left[ (D_s a_{s_1}) a_{s_2} \cdots a_{s_m m} + a_{s_1} (D_s a_{s_2}) \cdots a_{s_m m} + \cdots \right],
\]

where the summation on the right runs over all permutations

\[
p = \begin{pmatrix} 0 & 1 & \cdots & m \\ s & s_1 & \cdots & s_m \end{pmatrix}.
\]

By hypothesis \(D_s a_{s_1} = D_{s_1} a_{s_1}\). Since \(\epsilon(s_1 \cdots s_m) = -\epsilon(s_1 s \cdots s_m)\), the two terms \(\epsilon(s_1 \cdots s_m) (D_s a_{s_1}) a_{s_2} \cdots a_{s_m m}\) and \(\epsilon(s_1 s \cdots s_m) (D_{s_1} a_{s_1}) a_{s_2} \cdots a_{s_m m}\) cancel each other. Similarly all the other terms are divided into such pairs. Thus we see that \(\sum D_s \bar{a}_{0s} = 0\). Similarly \(\sum D_s \bar{a}_{is} = 0\) for all \(s\). Thus Lemma 2.3 is proved.

Using Lemma 2.3, we can change (2.2.1) into a more convenient form. We set \(a_{ij} = c'_{ij}\). Then the formula corresponding to (2.0.2) shows that \(a_{ij}\) satisfy the condition of Lemma 2.3. Let \(f = \det (c'_{ij})\). Then \(a_{ij} = fc_{ij}\). Hence by Theorem 2.2 we have \(\sum D_s (f c_{is}) = 0\) for all \(s\). Therefore \(f \sum D_s c_{is} + \sum c_{is} D_s f = 0\), and we obtain

(2.3.1) \[
\sum D_s c_{is} + \sum c_{is} (f^{-1} D_s f) = 0.
\]

From (2.3.1) and (2.2.1) we see that

(2.3.2) \[
a_i' = \sum c_{is} (a_s + f^{-1} D_s f), \quad f = \det (a_{ij})^{-1}, \text{ for all } i.
\]
3. Principal systems. A system \((D_i)\) is called principal if \(D_i \subseteq \Phi\) for all \(i\) implies \(f \subseteq \Phi\). Elements \(g_1, \cdots, g_n \in \mathfrak{A}\) are said to form a set of principal generators of \(\mathfrak{A}\) if \(g_i = 1\) for all \(i\) and if the \(p^n\) elements \(g_1^{u_1} \cdots g_n^{u_n}\), where \(0 \leq u_i < p\), \(g_1^0 = 1\), form a basis of \(\mathfrak{A}\) over \(\Phi\). The following Lemmas 3.1 and 3.2 are proved in [4].

**Lemma 3.1.** Any system is equivalent to a principal system.

**Lemma 3.2.** For any principal system \((D_i)\), there exists a set of principal generators \(g_1, \cdots, g_n\) of \(\mathfrak{A}\) such that

\[
(3.2.1) \quad D_i = \sum_{i=1}^{n} \alpha_{ij} G_j
\]

for all \(i\), where \(\alpha_{ij} \in \Phi\) and where \(G_i = g_i \partial / \partial g_i\) are derivations of \(\mathfrak{A}\) such that \(G_i g_j = \delta_{ij} g_j\) for all \(i, j\) and \(\delta_{ij}\) is the Kronecker delta. The principal generators \((g_i)\) will be said to belong to the principal system \((D_i)\).

From (1.0.3)–(1.0.4) we see easily that the \(\alpha_{ij}\) in (3.2.1) must satisfy (3.2.2)–(3.2.3) below:

\[
(3.2.2) \quad \text{If } u_1, \cdots, u_n \text{ are integers such that } \sum_i \alpha_{ij} u_j = 0 \text{ for all } i, \text{ then } u_i \equiv 0 \pmod{p} \text{ for all } i;
\]

\[
(3.2.3) \quad \text{If } \xi_0, \cdots, \xi_m \in \Phi \text{ are such that } \sum_{i=0}^{m} \xi_i \alpha_{si} = 0 \text{ for all } i, \text{ then } \xi_i = 0 \text{ for all } i.
\]

Conversely if elements \(\alpha_{ij} \in \Phi\) satisfy (3.2.2)–(3.2.3) and if \(D_i\) are defined by (3.2.1) with an arbitrary set of principal generators \(g_1, \cdots, g_n\) of \(\mathfrak{A}\), then \((D_i)\) is a system, as is proved in §9 of [4]. We shall now show that the system \((D_i)\) is principal. Let \(D_i \subseteq \Phi\) for all \(i\), where \(f = \sum \gamma_u g^u\), \(\gamma_u \in \Phi\). Then \(\gamma_u(e_1 \cdot u) = 0\) for all \(u \neq 0\) and \(i\), and hence by (3.2.4) we have \(\gamma_u = 0\) for all \(u \neq 0\). Therefore \(f \subseteq \Phi\), and hence \((D_i)\) is shown to be principal.

For any integers \(m\) and \(n\) such that \(0 \leq m < n\), there exist \(\alpha_{ij} \in \Phi\) such that (3.2.2)–(3.2.3) hold, since \(\Phi\) is assumed to be algebraically closed and hence infinite.

Suppose that the system \((D_i)\) is given by (3.2.1). Consider the \((m+1)\)-dimensional vector space \(\mathfrak{H}\) over \(\Phi\) consisting of all \((m+1)\)-tuples \(x = (\xi_0, \cdots, \xi_m), \xi_i \in \Phi\), and also the \(n\)-dimensional vector space \(\mathfrak{B}\) over \(\Phi\) consisting of all \(n\)-tuples \(u = (u_1, \cdots, u_n), u_i \in \Phi\). Let \(\mathfrak{B}\) be the subset of \(\mathfrak{B}\) consisting of all \(u\) such that \(u_i \in GF(p) \subseteq \Phi\) for \(i = 1, 2, \cdots, n\). Define a bilinear function \(x \cdot u\), where \(x \in \mathfrak{H}, u \in \mathfrak{B}\), with values in \(\Phi\) by setting \(x \cdot u = \sum_{i,j} \xi_i \alpha_{ij} u_j\). Then (3.2.2) and (3.2.3) are equivalent to (3.2.4) and (3.2.5), below, respectively:
(3.2.4) If $x \cdot u = 0$ for all $x \in R$ and if $u \in B$ then $u = 0$;
(3.2.5) $x \cdot u = 0$ for all $u \in B$ implies $x = 0$.

Suppose now that $g_1, \ldots, g_n$ are principal generators belonging to the principal system $(D_i)$. For $u = (u_1, \ldots, u_n) \in B$ we shall write $g^u = g_1^{u_1} \cdots g_n^{u_n}$. Let $e_i \in R$ be a vector whose $(i+1)$th component is 1 and whose other components are all 0. Then $D_ig^u = (e_i \cdot u)g^u$, and, more generally,

$$
(3.2.6) \quad (\xi_0D_0 + \cdots + \xi_mD_m)g^u = (x \cdot u)g^u, \quad \text{where} \quad x = (\xi_0, \ldots, \xi_m) \in R.
$$

The notations introduced in this section will be preserved in what follows.

4. Type and dimension of $\mathcal{L}$. For a derivation $D$ and an element $a \in \mathcal{A}$ we define a linear mapping $D-a$ of $\mathcal{A}$, regarded as a vector space over $\Phi$, into itself by $(D-a)f = Df - af$. Then the condition $D_ia_j = D_ja_i$ is equivalent to saying that the linear mappings $D_i-a_i$ and $D_j-a_j$ are commutative. Therefore, if $\mathcal{L} = \mathcal{L}(D_i; a_i) \in \mathcal{F}$, then there exist a nonzero element $b \in \mathcal{A}$ and $\alpha_i \in \Phi$ such that

$$
(4.0.1) \quad (D_i - a_i)b = \alpha_ib
$$

for all $i$; $b$ will be called a proper element of $(D_i; a_i)$ and $(\alpha_0, \ldots, \alpha_m)$ proper root belonging to $b$.

**Lemma 4.1.** If $(D_i)$ is a principal system and if $b$ is a proper element of $(D_i; a_i)$, then $b$ is a unit in $\mathcal{A}$ and all the other proper elements of $(D_i; a_i)$ are, up to a constant factor, of the form $bg^u$, where $g_1, \ldots, g_n$ is any fixed set of principal generators of $\mathcal{A}$ belonging to $(D_i)$ and where $u$ runs over $B$. If $(\alpha_0, \ldots, \alpha_m)$ is the proper root belonging to $b$, then $(\alpha_0 - (e_0 \cdot u), \ldots, \alpha_m - (e_m \cdot u))$ is the proper root belonging to $bg^u$.

**Proof.** The fact that $b$ is a unit follows immediately from Lemma 1.1, since $D_ib = (a_i - \alpha_i)b$ for all $i$.

Let $D_ib' = (a_i - \alpha_i')b'$ for all $i$. Then $D_i(b^{-1}b') = (\alpha_i - \alpha_i')b^{-1}b'$. We may suppose that $b^{-1}b' = \sum_{u \in B} \gamma_ug^u$, where $\gamma_u \in \Phi$. Then $(e_i \cdot u)\gamma_u = (\alpha_i - \alpha_i')\gamma_u$ for all $i$. Therefore if $\gamma_u \neq 0$ then $e_i \cdot u = \alpha_i - \alpha_i'$ for all $i$. Furthermore, if $\gamma_u \neq 0$ then $e_i \cdot u = \alpha_i - \alpha_i' = e_i' \cdot u'$, and hence $(e_i \cdot u - u') = 0$ for all $i$. Hence we have $u = u'$. Therefore, any proper element of $(D_i; a_i)$ is, up to a constant factor, of the form $bg^u$, and the proper root belonging to $bg^u$ is $(\alpha_0 - (e_0 \cdot u), \ldots, \alpha_m - (e_m \cdot u))$.

It is easily seen that $bg^u$ is a proper element of $(D_i; a_i)$ for any $u \in B$. Thus Lemma 4.1 is proved.

By Theorem 2.2 and Lemma 3.1, every $\mathcal{L} \in \mathcal{F}$ can be expressed as $\mathcal{L} = \mathcal{L}(D_i; a_i)$ with some principal system $(D_i)$. If there exists a proper element $b$ of $(D_i; a_i)$ such that the proper root belonging to $b$ is zero, i.e. $\alpha_i = 0$ for all $i$, then we shall say that $\mathcal{L}$ is of type I. Otherwise, $\mathcal{L}$ is said to be of type II. We will show that the above definition of the type of $\mathcal{L} = \mathcal{L}(D_i; a_i)$ is independent
of the principal system \((D_i)\) used to form \(\mathfrak{L}\). This will be done by computing the dimension of \(\mathfrak{L}\) over \(\Phi\) as follows.

Let \(b\) be a proper element of \(\mathfrak{L} = \mathfrak{L}(D; a_i)\) and let \((\alpha_0, \ldots, \alpha_m)\) be the proper root belonging to \(b\). Since \(b\) is a unit in \(\mathfrak{A}\), every element \(D \in \mathfrak{L}\) can be written in the form \(D = b \sum f_i D_i\), with \(f_i \in \mathfrak{A}\). An elementary computation shows that the condition \(\sum D_i(b f_i) = \sum a_i b f_i\) is equivalent to \(\sum D_i f_i = \sum \alpha_i f_i\). Hence we have \(\mathfrak{L}(D; a_i) = \mathfrak{L}(D; a_i)\) where \(b = \{b D | D \in \mathfrak{L}\}\). In particular, \(\dim \mathfrak{L}(D; a_i) = \dim \mathfrak{L}(D; a_i)\). Suppose now that \((D_i)\) is a principal system and the \(g_1, \ldots, g_n\) form a set of principal generators belonging to \((D_i)\). Consider \(D = \sum f_i D_i \in \mathfrak{L}(D; a_i)\). We may write \(f_i = \sum u_{i} \in \mathfrak{A} \phi_i u\), where \(\phi_i u \in \Phi\). Then the condition \(\sum f_i D_i = \sum \alpha_i f_i\) is easily seen to be equivalent to

\[
\sum_i (e_i u) \phi_i u = \sum \alpha_i \phi_i u \quad \text{(for all } u)\]

We set \(D_u = g^u \sum_i \phi_i u D_i\). Then \(D = \sum D_u, D_u \in \mathfrak{L}(D; a_i)\). Thus the vector space \(\mathfrak{L}(D; a_i)\) over \(\Phi\) is a direct sum of the vector spaces \(\mathfrak{L}_u, u \in \mathfrak{A}\), where \(\mathfrak{L}_u\) consists of elements of the form \(g^u \sum_i \xi_i D_i, \xi_i \in \Phi\). Now \(g^u \sum \xi_i D_i \in \mathfrak{L}_u\) if and only if

\[
\sum_i (e_i u) \xi_i = \sum \alpha_i \xi_i \tag{4.2.1}\]

Suppose that \(\mathfrak{L} = \mathfrak{L}(D; a_i)\) is of type I. Then we may assume \(\alpha_i = 0\) for all \(i\). From (3.2.4) and (4.2.1) it follows easily that \(\dim \mathfrak{L}_u = m\) for \(u \neq 0\) and that \(\dim \mathfrak{L}_0 = m + 1\). Hence \(\dim \mathfrak{L} = mp^n + 1\).

Suppose that \(\mathfrak{L} = \mathfrak{L}(D; a_i)\) is of type II. By (3.2.5), we may set \(\alpha_i = e_i \cdot k\), where \(k \in \mathfrak{A}\). Then by Lemma 4.1 we see that

\[
((e_0 \cdot k - u), \ldots, (e_m \cdot k - u)) \neq 0 \tag{4.2.2}\]

for all \(u \in \mathfrak{A}\). Now (4.2.1) can be expressed in the form \((x \cdot k - u) = 0\), where \(x = (\xi_0, \ldots, \xi_m)\). Therefore, because of (4.2.2), we have \(\dim \mathfrak{L}_u = m\) for all \(u \in \mathfrak{A}\). Hence \(\dim \mathfrak{L} = mp^n\). Thus we have proved

**Theorem 4.2.** If \(\mathfrak{L}\) is of type I, then \(\dim \mathfrak{L} = mp^n + 1\). If \(\mathfrak{L}\) is of type II, then \(\dim \mathfrak{L} = mp^n\).

5. **Another characterization of \(\mathfrak{L}\).** Let \(\mathfrak{L} = \mathfrak{L}(D; a_i) \subset \mathfrak{A}\) be defined by a principal system \((D_i)\). Let \(b\) be a proper element, and \((\alpha_0, \ldots, \alpha_m)\) the proper root belonging to \(b\). We set \(\alpha_i = e_i \cdot k, k \in \mathfrak{A}\), as before. (If \(L\) is of type I, then, by Lemma 4.1, we may take \(k \in \mathfrak{B}^{(3)}\).) It was shown in the course of the proof of Theorem 4.2 that \(\mathfrak{L}\) is spanned by the elements of the form \(b g^u(\sum \xi_i D_i)\), where \((x \cdot u - k) = 0, x = (\xi_0, \ldots, \xi_m)\).

Introduce the symbol \((x, u) = b g^u(\sum \xi_i D_i)\). Then:

(1) The idea of considering the case \(k \neq 0\) for algebras of type I will become clear when the reader reaches §7.
(5.0.1) \( \mathfrak{L} \) consists of elements of the form \( \sum_{u \in \mathcal{B}} (x_u, u) \), where \( (x_u \cdot u - k) = 0 \) for all \( u \in \mathfrak{B} \);

(5.0.2) \( \sum (x_u, u) = \sum (y_u, u) \) if and only if \( x_u = y_u \) for all \( u \in \mathfrak{B} \);

(5.0.3) \( \lambda \sum (x_u, u) = \sum (\lambda x_u, u) \) if \( \lambda \in \Phi \);

(5.0.4) \( \sum (x_u, u) + \sum (y_u, u) = \sum (x_u + y_u, u) \);

(5.0.5) \( (x, u) \circ (y, v) = \sum_{w \in \mathcal{B}} \beta_w((x \cdot v + w)y - (y \cdot u + w)x, u + v + w) \).

The coefficients \( \beta_w \) in (5.0.5) are those in the representation \( b = \sum_{w \in \mathcal{B}} \beta_w g^w \). Note that \( \sum \beta_w \neq 0 \) since \( b \) is a unit in \( \mathfrak{A} \). Note also that \( (x \cdot u - k) = (y \cdot v - k) = 0 \) implies \( (((x \cdot v + w)y - (y \cdot u + w)x) \cdot (u + v + w - k)) = 0 \). Conversely if we start with a bilinear function \( x \cdot u, x \in \mathfrak{R}, u \in \mathfrak{B} \), satisfying (3.2.4)-(3.2.5), an element \( k \in \mathfrak{B} \), and arbitrary elements \( \beta_u \in \Phi \), then by (5.0.1)-(5.0.5) we can define an algebra \( \mathfrak{L} \) over \( \Phi \). It can be easily verified that the multiplication \( \circ \) is skew-symmetric and satisfies the Jacobi-identity. Therefore \( \mathfrak{L} \) is a Lie algebra. If \( \sum_{w \in \mathcal{B}} \beta_w \neq 0 \) then \( \mathfrak{L} \) is isomorphic to an algebra in \( \mathfrak{F} \). This can be seen as follows: Let \( g_1, \ldots, g_m \) be a set of principal generators of \( \mathfrak{A} \). We define linear mappings \( D_i, 0 \leq i \leq m \), by \( D_i g^w = (e_i \cdot u) g^w \). It is easily verified that \( D_i \) are derivations of \( \mathfrak{A} \) and that \( (D_0, \ldots, D_m) \) is a system. If \( b = \sum_{w \in \mathcal{B}} \beta_w g^w \), then \( \sum \beta_w \neq 0 \) implies that \( b \) is a unit in \( \mathfrak{A} \). Set \( a_i = b^{-1} D_i b + e_i \cdot k \) for all \( i \). Then \( D_i a_i = D_i b \), and we have \( \mathfrak{L} \cong \mathfrak{L}(D_i; a_i) \), where \( (x, u) \) corresponds to \( b g^u \sum \xi_i D_i, x = (\xi_0, \ldots, \xi_m) \).

In the above formulation (5.0.1)-(5.0.5), \( \mathfrak{L} \) is of type I if and only if there exists \( k' \in \mathfrak{B} \) such that \( x \cdot k = x \cdot k' \) for all \( x \in \mathfrak{R} \).

Suppose that \( \mathfrak{L} \) is of type I. Then we may assume \( k \in \mathfrak{B} \). Consider the first derived algebra \( \mathfrak{L}' \) of \( \mathfrak{L} \). In the right hand side of (5.0.5), if \( u + v + w = k \), then for \( x \in \mathfrak{R} \) and \( y \in \mathfrak{Q} \), \( (x \cdot v + w) = -(x \cdot u - k) = 0 \), \( (y \cdot u + w) = -(y \cdot v - k) = 0 \). Therefore, if \( \sum (x_u, u) \in \mathfrak{L}' \) then \( x_k = 0 \). Thus we have proved

**Theorem 5.1.** If the algebra \( \mathfrak{L} \subseteq \mathfrak{F} \) is of type I, then \( \mathfrak{L}' \) is contained, as an ideal, in the subalgebra of \( \mathfrak{L} \) consisting of all \( \sum (x_u, u) \in \mathfrak{L} \) such that \( x_k = 0 \). In particular, \( \dim \mathfrak{L}' \leq m(\phi^n - 1) \).

Consider now the special case where \( m = 1, 0 \neq k \in \mathfrak{B}, \beta_0 = 1, \beta_0 = 0 \) for all \( w \neq 0 \). If \( \mathfrak{L} \) is of type I, and if \( \sum (x_u, u) \in \mathfrak{L}' \) then \( x_k = 0 \), so that (5.0.5) becomes

(5.2.1) \( (x, u) \circ (y, v) = ((x \cdot v) - (y \cdot u), x + u) \).

Suppose \( u + v = 2k \). Then \( (x \cdot u - k) = (y \cdot u - k) = 0 \). Therefore, if \( u \neq k \), \( x \neq 0 \), then \( y = \lambda x \) with \( \lambda \in \Phi \) since \( m = 1 \). Hence

\[
(x \cdot v)y - (y \cdot u)x = \lambda(x \cdot v)x - \lambda(x \cdot u)x = 0.
\]

Thus we see that if \( \sum (x_u, u) \in \mathfrak{L}'' \), the second derived algebra of \( \mathfrak{L} \), then \( x_k = x_{2k} = 0 \). In other words, \( \mathfrak{L}'' \) is contained, as an ideal, in the subalgebra con-
6. Reduction theorems. We define a subfamily \( \mathfrak{F} \) of \( \mathfrak{L} \) as follows: \( \mathfrak{L} \subseteq \mathfrak{F} \) if and only if there exists a principal system \( (D_i) \) and elements \( \lambda_i \in \Phi \) such that \( \mathfrak{L} = \mathfrak{L}(D_i; \lambda_i) \). As we shall see later, algebras in \( \mathfrak{F} \) can be discussed fairly easily. It is an open question whether \( \mathfrak{F} = \mathfrak{F}^c \) or not.

**Theorem 6.1.** Let \( \mathfrak{L} = \mathfrak{L}(D_i; \lambda_i) \) be defined by a principal system \( (D_i) \). Then \( \mathfrak{L} \subseteq \mathfrak{F} \) if and only if there exists \( a_0, \ldots, a_m \in \mathfrak{A} \) and \( \gamma_0, \ldots, \gamma_m \in \Phi \) such that \( f = \det (\delta_{ij} + D_i \gamma_j) \) is a unit in \( \mathfrak{A} \) and such that

\[
a_i = -f^{-1}D_if + D_i(\sum \lambda_i c_i) + \lambda_i \quad \text{for all } i = 0, \ldots, m.
\]

For the proof of Theorem 6.1 we need the following

**Lemma 6.2.** Suppose \( (D_i) \) is a principal system. If \( h_0, \ldots, h_m \in \mathfrak{A} \) are such that \( D_i h_j = D_j h_i \) for all \( i, j \), then there exist \( h \in \mathfrak{A} \) and \( \gamma_0, \ldots, \gamma_m \in \Phi \) such that \( h_i = D_i h + \gamma_i \) for all \( i \).

**Proof of Lemma 6.2.** Let \( g_1, \ldots, g_n \) be a set of principal generators of \( \mathfrak{A} \) belonging to \( (D_i) \), and let \( h_i = \sum u \in \mathfrak{B} \eta_{iu} g^n \), \( \eta_{iu} \in \Phi \). Then \( D_i h_j = D_j h_i \) implies \( (e_i \cdot u) \eta_{ju} = (e_j \cdot u) \eta_{iu} \) for all \( u \in \mathfrak{B} \). From (3.2.4) we have \( ((e_0 \cdot u), \ldots, (e_m \cdot u)) \neq 0 \) if \( u \neq 0 \). Hence there exists \( \rho_u \in \Phi \), for all \( u \neq 0 \), such that \( \eta_{iu} = (e_i \cdot u) \rho_u \) for all \( i \). Put \( h = \sum u \in \mathfrak{B} \rho_u g^n \), \( \gamma_i = \eta_{i0} \). Then \( h_i = D_i h + \gamma_i \) for all \( i \), as required.

**Proof of Theorem 6.1.** Suppose \( \mathfrak{L} \subseteq \mathfrak{F} \). Then there exist a principal system \( (D'_i) \) equivalent to \( (D_i) \) and a \( \lambda_i \in \Phi \) such that \( \mathfrak{L} = \mathfrak{L}(D'_i; \lambda_i) \). Let \( (D_i) \) and \( (D'_i) \) be related as in (2.0.1). Then (2.3.2) yields

\[
\lambda_i = \sum c_{is}(a_s + f^{-1}D_s f), \quad f = \det (c_{ij}).
\]

By a formula corresponding to (2.0.2) and Lemma 6.2, we see that there exist \( c_i \in \mathfrak{A} \) and \( \gamma_{ij} \in \Phi \) such that

\[
c_{ij} = D_i c_j + \gamma_{ij}, \quad i, j = 0, \ldots, m,
\]

where \( \gamma_{ij} \) are uniquely determined by \( c_{ij} \), since \( (D_i) \) is principal. We shall show that \( \det (\gamma_{ij}) \neq 0 \). Suppose \( \xi_i \in \Phi \) are such that \( \sum_{i=0}^n \gamma_i \xi_i = 0 \) for all \( i \). Then (6.1.3) yields \( \sum c_i \xi_i = D_i c \), where \( c = \sum c_i \xi_i \), and hence \( D'_i c = \xi_i \in \Phi \) for all \( i \). Since \( (D'_i) \) is principal, we have \( c \in \Phi \), and hence \( \xi_i = 0 \) for all \( i \). Thus \( \det (\gamma_{ij}) \neq 0 \) is proved. Let \( (\gamma'_{ij}) \) be the inverse matrix of \( (\gamma_{ij}) \), and let \( \bar{\lambda}_i = \sum \gamma_i \lambda_s \), \( \bar{\gamma}_i = \sum c_i \gamma'_{is} \), \( \bar{f} = \det (D_i \bar{c}_j + \delta_{ij}) \), \( \gamma = \det (\gamma_{ij}) \). Then \( \bar{f} = f \gamma \), and from (6.1.2) and (6.1.3) we have easily \( a_i = -f^{-1}D_if + D_i(\sum \bar{\lambda}_i \bar{c}_i) + \bar{\lambda}_i \) for all \( i \).

Suppose conversely, that there exist \( c_i \in \mathfrak{A} \) and \( \lambda_i \in \Phi \) such that \( f = \det (D_i \bar{c}_j + \delta_{ij}) \) is a unit in \( \mathfrak{A} \) and such that (6.1.1) holds. We set \( c'_{ij} = D_i c_j + \delta_{ij} \), \( (c_{ij}) = (c'_{ij})^{-1} \), \( D'_i = \sum c'_{ij} D_s \). First, we shall show that \( (D'_i) \) is a system. Since \( (D_i) \) is already a system, by Lemma 2.1 it is sufficient to show that \( D'_i \circ D'_j = 0 \) for all \( i, j \). Since \( D_i = \sum c'_{ij} D'_j \) for all \( i \), we have
\[ 0 = D_i \circ D_j = \sum_{s, t} (c_{is} D_{ij} c_{jt}) D_i' - \sum_{s, t} (c_{ij} D_{is} c_{ts}) D_j' + \sum_{s, t} c_{is} c_{jt} (D_i' \circ D_j') \]

\[ = \sum_{s, t} [(D_j c_{ij}) D_i' - (D_j c_{ij}) D_i'] + \sum_{s, t} c_{is} c_{jt} (D_i' \circ D_j'). \]

Now \( D_i c_{ij} = D_j c_{ij} \) for all \( i, j, t \), so that \( \sum_{s, t} c_{is} c_{jt} (D_i' \circ D_j') = 0 \) for all \( i, j \).

Finally since \( \det (c_{ij}) \) is a unit in \( \mathbb{A} \), we have \( D_i' \circ D_j' = 0 \) for all \( i, j \). Thus \( (D_i') \) is proved to be a system. We shall show that \( (D_i') \) is principal. Suppose \( D_i f = \xi_i \in \Phi \) for all \( i \). Then \( D_i = \sum_s c_{is} D_i' \) implies \( D_i (f - \sum \xi_s c_s) = \xi_i \in \Phi \) for all \( i \). Since \( (D_i) \) is principal we have \( f - \sum \xi_s c_s \in \Phi, \xi_i = 0 \) for all \( i \), and hence \( f \in \Phi \). Thus \( (D_i') \) is a principal system. The fact that \( \mathfrak{L} = \mathfrak{L}(D_i'; \lambda_i) \) follows easily from (6.1.1) and (2.3.2), and Theorem 6.1 is proved.

Define a subfamily \( \mathfrak{H}_0 \) of \( \mathfrak{H} \) as follows: \( \mathfrak{L} \in \mathfrak{H}_0 \) if and only if there exists a principal system \( (D_i) \) such that \( \mathfrak{L} = \mathfrak{L}(D_i; 0) \). Clearly every algebra in \( \mathfrak{H}_0 \) is of type I. Later we shall show that the first derived algebras \( \mathfrak{L}' \) of \( \mathfrak{L} \) in \( \mathfrak{H}_0 \) are simple for any prime \( p > 0 \). The following theorem may be proved just like Theorem 6.1.

**Theorem 6.3.** Let \( \mathfrak{L} = \mathfrak{L}(D_i; a_i) \) be defined by a principal system \( (D_i) \). Then \( \mathfrak{L} \in \mathfrak{H}_0 \) if and only if there exist \( c_0, \ldots, c_m \in \mathbb{A} \) such that \( f = \det (D_i c_j + \delta_{ij}) \) is a unit in \( \mathbb{A} \) and such that \( a_i = -f^{-1} D_i f \) for all \( i \).

Let \( (D_i) \) be a principal system, and \( (g_1, \ldots, g_n) \) a set of principal generators belonging to \( (D_i) \). For convenience an element \( h \in \mathbb{A} \) will be called "unitary" with respect to \( (D_i) \) if \( \eta_0 \) in the expression \( h = \sum_{u \in V} \gamma_u g^u, \eta_u \in \Phi, \) is not zero. This property does not depend on the choice of principal generators belonging to \( (D_i) \).

**Corollary 6.4.** Let \( (D_i) \) be a principal system, and let \( f \) be a unit in \( \mathbb{A} \) which is unitary with respect to \( (D_i) \). Then \( \mathfrak{L}(D_i; -f^{-1} D_i f) \in \mathfrak{H}_0 \).

**Proof.** In view of Theorem 6.3 it is sufficient to show that there exist \( c_0, \ldots, c_m \in \mathbb{A} \) such that \( f = \gamma \det (D_i c_j + \delta_{ij}) \) with a nonzero element \( \gamma \) in \( \Phi \).

It was proved in §9 of [4] that for any principal system \( (D_i) \), there exist elements \( \alpha_i \in \Phi \) such that the derivation \( D = \sum \alpha_i D_i \) satisfy the condition:

\[(6.4.1) \quad D h = 0 \text{ implies } h \in \Phi.\]

Let \( (g_1, \ldots, g_n) \) be a set of principal generators belonging to \( (D_i) \), and \( D g^u = \delta_u g^u, \delta_u \in \Phi \). Then (6.4.1) yields \( \delta_u \neq 0 \) for all \( u \neq 0 \). Now let \( f = \sum_{u \in V} \gamma_u g^u, \gamma_u \in \Phi, \) where \( \gamma_0 \neq 0 \) by hypothesis. Put \( c = \gamma_0^{-1} \sum_{u \neq 0} \gamma_u \delta_u^{-1} g^u, c_i = \alpha_i c. \) Then \( f = \gamma_0 (1 + D c) \), and we have det \( (D_i c_j + \delta_{ij}) = 1 + \sum D_i c_i = 1 + D c, \) and hence \( f = \gamma_0 \det (D_i c_j + \delta_{ij}) \). Thus Corollary 6.4 is proved.

7. Some lemmas. Algebras in \( \mathfrak{H}_e \) are those obtained by setting \( b = \sum \beta_{ug^u} = 1 \) in the characterization (5.0.1)-(5.0.5), and will be considered in this section and the one following. For our purposes, however, it is more convenient to consider the algebra \( \tilde{\mathfrak{L}} \) which is defined as follows: Assuming always that
\( \beta_0 = 1, \beta_w = 0 \) for \( w \neq 0 \) in \((5.0.1)-(5.0.5)\), then

(i) if \( \xi \in \mathfrak{g}_c \) is of type II, then we set \( \mathfrak{g} = \mathfrak{g} \);

(ii) if \( \xi \in \mathfrak{g}_c \) is of type I and if either \( m > 1 \) or \( k = 0 \), then we set \( \mathfrak{g} \) to be the algebra consisting of all \( \sum (x_u, u) \in \mathfrak{g} \) such that \( x_k = 0 \);

(iii) if \( \xi \in \mathfrak{g}_c \) is of type I, if \( m = 1 \), and if \( k \neq 0 \), then we set \( \mathfrak{g} \) to be the algebra consisting of all \( \sum (x_u, u) \in \mathfrak{g} \) such that \( x_k = x_{2k} = 0 \).

We shall assume \( p \neq 2 \) in case (iii) and also in case (i) if \( m = 1 \). With this assumption we shall prove that \( \mathfrak{g} \) is simple. Then we see from the result in §5 that \( \mathfrak{g} \) in case (i), \( \mathfrak{g}' \) in case (ii), and \( \mathfrak{g}'' \) in case (iii) are simple and of dimensions \( mp^n \), \( m(p^n - 1) \), and \( p^n - 2 \), respectively. In this section we shall prepare for the proof of the simplicity of \( \mathfrak{g} \).

**Lemma 7.1.** If nonzero elements \( u, v \) in \( \mathfrak{g} \) are such that \( xu = 0 \), where \( x \in \mathfrak{g}_t \), implies \( xv = 0 \), and vice versa, then there exists a nonzero \( \lambda \neq 0 \) in \( \Phi \) such that \( x \cdot u = \lambda x \cdot v \) for all \( x \in \mathfrak{g} \).

**Proof.** There exist \( \alpha_{ij} \in \Phi \) such that \( x \cdot u = \sum_{i=0}^{m} \sum_{j=1}^{n} \xi_i \alpha_{ij} u_j \), where \( x = (\xi_0, \cdots, \xi_m) \), \( u = (u_1, \cdots, u_n) \). Set \( \beta_i = \sum_j \alpha_{ij} u_j \), \( \gamma_i = \sum_j \alpha_{ij} \). Then our hypothesis implies that \( \xi_0 \beta_0 + \cdots + \xi_m \beta_m = 0 \) if and only if \( \xi_0 \gamma_0 + \cdots + \xi_m \gamma_m = 0 \). Therefore, there exists a nonzero \( \lambda \in \Phi \) such that \( \beta_i = \lambda \gamma_i \), for all \( i \), so that \( x \cdot u = \lambda x \cdot v \) for all \( x \in \mathfrak{g} \).

An element \( (x, u) \in \mathfrak{g} \) will be called a \( u \)-term or simply a term. Let \( \mathfrak{g} \) be a nonzero ideal of \( \mathfrak{g} \), and let \( A = \sum_{i=1}^{r} (x_i, u_i) \), where \( x_i \neq 0 \), \( i = 1, \cdots, r \), and where \( u_1, \cdots, u_r \) are distinct, be a nonzero element in \( \mathfrak{g} \) such that the number \( r \) of nonzero terms is as small as possible. Such an element \( A \) will be called a minimal element in \( \mathfrak{g} \).

**Lemma 7.2.** Suppose \( k \neq 0 \). If \( A = \sum (x_i, u_i) \) is a minimal element in an ideal \( \mathfrak{g} \neq 0 \), then, for any distinct \( i \) and \( j \leq r \) there exists a nonzero \( \lambda \in \Phi \) such that \( x \cdot (u_j - u_i) = \lambda x \cdot k \) for all \( x \in \mathfrak{g} \).

**Proof.** By Lemma 7.1, it is sufficient to show that \( y \cdot k = 0 \) implies \( y \cdot (u_i - u_j) = 0 \). Consider \( A' = A \circ (y, 0) = \sum_{i=1}^{r} ((y \cdot u_j) x_i, u_i) \). Since \( A' \in \mathfrak{g} \) and \( A' = (y \cdot u_j) A \) is also in \( \mathfrak{g} \) and has less than \( r \) nonzero terms. Hence \( A' = (y \cdot u_j) A \), from which it follows that \( (y \cdot u_j) x_i - (y \cdot u_j) x_i = 0 \). Therefore \( y \cdot (u_i - u_j) = 0 \).

**Lemma 7.3.** Suppose \( k = 0 \). If \( A = \sum (x_i, u_i) \) is a minimal element in \( \mathfrak{g} \), then, for any \( i \) and \( j \), there exists a nonzero \( \lambda \in \Phi \) such that \( x \cdot u_i = \lambda x \cdot u_j \) for all \( x \in \mathfrak{g} \).

**Proof.** By Lemma 7.1, it is sufficient to show that \( y \cdot u_i = 0 \) if and only if \( y \cdot u_i = 0 \). Let \( y \cdot u_i = 0 \). Then \( A' = A \circ (y, -u_i) \in \mathfrak{g} \), and \( A' \) contains less than \( r \) terms, so that \( A' = 0 \). Therefore

\[
(7.3.1) \quad (x_i \cdot u_i) y + (y \cdot u_i) x_i = 0
\]

for all \( i \). Since \( x_i \cdot u_i = 0 \), \((7.3.1)\) yields \((x_i \cdot u_i)(y \cdot u_i) = 0\). Suppose \( y \cdot u_i \neq 0 \).
Then \( x_i \cdot u_i = 0 \), and hence (7.3.1) yields \( y \cdot u_i = 0 \), a contradiction. Thus \( y \cdot u_i = 0 \), and Lemma 7.3 is proved.

**Lemma 7.4.** If \( A = \sum (x_i, u_i) \) is a minimal element in \( \mathcal{F} \), then \( x_i \cdot u_j = 0 \) for any \( i \neq j \).

**Proof.** Since \( A \circ (x_i, u_i) \) contains less than \( r \) terms, we have \( (x_j, u_j) \circ (x_i, u_i) = 0 \) for any \( i \) and \( j \). Hence

\[(7.4.1) \quad (x_i \cdot u_j)x_j - (x_j \cdot u_i)x_i = 0.
\]

Therefore \( (x_i \cdot u_j)(x_j \cdot u_j) - (x_j \cdot u_i)(x_i \cdot u_j) = 0 \). Suppose \( x_i \cdot u_j \neq 0 \). Then (7.4.1) yields

\[(7.4.2) \quad x_j \cdot (u_j - u_i) = 0.
\]

If \( k = 0 \) then Lemma 7.4 follows immediately from Lemma 7.3. Hence we assume \( k \neq 0 \). Then by Lemma 7.2 there exists \( \lambda \neq 0 \) such that \( x_j \cdot (u_j - u_i) = \lambda x_j \cdot k \). Therefore (7.4.2) gives \( x_j \cdot k = 0 \), and hence \( x_j \cdot u_j = 0 \). Then by (7.4.2) we have \( x_j \cdot u_i = 0 \). But then (7.4.1) yields \( x_i \cdot u_j = 0 \), since \( x_j \neq 0 \). This is a contradiction, and Lemma 7.4 is proved.

**Lemma 7.5.** If \( r > 1 \) for a minimal element in \( \mathcal{F} \), then \( \mathcal{F} \) contains a minimal element \( Y(x_i, M^i) \) such that \( u_1 \neq 0, u_2 \neq 0 \).

**Proof.** If \( k = 0 \), then every \( u_i \neq 0 \), and hence the lemma is clear. Suppose that \( k \neq 0 \), \( u_1 \neq 0 \), \( u_2 = 0 \). Since \( x_2 \neq 0 \), there exists \( v \in \mathbb{R} \) such that \( x_2 \cdot v \neq 0 \). If \( u_1 + v = 0 \) then \( x_2 \cdot v = -x_2 \cdot u_1 = 0 \) by Lemma 7.4, which is a contradiction. Hence

\[(7.5.1) \quad u_1 + v \neq 0, \quad v \neq 0.
\]

There exists a nonzero element \( y \in \mathbb{R} \) such that \( y \cdot (v - k) = 0 \). Consider \( A' = A \circ (y, v) \in \mathcal{F} \). Then \( A' = \sum (x'_i, u'_i) \) contains a term \( ((x_2 \cdot v)y, v) \neq 0 \). Therefore \( A' \) is a minimal element, and \( u'_1 = u_1 + v \neq 0, u'_2 = v \neq 0 \) by (7.5.1).

**Lemma 7.6.** Suppose \( m > 1 \). If \( A = \sum (x_i, u_i) \) is a minimal element in \( \mathcal{F} \), and if \( u_i \neq 0 \) for some \( i \), then \( x_j \cdot k = 0 \) for all \( j \neq i \).

**Proof.** The subspace \( \mathbb{R}' \) of \( \mathbb{R} \) consisting of all \( x' \) such that \( x' \cdot u_i = 0 \) is of dimension \( m > 1 \). Hence there exists \( y \in \mathbb{R}' \) such that \( y \) and \( x_j \) are linearly independent. The element \( A' = A \circ (y, k - u_i) \) is in \( \mathcal{F} \) and contains less than \( r \) terms. Hence \( A' = 0 \), and we have \( (x_j, u_j) \circ (y, k - u_i) = (x_j \cdot (k - u_i)y - (y \cdot u_j)x_j = 0 \) for \( j \neq i \). Since \( y \) and \( x_j \) are linearly independent, we have \( x_j \cdot (k - u_i) = 0 \). Then, by Lemma 7.4, we have \( x_j \cdot k = 0 \), as required.

**Lemma 7.7.** Suppose \( m = 1 \), \( p > 2 \), \( k \neq 0 \). If \( \sum_{i=1}^{p} (x_i, u_i) \) is a minimal element in \( \mathcal{F} \neq 0 \), and if \( r > 1 \), then \( x_i \cdot k = 0 \) for all \( i \).

**Proof.** We may assume \( i = 1 \). We have \( x_1 \cdot (u_1 - k) = 0 \), and \( x_1 \cdot u_2 = 0 \) by
Lemma 7.4. Hence $x_1 \cdot (u_1 - u_2 - k) = 0$. On the other hand, there exists a non-zero $\lambda \in \Phi$ such that

$$x \cdot (u_1 - u_2) = \lambda x \cdot k$$

for all $x \in \mathfrak{g}$. By setting $x = x_1$ in (7.7.1), we have $(\lambda - 1)x_1 \cdot k = 0$. If $\lambda \neq 1$ then $x_1 \cdot k = 0$, as required. Suppose $\lambda = 1$. Then by (7.7.1) we have $x \cdot (u_1 - u_2) = x \cdot k$ for all $x \in \mathfrak{g}$. Therefore $\mathfrak{g}$ is of type $I$, and we may assume $u_1 - u_2 = k$. Hence $u_2 \neq 0$, and we have $x_2 \cdot (u_2 + k) = 0$, $x_2 \cdot (u_2 - k) = 0$. Since $p \neq 2$, we have $x_2 = u_2 = x_2 \cdot k = 0$. By Lemma 7.4, $x_1 = u_2 = 0$. Now the subspace $\mathfrak{g}'$ consisting of all $x'$ such that $x' \cdot u_2 = 0$ is of dimension $m = 1$, since $0 \neq u_2 \in \mathfrak{B}$. Hence $x_1 = \mu x_2$ with some $\mu \in \Phi$. Then $x_1 \cdot k = \mu x_2 \cdot k = 0$, as required.

Lemma 7.8. If $A = \sum_{i=1}^{r} (x_i, u_i)$, $x_i \neq 0$, is a minimal element in a nonzero ideal $\mathfrak{g}$ in $\mathfrak{g}$, where $p$ is assumed $\neq 2$ if both of $k \neq 0$ and $m = 1$ hold, then $r = 1$.

**Proof.** Suppose $r > 1$. We shall derive a contradiction.

First consider the case $k \neq 0$. By Lemma 7.5, we may assume that $u_i \neq 0$, $u_2 \neq 0$. Then, by Lemmas 7.6 and 7.7, we have $x_i \cdot u_i = x_i \cdot k = 0$ for all $i = 1, \ldots, r$. Since $x_i \neq 0$, there exists an element $v \in \mathfrak{B}$ with $x_i \cdot v \neq 0$. Then $x_i \cdot (v - k) \neq 0$, since $x_i \cdot k = 0$. The subspaces $\mathfrak{R}' = \{x' | x' \cdot (v - k) = 0\}$ and $\mathfrak{R}'' = \{x'' | x'' \cdot k = 0\}$ are both of dimension $m$. Since $x_1 \in \mathfrak{R}'$, $x_i \in \mathfrak{R}''$ we have $\mathfrak{R} \neq \mathfrak{R}'$. Let $y \in \mathfrak{R}'$, $y \in \mathfrak{R}''$. Then $y \cdot (v - k) = 0$, $y \cdot k \neq 0$, and also $u_i + v \neq 0$ for all $i$. Since

$$(7.8.1) \quad A' = A \circ (y, v) = \sum ((x_i \cdot v)y - (y \cdot u_i)x_i, u_i + v)$$

is a minimal element, by Lemmas 7.6 and 7.7, we have $(x_i \cdot v)(y \cdot k) - (y \cdot u_i)(x_i \cdot k) = 0$ for all $i$. Since $x_i \cdot k = 0$, $y \cdot k \neq 0$, we have $x_i \cdot v = 0$ for all $i = 1, \ldots, r$, a contradiction. Therefore $r = 1$, as required.

Next consider the case $k = 0$. Choose $v \in \mathfrak{B}$, as before, such that $x_1 \cdot v \neq 0$, and $y \in \mathfrak{R}$ such that $y \cdot v = 0$, $y \cdot u_i \neq 0$. Consider $A'$ given by (7.8.1). By Lemma 7.4, we have $(x_1 \cdot v)y - (y \cdot u_1)x_1 \cdot (u_i + v) = 0$ for all $i$, and hence $(x_1 \cdot v)(y \cdot u_i) = (y \cdot u_i)(x_1 \cdot v)$, which yields $y \cdot (u_i - u_1) = 0$, since $x_1 \cdot v \neq 0$. By Lemma 7.3, there exists a nonzero $\lambda \in \Phi$ such that $y \cdot u_i = \lambda y \cdot u_i$. Then $(\lambda - 1)(y \cdot u_i) = 0$. Since $y \cdot u_i \neq 0$, $\lambda = 1$. Then $x \cdot u_i = x \cdot u_1$ for all $x \in \mathfrak{R}$, and hence $u_i = u_1$, $r = 1$. Thus Lemma 7.8 is proved.

In the following, we shall denote by $\mathfrak{R}(u)$, where $u \in \mathfrak{B}$, the subspace $\mathfrak{R}' = \{x' | x' \cdot u = 0\}$ of $\mathfrak{R}$, provided there exists at least one element $x \in \mathfrak{R}$ such that $x \cdot u \neq 0$. Note that $\mathfrak{R}(u)$, if it exists, is always of dimension $m$. If $\mathfrak{g}$ is of type $II$ and if $u \in \mathfrak{B}$ then by (4.2.2) there exists $x \in \mathfrak{R}$ such that $x \cdot (u - k) \neq 0$, and hence we can always define $\mathfrak{R}(u - k)$.

Lemma 7.9. If $0 \neq (x, u) \in \mathfrak{g}$, an ideal of $\mathfrak{g}$, and if $x \in \mathfrak{R}(v - k)$, $x \in \mathfrak{R}(v - 2k)$, then all $v$-terms are contained in $\mathfrak{g}$.

**Proof.** Since $x \in \mathfrak{R}(v - 2k)$, we have $v - u \neq k$. Let $y_1, \ldots, y_m$ be a basis
of \( R(v-u-k) \). Then \((z_i, v) = (x, u) \circ (y_i, v-u) \in \mathfrak{F}\), where \( z_i = (x \cdot v - u) y_i - (y_i \cdot u)x \). It is sufficient to show that \( z_1, \ldots, z_m \) are linearly independent. Suppose \( \sum \lambda_i z_i = 0 \) with \( \lambda_i \in \Phi \). Then

\[
(7.9.1) \quad (x \cdot v - u) \sum \lambda_i y_i - (\sum \lambda_i y_i \cdot u) x = 0.
\]

Since \( y_i \in R(v-u-k) \), (7.9.1) yields \((\sum \lambda_i y_i \cdot u)(x \cdot v - u - k) = 0\). However, \((x \cdot v - u - k) = (x \cdot v - 2k) \neq 0\). Hence \( \sum \lambda_i y_i \cdot u = 0 \). Then (7.9.1) gives \( \sum \lambda_i y_i = 0 \), because \((x \cdot v - u) = (x \cdot v - k) \neq 0\), and since \( y_1, \ldots, y_m \) are linearly independent, \( \lambda_i = 0, i = 1, \ldots, m \).

**Lemma 7.10.** If all \( u \)-terms are contained in \( \mathfrak{F} \) and if \( R(u-k) \neq R(v-k) \), then all \( v \)-terms are contained in \( \mathfrak{F} \).

**Proof.** By Lemma 7.9, it is sufficient to show that there exists \( x \in R(u-k) \) such that \( x \in R(v-k) \), \( x \in R(v-2k) \). Suppose that every \( x \in R(u-k) \) is either in \( R(v-k) \) or in \( R(v-2k) \). Let \( x_i \in R(u-k) \) be such that \( x_i \in R(v-ik) \), \( i = 1, 2 \). Then \( x_1 \in R(v-2k) \) and \( x_2 \in R(v-k) \). Then \( x = x_1 + x_2 \in R(v-ik), i = 1, 2 \), and \( x \in R(u-k) \).

**Lemma 7.11.** Suppose \( k \neq 0 \). If \( 0 \neq (x, 0) \in \mathfrak{F} \) and if \( x \cdot v \neq 0 \), then all \( v \)-terms are contained in \( \mathfrak{F} \). If all \( 0 \)-terms are contained in \( \mathfrak{F} \) and if \( R(k) \neq R(v) \) then all \( v \)-terms are contained in \( \mathfrak{F} \).

**Proof.** Lemma 7.11 follows immediately from Lemmas 7.9 and 7.10, since \( x \cdot k = 0 \).

**Lemma 7.12.** Suppose \( p \neq 2 \). If \( 0 \neq x \in R \) then there exists \( u \in \mathcal{B} \) such that \( x \in R(u-k), x \in R(u-2k) \).

**Proof.** If \( x \cdot (u'-k) = 0 \) for all \( u' \in \mathcal{B} \), then \( x \cdot u' = 0 \) for all \( u' \in \mathcal{B} \), and hence \( x = 0 \). Therefore there exists \( u' \in \mathcal{B} \) such that \( x \cdot (u'-k) = 0 \). If \( x \cdot (u'-2k) \neq 0 \), then \( u = u' \) is the required element. Suppose \( x \cdot (u'-2k) = 0 \). Then \( x \cdot (u'-k) = x \cdot k \neq 0 \). Hence \( k \neq 0 \) and \( u = 0 \) is the required element of \( \mathcal{B} \), since \( x \cdot 2k \neq 0 \) follows from \( p \neq 2 \).

**Lemma 7.13.** Suppose that \( k \neq 0 \) and that \( p > 2 \) if \( m = 1 \). Then all \( 0 \)-terms are contained in any ideal \( \mathfrak{J} \neq 0 \) of \( \mathfrak{F} \).

**Proof.** First consider the case \( p \neq 2 \). By Lemma 7.8 there exists a nonzero element \((x', u') \) in \( \mathfrak{J} \). Since \( x' \neq 0 \), by Lemma 7.12 there exists \( u \in \mathcal{B} \) such that \( x' \in R(u-k), x' \in R(u-2k) \). Then, by Lemma 7.9, all \( u \)-terms are contained in \( \mathfrak{J} \). Let \( 0 \neq x \in R(u-k) \). Then, again by Lemma 7.12, there exists \( v \in \mathcal{B} \) such that \( x \in R(v-ik), i = 1, 2 \). Thus by Lemma 7.9 all \( v \)-terms are in \( \mathfrak{J} \), and clearly \( R(u-k) \neq R(v-k) \). Now \( R(-k) = R(-2k) \), since \( p \neq 2 \). Since \( R(u-k) \neq R(v-k) \), we see that either \( R(u-k) \) or \( R(v-k) \) is different from \( R(-k) = R(-2k) \). Then by Lemma 7.10 all \( 0 \)-terms are contained in \( \mathfrak{J} \).

Next consider the case \( p = 2, m > 1 \). Let \( 0 \neq (x, u) \in \mathfrak{J} \). If \( x \cdot k = 0 \) then take
v \in \mathcal{B} \text{ such that } x \cdot v \neq 0. \text{ Hence } x \cdot (v - k) \neq 0. \text{ Since } \mathcal{R}(k) \text{ and } \mathcal{R}(v - k) \text{ are different and both of dimension } m, \text{ there exists } y \in \mathcal{R}(v - k) \text{ such that } y \notin \mathcal{R}(k). \text{ Consider } (x', u + v) = (x, u) \circ (y, v) = ((x \cdot v)y - (y \cdot u)x, u + v). \text{ Then } (x', u + v) \in \mathfrak{Z}, \text{ and } x' \cdot k = ((x \cdot v)y - (y \cdot u)x) \cdot k = (x \cdot v)(y \cdot k) \neq 0. \text{ Therefore we may assume that there exists a nonzero element } (x, u) \text{ in } \mathfrak{Z} \text{ such that } x \cdot k \neq 0. \text{ Let } x_1 = x, x_2, \ldots, x_m \text{ be a basis of } \mathcal{R}(u - k). \text{ Put } (y_1, 0) = (x_1, u) \circ (x_1, u). \text{ Then } (y_1, 0) \in \mathfrak{Z} \text{ and } y_1 = (x_1 \cdot k)x_i - (x_i \cdot k)x_1. \text{ Since } x_1 \cdot k \neq 0, \text{ the elements } y_2, \ldots, y_m \text{ form a basis of } \mathcal{R}(u - k) \cap \mathcal{R}(k). \text{ Set } y_2 = y. \text{ Then there exists } v \in \mathcal{B} \text{ such that } y \cdot v \neq 0. \text{ Since } y \cdot k = 0, \text{ we have } y \cdot (v - k) \neq 0. \text{ Since } \mathcal{R}(k) \neq \mathcal{R}(v - k), \text{ there exists } z \in \mathcal{R}(v - k) \text{ such that } z \notin \mathcal{R}(k). \text{ Now } (y, 0) \circ (z, v) = ((y \cdot v)z, v) \in \mathfrak{Z}. \text{ Since } y \cdot v \neq 0, \text{ we have } (z, v) \in \mathfrak{Z}. \text{ Now } z \cdot k \neq 0 \text{ implies, as before, that } (z', 0) \in \mathfrak{Z} \text{ for any } z' \in \mathcal{R}(v - k) \cap \mathcal{R}(k). \text{ We have } (y, 0) \in \mathfrak{Z} \text{ with } y \in \mathcal{R}(v - k) \cap \mathcal{R}(k). \text{ Since } \mathcal{R}(v - k) \cap \mathcal{R}(k) \text{ is of dimension } m - 1, \text{ we see that all 0-terms are contained in } \mathfrak{Z}.

8. Simplicity of \( \mathfrak{Z} \). We are now ready to prove the following

**Theorem 8.1.** If \( \mathfrak{L} \in \mathfrak{F}_0 \), then the first derived algebra \( \mathfrak{L}' \) is simple for any prime \( p > 0 \). \( \mathfrak{L}' \) is of dimension \( m(p^n - 1) \), where \( 1 \leq m < n \).

**Proof.** If \( \mathfrak{L} \in \mathfrak{F}_0 \) then \( \mathfrak{L} \) belongs to the case (ii) of \( \S 7 \) with \( k = 0 \). Therefore, by Theorem 5.1, it is sufficient to show that \( \mathfrak{L} \) is simple for this case.

Let \( \mathfrak{Z} \) be a nonzero ideal of \( \mathfrak{L} \). By Lemma 7.8, \( \mathfrak{Z} \) contains an element of the form \( (x, u) \neq 0 \). Since \( x \neq 0 \) there exists \( v \in \mathcal{B} \) such that \( x \cdot v \neq 0 \). Then by Lemma 7.9 all \( v \)-terms are contained in \( \mathfrak{Z} \). Now, let nonzero \( w \in \mathcal{B} \) be such that \( x \cdot w = 0 \). Since \( x \cdot v \neq 0 \), we have \( \mathcal{R}(w) \neq \mathcal{R}(v) \). Hence there exists \( y \in \mathcal{R}(w) \) such that \( y \notin \mathcal{R}(v) \). Since \( (y, v) \) is a \( \mathcal{R} \)-term, we have \( (y, v) \in \mathfrak{Z} \). Then, by Lemma 7.9, \( y \in \mathcal{R}(w) \) implies that all \( w \)-terms are contained in \( \mathfrak{Z} \). Therefore \( \mathfrak{Z} = \mathfrak{L} \), and hence \( \mathfrak{L} = \mathfrak{L}' \) is simple.

In the following, we shall denote by \( \mathfrak{F}_1 \) and \( \mathfrak{F}_{11} \), the subfamilies of \( \mathfrak{F} \) consisting of all algebras of types I and II respectively. Then \( \mathfrak{F}_0 \subset \mathfrak{F}_1 \). Let \( \mathfrak{F}_1 - \mathfrak{F}_0 \) be the set-theoretical difference of \( \mathfrak{F}_1 \) and \( \mathfrak{F}_0 \).

**Theorem 8.2.** If \( m > 1 \) then the first derived algebra \( \mathfrak{L}' \) of any algebra \( \mathfrak{L} \) in \( \mathfrak{F}_0 \cap (\mathfrak{F}_1 - \mathfrak{F}_0) \) is simple and of dimension \( m(p^n - 1) \), where \( 1 < m < n \), for any prime \( p > 0 \).

**Proof.** As in the proof of Theorem 8.1, it is sufficient to show that \( \mathfrak{F} \) is simple for the case (ii) of \( \S 7 \) when \( k \neq 0 \).

Let \( \mathfrak{Z} \) be a nonzero ideal of \( \mathfrak{F} \). By Lemma 7.13, all 0-terms are contained in \( \mathfrak{Z} \). Hence by Lemma 7.11, if \( \mathcal{R}(u) \neq \mathcal{R}(k) \) then all \( u \)-terms are contained in \( \mathfrak{Z} \).

Suppose that \( \mathcal{R}(u) = \mathcal{R}(k) \), with \( u \neq k, 2k \). Then \( \mathcal{R}(u - k) = \mathcal{R}(u - 2k) = \mathcal{R}(k) \). Let \( 0 \neq x \in \mathcal{R}(k), x \cdot v \neq 0, v \in \mathcal{B} \). Then \( \mathcal{R}(k) \neq \mathcal{R}(v) \) and hence by Lemma 7.11 all \( v \)-terms are contained in \( \mathfrak{Z} \). We have \( x \cdot (v - k) = x \cdot (v - 2k) = x \cdot v \neq 0 \). Hence \( \mathcal{R}(v - k) \neq \mathcal{R}(u - k) = \mathcal{R}(u - 2k) \). Then by Lemma 7.10 all \( u \)-terms are contained in \( \mathfrak{Z} \).
Suppose now \( u = 2k \neq 0 \). Then \( p \neq 2 \). Choose \( v \in \mathfrak{B} \) such that \( \mathfrak{K}(v) \neq \mathfrak{K}(k) \). Then \( \mathfrak{K}(2k - v) \neq \mathfrak{K}(k) \). Therefore by Lemma 7.11 all \( v \)-terms and all \( 2k - v \)-terms are contained in \( \mathfrak{I} \). Let \( x_1, \ldots, x_m \) be a basis of \( \mathfrak{K}(v - k) \), and let \( x_1 \cdot k \neq 0 \). We set \((y_1, 2k) = (x_1, v) \circ (x_2, 2k - v) \). Then \((y_2, 2k) \in \mathfrak{I} \) and \( y_2, \ldots, y_m \) are linearly independent. Hence \((y, 2k) \in \mathfrak{I} \) for any \( y \in \mathfrak{K}(v - k) \cap \mathfrak{K}(k) \). Let \( 0 \neq y \in \mathfrak{K}(v - k) \cap \mathfrak{K}(k) \), which is possible since \( m > 1 \), and let \( y \cdot v \neq 0 \). Then \( \mathfrak{K}(v') \neq \mathfrak{K}(k) \), and as before \((y', 2k) \in \mathfrak{I} \) for any \( y' \in \mathfrak{K}(v' - k) \cap \mathfrak{K}(k) \). Since \( y \in \mathfrak{K}(v' - k) \cap \mathfrak{K}(k) \), all \( 2k \)-terms are contained in \( \mathfrak{I} \). Thus \( \mathfrak{I} = \mathfrak{I}' \), which proves the simplicity of \( \mathfrak{I} = \mathfrak{I}' \).

The following two theorems may be proved similarly.

**Theorem 8.3.** Suppose \( m = 1, p > 2 \). Then the second derived algebra \( \mathfrak{L}'' \) of any algebra \( \mathfrak{L} \) in \( \mathfrak{F}_c \cap (\mathfrak{F}_1 - \mathfrak{F}_0) \) is simple and of dimension \( p^n - 2 \), where \( n > 1 \).

**Theorem 8.4.** Suppose \( p > 2 \) if \( m = 1 \). Then any algebra \( \mathfrak{L} \) in \( \mathfrak{F}_c \cap \mathfrak{F}_II \) is simple and of dimension \( mp^n \), where \( 1 \leq m < n \).

**9. Remarks.** Let \( g_1, \ldots, g_n \) be a set of principal generators of \( \mathfrak{A} \). The algebra considered by M. S. Frank [2] is obtained as \( \mathfrak{A} = \mathfrak{A}(D_1, \ldots, D_n; a_1, \ldots, a_n) \) by setting \( D_i = \partial / \partial g_i, a_i = \cdots = a_n = 0 \). Put \( D'_i = g_i \partial / \partial g_i \). Then \( (D'_i) \) is a principal system equivalent to \( (D_i) \), and \( \mathfrak{L}(D_i; 0) = \mathfrak{L}(D'_i; a'_i) \), where \( a'_i = \cdots = a'_n = -1 \), as is easily seen from (2.2.3). Put \( k = (-1, \ldots, -1) \in \mathfrak{B} \). Then \( a'_i = e_i \cdot k \) for all \( i \). Hence \( \mathfrak{L} \) falls into the family considered in Theorem 8.2. \( \mathfrak{L}' \) is simple and of dimension \( (n - 1)(p^n - 1) \) if \( n > 2 \).

The algebra denoted by the notation \( \mathfrak{L}_n \) in [1] is obtained as \( \mathfrak{L}(D_i, a_i) \) by setting \( D_i = \partial / \partial g_i, a_i = 1 \) for \( i = 1, 2, \ldots, n \). Set \( D'_i = g_i \partial / \partial g_i \) as before. Then (2.2.3) yields \( a'_i = g_i - 1 \). Suppose that \( \mathfrak{L} = \mathfrak{L}(D'_i, a'_i) \) is of type I. Then there exists a nonzero \( b \in \mathfrak{A} \) such that \( (D'_i - a'_i)b = 0 \) for all \( i \), from which it follows easily that \( \partial (bg_i) / \partial g_i = bg_i \) for all \( i \). Hence we have \( bg_i = 0, b = 0 \), a contradiction. Thus \( \mathfrak{L}_n \) is of type II, and hence of dimension \( (n - 1)p^n \). The authors have been unable to decide whether or not \( \mathfrak{L} \in \mathfrak{F}_c \). If \( \mathfrak{L} \in \mathfrak{F}_c \) then \( \mathfrak{L} \) will fall into the family considered in Theorem 8.4.

Consider now any simple algebra \( \mathfrak{L} \) of dimension \( p^n - 1 \) obtained by setting \( m = 1 \) in our Theorem 8.1. It is spanned by elements of the form \( g^u(\xi_0 D_0 + \xi_1 D_1) \), where \( g_1, \ldots, g_n \) is a set of principal generators belonging to the principal system \( (D_0, D_1) \) and where \( \xi_0, \xi_1 \in \Phi \) are such that \( \xi_0 D_0 g^u + \xi_1 D_1 g^u = 0 \). Therefore we may take as a basis of \( \mathfrak{L} \) the form of the field \( \mathfrak{L}_e = (D_1 g^u) D_0 - (D_0 g^u) D_1, \) \( u \) running over all elements \( \neq 0 \) in \( \mathfrak{B} \). Set

\[
D_1 g^u = \phi_1(u) g^u, \quad i = 0, 1; \quad \phi(u, v) = \phi_1(u) \phi_0(v) - \phi_0(u) \phi_1(v).
\]

Then it is easily seen that \( e_u \circ e_v = \phi(u, v) e_{u+v} \) for all \( u \) and \( v \). The function \( \phi(u, v) \) is a skew-symmetric bilinear form with respect to \( u \) and \( v \). Therefore the algebra \( \mathfrak{L} \) becomes a special case of the algebras considered in Theorem 11 of [1] if \( \phi(u, v) \) satisfies the condition:
\(\phi(u, v) = 0\) if and only if \(u\) and \(v\) are linearly dependent over \(GF(p)\).

However, an arbitrary principal system \((D_0, D_1)\), which can be used to define a simple algebra of dimension \(p^n - 1\) as in Theorem 8.1, does not always satisfy the condition (9.0.1).

Similar remarks may be made about the connection between simple algebras of dimension \(p^n - 2\) given in our Theorem 8.3 and those in Theorem 12 of [1].

References


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