ON QUASI-CONFORMALITY AND PSEUDO-ANALYTICITY[1]

BY

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1. **Introduction.** There exist various definitions of quasi-conformality and pseudo-analyticity. The following definition seems to be most basic: *A topological mapping* $w = T(z) = u(x, y) + iv(x, y)$ *of a plane region* D *of the* $z = x + iy$-*plane onto a region in the* $w = u + iv$-*plane is called quasi-conformal if the following conditions are satisfied*:

(i) *the partial derivatives* u_x, u_y, v_x, *and* v_y *exist and are continuous*;

(ii) $J(z) = u_x v_y - u_y v_x > 0$; *and*

$$\text{(iii)} \qquad Q(z) = \frac{\max\limits_{\theta} |D_\theta w|}{\min\limits_{\theta} |D_\theta w|} \leqq K$$

for a constant $K \geqq 1$, *where*

$$D_\theta w = (u_x \cos\theta + u_y \sin\theta) + i(v_x \cos\theta + v_y \sin\theta), \qquad 0 \leqq \theta < 2\pi.$$

A pseudo-analytic function $w = f(z)$ in D is defined as an interior mapping of D into the w-sphere[2], which is a quasi-conformal mapping of D onto the Riemann image of D under $f(z)$.

The most essential property of a quasi-conformal mapping T is that the *moduli* of a quadrilateral Ω in D and its T-image satisfy the relation

$$\text{(1)} \qquad K^{-1} \bmod \Omega \leqq \bmod T(\Omega) \leqq K \bmod \Omega;$$

(for the definition of the terms, cf. §2).

For certain reasons[3], we usually allow, in the above definition, the presence of an exceptional set consisting of isolated points in D, where the existence or the continuity of the partial derivatives may be violated. Naturally, (1) holds also in such cases.

This definition, however, has some disadvantages, e.g., the limit of a

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(1) Professor Akira Mori died after this paper was submitted to the editors. At the request of the editors the undersigned edited the paper. The editing consisted only in minor stylistic changes and in correction of obvious misprints. A few footnotes and additional references have been added. L. Bers

(2) Some mathematicians prefer to deduce this property of $f(z)$ from other conditions imposed on it.

(3) Especially, in order to include some extremal mappings; cf. Grötzsch [8], and Theorem 5 in §5 of the present paper.

uniformly convergent sequence of uniformly quasi-conformal(4) mappings is in general not quasi-conformal in the sense stated above.

Yûjôbô [18] gave a rather general definition of quasi-conformality and pseudo-analyticity, for which the usual proof of (1) (cf. that of Theorem 2, §4) is still valid, and developed his theory to a considerable extent. Though his arguments go deep into the matter, his definition has a further disadvantage: it does not assure the quasi-conformality of the inverse to a quasi-conformal mapping.

In the present paper, we shall start from the most general definition, based directly on (1), and shall deduce differentiability properties sufficient to carry out the usual proofs. Our definition also has the following advantages: (i) if T is quasi-conformal, so is its inverse T^{-1}; (ii) the family of uniformly pseudo-analytic(5) functions in D is closed with respect to uniform convergence on compact subsets; and (iii) there exists a "pseudo-analytic continuation."

Essentially the same definition was suggested by Pfluger [12]; simple consequences were stated by Hersch and Pfluger [9]. In a recent paper [1] of Ahlfors, the same definition of quasi-conformality is used, and some of our results (especially Theorems 2′ and 7) are stated in a weaker form, the proofs being quite different.

2. Definitions.

Quadrilateral. Let Ω be a simply connected region on the z-plane bounded by a Jordan curve, and let z_1, z_2, z_3, z_4 be four distinct boundary points of Ω, which lie in this order on the positively oriented boundary curve. We call such a configuration a "quadrilateral," and denote it by $\Omega(z_1, z_2, z_3, z_4)$. An orientation preserving topological mapping of the plane transforms quadrilaterals into quadrilaterals.

Modulus of a quadrilateral. Map the region Ω conformally onto a rectangle: $0<\xi<1$, $0<\eta<h$, in the $\zeta=\xi+i\eta$-plane, in such a manner that z_1, z_2, z_3, z_4 correspond to the vertices $\zeta=0, 1, 1+ih, ih$ respectively. We call the positive number h the modulus of the quadrilateral $\Omega(z_1, z_2, z_3, z_4)$, and denote it by $\operatorname{mod}\Omega(z_1, z_2, z_3, z_4)$(6). Obviously,

$$\operatorname{mod}\Omega(z_2, z_3, z_4, z_1) = 1/\operatorname{mod}\Omega(z_1, z_2, z_3, z_4). \tag{2}$$

DEFINITION. *A topological mapping $w=T(z)$ of a plane region D onto another such region Δ is called quasi-conformal with the parameter K, or, simply, a K-QC mapping, if*

(i) *$w=T(z)$ preserves orientation of the plane, and*

(ii) *for any quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ contained in D together with its boundary,*

(4) I.e., with uniformly bounded dilation quotients $Q(z)$.

(5) I.e., with one and the same parameter of pseudoanalyticity K; cf. §2.

(6) When no confusion can arise, we write simply mod Ω.

(3) $$\operatorname{mod} T(\Omega(z_1, z_2, z_3, z_4)) \leqq K \operatorname{mod} \Omega(z_1, z_2, z_3, z_4),$$

where K is a constant $\geqq 1$.

Condition (ii) states a *global* property of T. We shall later prove that it follows from the corresponding *local* properties; cf. Theorem 2, §4.

Applying (3) to $\Omega(z_2, z_3, z_4, z_1)$ and using (2), we see that also

$$K^{-1} \operatorname{mod} \Omega(z_1, z_2, z_3, z_4) \leqq \operatorname{mod} T(\Omega(z_1, z_2, z_3, z_4)).$$

Hence, *if T is a K-QC mapping, its inverse T^{-1} is also K-QC.* Further, we note the following obvious facts: *if $1 \leqq K \leqq K'$, a K-QC mapping is a K'-QC mapping; a mapping, which is K'-QC for any $K' > K \geqq 1$, is K-QC*; and *a K-QC mapping followed by a K'-QC mapping is a (KK')-QC mapping.*

Since any conformal mapping is a 1-QC mapping(7), *the composition of three mappings, of which the second is K-QC and the first and the third are conformal, is a K-QC mapping.*

In order to define the pseudo-analyticity of a function, we remark first that the notion of a K-QC mapping between plane regions can be immediately carried over, at least locally, by means of local parametrizations, to the case of a topological mapping of a Riemann surface onto another.

Let $w-f(z)$ be a complex-valued(8) function defined in a plane region D, and suppose that it is an *interior mapping* of D into the w-sphere in the sense of Stoïlow [16]. Then, as is well known, the set of all pairs of points $(z, f(z))$, z in D, forms a Riemann surface F, the "Riemann image" of D under $f(z)$. The transformation $z \leftrightarrow (z, f(z))$ is a topological mapping between D and F.

Definition. *A complex-valued function $w=f(z)$ defined in a plane region D is called pseudo-analytic with the parameter K, or, simply, a K-PA function, if it is either identically equal to a constant or satisfies the following conditions:*

(i) *$w=f(z)$ is an interior mapping of D into the w-sphere*; *and*

(ii) *the transformation $z \leftrightarrow (z, f(z))$ is a (locally) K-QC mapping of D onto the Riemann image F of D by $f(z)$.*

Since the Riemann surface F can be mapped conformally onto a plane region, we have the following proposition: *$w=f(z)$ is a K-PA function if and only if it can be represented in the form $w=\phi(T(z))$, where $\zeta=T(z)$ is (locally) a plane K-QC mapping and $\phi(\zeta)$ is an analytic function.*

3. Lemmas.

Lemma 1. *Let $\Omega^{(n)}(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)})$, $n=1, 2, \cdots$, be a sequence of quadrilaterals, and let $\Omega(z_1, z_2, z_3, z_4)$ be a quadrilateral. Suppose that the boundary arcs $[z_1^{(n)}z_2^{(n)}]^\frown$, $[z_2^{(n)}z_3^{(n)}]^\frown$, $[z_3^{(n)}z_4^{(n)}]^\frown$, and $[z_4^{(n)}z_1^{(n)}]^\frown$ of $\Omega^{(n)}$ converge in the Fréchet sense to the boundary arcs $[z_1z_2]^\frown$, $[z_2z_3]^\frown$, $[z_3z_4]^\frown$, and $[z_4z_1]^\frown$ of Ω. Then* $\lim_{n\to+\infty} \operatorname{mod} \Omega^{(n)} = \operatorname{mod} \Omega$.

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(7) As for the converse, cf. Corollary 1 of Theorem 3, §4.

(8) The value $w=\infty$ is admitted.

Hence: if a sequence of K-QC mappings of D converges to a topological mapping of D uniformly on compact subsets, the limit mapping is K-QC(9).

Proof. Without loss of generality assume that $z=0$ belongs to all $\Omega^{(n)}$ and Ω. Let $z=f_n(\zeta)$ be the function which maps U: $|\zeta|<1$ conformally onto $\Omega^{(n)}$, with the normalization $f_n(0)=0, f_n'(0)>0$. Let $z=f(\zeta)$ be the corresponding mapping function for Ω. By Courant's theorem [4], $f_n(\zeta)$ converges to $f(\zeta)$ uniformly on the closure of U. Let $\zeta_i^{(n)}$, $i=1, 2, 3, 4$, be the image of $z_i^{(n)}$ under f_n^{-1}, ζ_i the image of z_i under f^{-1}. Since $\lim_n f_n(\zeta_i^{(n)})=z_i=f(\zeta_i)$ and $f(\zeta)$ is univalent on the closure of U, we have $\lim_n \zeta_i^{(n)}=\zeta_i$, $i=1, 2, 3, 4$. Since $\operatorname{mod} U(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$ is a continuous function of the four variables $\theta_1<\theta_2<\theta_3<\theta_4<\theta_1+2\pi$, the lemma follows.

We state without proof the following theorem of Rengel [14].

LEMMA 2. *Let $\Omega(z_1, z_2, z_3, z_4)$ be a quadrilateral, let S be the area of Ω, and λ the infimum of the lengths of curves in Ω, which connect the boundary arcs $[z_4z_1]^\frown$ and $[z_2z_3]^\frown$ of Ω. Then* $\operatorname{mod} \Omega \leqq S/\lambda^2$.

A doubly connected plane region bounded by two continua will be called an *annulus*. For any annulus A, we can find a circular ring $0<q<|z|<1$ conformally equivalent to A. We call the positive number $\log(1/q)$ the *modulus* of A, and denote it by mod A.

LEMMA 3. *Let T be a K-QC mapping of D. Then, for any annulus A, which is contained in D and is bounded by two Jordan curves in D*(10),

$$K^{-1} \operatorname{mod} A \leqq \operatorname{mod} T(A) \leqq K \operatorname{mod} A. \tag{4}$$

Hence, by a simple approximation process, we see immediately that a K-QC mapping $w=T(z)$ never maps $|z|<1$ onto $|w|<+\infty$, nor $0<q<|z|<1$ onto $0<|w|<1$.

Proof. Since also T^{-1} is K-QC, it suffices to prove the second inequality. Making auxiliary conformal mappings, we may assume that A is the circular ring $q<|z|<1$, that $T(A)$ is $r<|w|<1$, and that T is defined and topological on the closure of A. We cut A along the positive real axis, so that, by $z'=\log z$, it is transformed into a rectangle Ω in the z'-plane. Then

$$\operatorname{mod} \Omega(\log q, 0, 2\pi i, \log q + 2\pi i) = 2\pi \Big/ \log \frac{1}{q} = 2\pi/\operatorname{mod} A.$$

By the mapping $w'=T'(z')=\log T(e^{z'})$, the quadrilateral Ω is transformed into a quadrilateral in the $w'=\log w$-plane. Applying Lemma 2 to $T'(\Omega)$, we obtain

(9) Compare with Corollary 1 of Theorem 10, §6.

(10) This second condition can be easily removed by a simple approximation process.

$$\text{mod } T'(\Omega) \leqq 2\pi \log \frac{1}{r} \Big/ \left\{\log \frac{1}{r}\right\}^2 = 2\pi/\text{mod } T(A).$$

Now, T' is topological on the closure of Ω, and K-QC in Ω. Hence, by an approximation process used in proving Lemma 1, we obtain easily the inequality

$$K^{-1} \text{mod } \Omega \leqq \text{mod } T'(\Omega),$$

and the desired inequality follows.

The following lemma plays a fundamental role in the sequel.

LEMMA 4. *Let $w = T(z)$ be a K-QC mapping of a plane region D onto another such region Δ. Suppose that a disc $|z - z_0| \leqq r$ is contained in D, and let $m(r)$, $M(r)$ denote the minimum and the maximum of $|T(z) - T(z_0)|$ on the circumference $|z - z_0| = r$. Then,*

$$M(r) \leqq e^{\pi K} m(r),$$

provided that the disc $|w - T(z_0)| \leqq M(r)$ is contained in Δ.

Proof. If $m(r) = M(r)$, there is nothing to prove. If $m(r) < M(r)$, denote by A the annulus $m(r) < |w - T(z_0)| < M(r)$ contained in Δ. Its inverse image $T^{-1}(A)$ is a plane annulus; one of its complementary continua contains $z = z_0$ and a point on $|z - z_0| = r$, the other contains the point at infinity and a point on the same circumference. By Teichmüller's theorem [17], the modulus of such an annulus does not exceed that of the $x + iy$-plane slit along the segment $-r \leqq x \leqq 0$ and the ray $r \leqq x < +\infty$. The latter is known to be $= \pi$ (cf. [17]). Hence, by Lemma 3,

$$\log \frac{M(r)}{m(r)} = \text{mod } A \leqq K \text{ mod } T^{-1}(A) \leqq \pi K,$$

which implies the result.

LEMMA 5. *Let C: $w = w(s) = u(s) + iv(s)$, $0 \leqq s \leqq L$, be a simple rectifiable curve in the $w = u + iv$-plane, s being the arc length of C measured from $w(0)$ to $w(s)$. Let E be a closed set of positive measure mE on the s-interval, and let E_w be its image in the w-plane. Then, at least one of the projections of E_w on the u- and v-axes is of positive linear measure.*

Proof. $u(s)$ and $v(s)$ are absolutely continuous in s, and $(du/ds)^2 + (dv/ds)^2 = 1$ almost everywhere in $0 < s < L$, so that

$$mE = \int_E \{(du/ds)^2 + (dv/ds)^2\}^{1/2} ds \leqq \int_E |du/ds| \, ds + \int_E |dv/ds| \, ds.$$

Hence, at least one of the last two integrals is $\geqq mE/2 > 0$. We assume that the first one is positive, and prove that the set of values $u(s)$ for $s \in E$ is of positive linear measure.

Cover the closed set E by an open set G consisting of a finite number of nonoverlapping open intervals I_n. For any $-\infty < u < +\infty$, let $N(u; I_n)$ denote the number of points s in I_n satisfying $u(s) = u$. Then, by Banach's theorem (cf. [15, p. 280]), $N(u; I_n)$ is Borel measurable, and its integral over $-\infty < u < +\infty$ equals the absolute variation of $u(s)$ on I_n, i.e.

$$\int_{-\infty}^{+\infty} N(u; I_n) du = \int_{I_n} \left| \frac{du}{ds} \right| ds.$$

Putting $N(u; G) = \sum_n N(u; I_n)$, we have

$$\int_{-\infty}^{+\infty} N(u;G) du = \int_{G} \left| \frac{du}{ds} \right| ds.$$

Now, we let G tend to E from above, so that the right hand side tends to $\int_E |du/ds| ds$. Since $N(u; G)$ decreases with G for each u, the left hand side tends to the integral of $\lim_{G \to E} N(u; G) = N(u)$.

Let $N(u; E)$ be the number of points s on E satisfying $u(s) = u$. Clearly, $N(u; E) \leqq N(u)$. We shall prove that the equality holds whenever $N(u)$ is finite. Since, for each u, $N(u; G)$ is an integer if finite, we have $N(u) = N = N(u; G)$ for G sufficiently close to E. Let s_i, $i = 1, \cdots, N$, be the N points in G satisfying $u(s_i) = u$. Since $N(u; G) = N$ for any G close to E, each s_i must be contained in any such G, so that $s_i \in E$, $i = 1, \cdots, N$. Hence, we have $N(u; E) \geqq N = N(u)$.

Since $N(u)$ is integrable over $-\infty < u < +\infty$, it is finite for almost all u. Hence, $N(u; E)$ is Lebesgue measurable and integrable, and we have

$$\int_{-\infty}^{+\infty} N(u; E) du = \int_{E} \left| \frac{du}{ds} \right| ds > 0.$$

Hence, $N(u; E) > 0$ for a set of u of positive measure, q.e.d.

4. **Fundamental properties of K-QC mappings.**

THEOREM 1([11]). *Let $w = T(z) = u(x, y) + iv(x, y)$ be a K-QC mapping of a plane region D in the $z = x + iy$-plane onto another such region Δ in the $w = u + iv$-plane. Then,*

(i) *$w = T(z)$ is totally differentiable*([12]) *almost everywhere in D;*

(ii) *at each point z, at which $w = T(z)$ is totally differentiable,*

([11]) Theorem 1 can be used in order to prove, in a few lines, that the definition of quasi-conformality used by Mori is equivalent to the analytic definition, more precisely, that every quasi-conformal mapping in the sense of Mori has generalized L_2 derivatives satisfying the differential inequality (ii), cf. [20]. Statement (iii) of Theorem 1 has been proved independently by K. Strebel [25]. L. B.

([12]) I.e., both $u(x, y)$ and $v(x, y)$ are totally differentiable.

$$\max_{\theta} |D_\theta w|^2 \leqq K \cdot J(z),$$

where

$$D_\theta w = (u_x \cos\theta + u_y \sin\theta) + i(v_x \cos\theta + v_y \sin\theta),$$
$$J(z) = u_x v_y - u_y v_x \geqq 0;$$

(iii) *for almost all* $y=y_0$, $w=T(x, y_0)$ *is absolutely continuous*([13]) *in* x *on any closed interval contained in the intersection of* $y=y_0$ *and* D.

In virtue of these properties, we can extend almost all known results on continuously differentiable quasi-conformal mappings to the class of K-QC mappings, by simple recapitulation of the original proofs.

Proof of (i). In view of Rademacher-Stepanoff's theorem (cf. [15, p. 310]), it suffices to show that, at almost every z in D,

$$(5) \qquad \limsup_{\Delta z \to 0} \frac{|T(z+\Delta z) - T(z)|}{|\Delta z|} < +\infty.$$

For any Borel set E in D, we denote by $S(E)$ the two-dimensional measure of its image $T(E)$ in Δ, which also is a Borel set. Clearly, $S(E)$ is a non-negative additive function of Borel sets, so that, by Lebesgue's theorem it is differentiable in the general sense almost everywhere in D.

For any fixed z and sufficiently small $r>0$, denote by $m(r)$ and $M(r)$ the minimum and the maximum of $|\Delta T| = |T(z+\Delta z) - T(z)|$ for $|\Delta z| = r$. Let d_r denote the closed disc of radius r about z. Since, by Lemma 4, $|\Delta T| \leqq M(r) \leqq e^{\pi K} m(r)$, we have

$$\left(\frac{|\Delta T|}{|\Delta z|}\right)^2 \leqq e^{2\pi K} \frac{m(r)^2}{r^2} \leqq e^{2\pi K} \frac{S(d_r)}{\pi r^2}.$$

For $r \to 0$, the last quotient tends to the derivative $DS(z)$ of $S(E)$, if S is derivable at z. Hence, (5) holds almost everywhere in D, q.e.d.

Proof of (ii). Suppose that $u(x, y)$ and $v(x, y)$ are totally differentiable at z_0. By translations and suitable rotations of the z- and w-planes, which clearly leave $J(z_0)$ and $\max_\theta |D_\theta w|^2$ invariant, we get $z_0 = T(z_0) = 0$ and

$$u(x, y) = ax + o(|z|), \qquad v(x, y) = by + o(|z|).$$

Since T preserves the orientation of the plane, $J(0) = ab$ cannot be negative, so that we may assume $0 \leqq b \leqq a$.

Suppose first that $J(0) = ab > 0$. For small $\delta > 0$, let $\Omega(\delta(-1-i), \delta(1-i), \delta(1+i), \delta(-1+i))$ be the square $-\delta < x < \delta$, $-\delta < y < \delta$. Since mod $\Omega = 1$, we have mod $T(\Omega) \geqq K^{-1}$. As is easily seen from Lemma 1, mod $T(\Omega)$ tends to b/a for $\delta \to 0$, so that $a \leqq Kb$. Hence,

([13]) I.e., both $u(x, y_0)$ and $v(x, y_0)$ are absolutely continuous.

$$|D_\theta w|^2 = a^2\cos^2\theta + b^2\sin^2\theta < a^2 \leqq Kab = K\cdot J(0)$$

for any $0\leqq\theta<2\pi$.

Next, suppose that $b=0$. Then, $J(0)=ab=0$ and $T(0, y)=o(|y|)$. By Lemma 4, we have

$$|u(x, y)| \leqq |T(x, y)| \leqq e^{\pi K}|T(0, |z|)| = o(|z|).$$

Hence, $ax=u(x, y)+o(|z|)=o(|z|)$, so that $a=0$. Then, $|D_\theta w|^2=a^2\cos^2\theta+b^2\sin^2\theta=0$ for any θ.

Proof of (iii). Let $x_1<x<x_2$, $y_1<y<y_2$ be a rectangle contained in D together with its boundary. Since D can be covered by a denumerable infinity of such rectangles, it suffices to prove that, for almost all $y=y_0$, $y_1<y_0<y_2$, $T(x, y_0)$ is absolutely continuous in x on the interval $x_1\leqq x\leqq x_2$.

For any $y_1<y_0<y_2$, let $A(y_0)$ denote the area of the T-image of the subrectangle $x_1<x<x_2$, $y_1<y<y_0$. Since $A(y_0)$ is a monotone increasing function of y_0, it has a finite derivative $A'(y_0)$ at almost every y_0. We choose such a y_0.

For any small $\Delta y>0$, let Ω denote the rectangle $x_1<x<x_2$, $y_0<y<y_0+\Delta y$. Its image $T(\Omega)$ in Δ has the area $A(y_0+\Delta y)-A(y_0)$[14]. Let $\lambda(y_0, \Delta y)$ be the infimum of the lengths of continuous curves in $T(\Omega)$, which connect the T-image of $x=x_1$, $y_0<y<y_0+\Delta y$ to that of $x=x_2$, $y_0<y<y_0+\Delta y$. Applying Lemma 2 to the T-image of the quadrilateral

$$\Omega(x_1+iy_0,\ x_2+iy_0,\ x_2+i(y_0+\Delta y),\ x_1+i(y_0+\Delta y)),$$

we have

$$\operatorname{mod} T(\Omega) \leqq \{A(y_0+\Delta y)-A(y_0)\}/(\lambda(y_0, \Delta y))^2.$$

Since $\Delta y/(x_2-x_1)=\operatorname{mod}\Omega\leqq K \operatorname{mod} T(\Omega)$, it follows that

$$\begin{aligned}(\lambda(y_0, \Delta y))^2 &\leqq K(x_2-x_1)\frac{A(y_0+\Delta y)-A(y_0)}{\Delta y}\\ &< K(x_2-x_1)\{A'(y_0)+1\} = \Lambda(y_0)\end{aligned}$$

for any sufficiently small $\Delta y>0$.

Let $x_1=\xi_0<\xi_1<\cdots<\xi_n=x_2$ be an arbitrary subdivision of the interval $x_1\leqq x\leqq x_2$. For any given $\epsilon>0$, take Δy so small that $|T(\xi_i, y)-T(\xi_i, y_0)|<\epsilon$, $i=0, 1, \cdots, n$, for $y_0<y<y_0+\Delta y$. For such a Δy, any curve C in Ω, which connects the opposite sides on $x=x_1$ and $x=x_2$, contains n nonoverlapping arcs, whose T-images connect a point in the disc $|w-T(\xi_{i-1}, y_0)|<\epsilon$ to a point in $|w-T(\xi_i, y_0)|<\epsilon$, $i=1, \cdots, n$. Hence, the length of the T-image of C is not less than $\sum_{i=1}^n\{|T(\xi_i, y_0)-T(\xi_{i-1}, y_0)|-2n\epsilon\}$. Among such C's, there is one, whose image in $T(\Omega)$ has a length $<\Lambda(y_0)$. Hence,

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(14) y_0 being a point of continuity of the function $A(y_0)$.

$$\sum_{i=1}^{n} | T(\xi_i, y_0) - T(\xi_{i-1}, y_0) | < \Lambda(y_0) + 2n\epsilon.$$

Letting first $\epsilon \to 0$, and next $n \to +\infty$ so as to make $\max_i (\xi_i - \xi_{i-1}) \to 0$, we see that $T(x, y_0)$ is of bounded variation on $x_1 \leqq x \leqq x_2$.

Let the rectifiable curve $w = T(x, y_0)$, $x_1 \leqq x \leqq x_2$, be represented by $w = w(s)$, $0 \leqq s \leqq L$, in terms of its arc length s from $w(0) = T(x_1, y_0)$; and let $x = x(s)$, $s = s(x)$ be the composed topological mappings between $0 \leqq s \leqq L$ and $x_1 \leqq x \leqq x_2$. The absolute continuity of $T(x, y_0)$ is equivalent to that of $s = s(x)$, and to prove the latter it suffices to show that any closed set E_x of measure zero in the open interval $x_1 < x < x_2$ is mapped by $s = s(x)$ onto a closed set E of measure zero in $0 < s < L$[15].

Suppose that $mE_x = 0$ and $mE > 0$. Then, $dx(s)/ds = 0$ must hold almost everywhere on E. Further, for $w = w(s)$, $\lim_{\Delta s \to 0} |w(s+\Delta s) - w(s)| / |\Delta s| = 1$ almost everywhere in $0 < s < L$. Hence, by Egoroff's theorem, we can find a closed subset E' of E, having positive measure such that for a sequence $\Delta s_n \to 0$

$$\{x(s + \Delta s_n) - x(s)\}/\Delta s_n \to 0 \text{ and } | w(s + \Delta s_n) - w(s) | / | \Delta s_n | \to 1$$

uniformly for $s \in E'$. We denote by E'_w the image of E' in the w-plane.

For any $\epsilon > 0$ choose $\Delta s > 0$ so small, that $x(s+\Delta s) - x(s) < \epsilon \Delta s$ and $r(s, \Delta s) = |w(s+\Delta s) - w(s)| > \Delta s/2$ for $s \in E'$. For a fixed $s \in E'$, let $m(r)$, $M(r)$ be the minimum and the maximum of $|T^{-1}(w) - (x(s) + iy_0)|$ on the circumference $|w - w(s)| = r = r(s, \Delta s)$. Then, since T^{-1} is a K-QC mapping, we have, by Lemma 4,

$$\begin{aligned} | T^{-1}(w) - (x(s) + iy_0) | &\leqq M(r) \\ &\leqq e^{\pi K} m(r) \leqq e^{\pi K} \{ x(s + \Delta s) - x(s) \} < e^{\pi K} \epsilon \Delta s \end{aligned}$$

for any w in the disc $|w - w(s)| \leqq r(s, \Delta s)$. Since E' is closed and contained in the open interval $0 < s < L$, any of the discs $|z - (x(s) + iy_0)| < e^{\pi K} \epsilon \Delta s$, $s \in E'$, is contained in the rectangle $x_1 < x < x_2$, $y_1 < y < y_2$, provided that Δs is sufficiently small. Hence, the T-image of the rectangle $x_1 < x < x_2$, $y_0 - e^{\pi K} \epsilon \Delta s < y < y_0 + e^{\pi K} \epsilon \Delta s$ contains all the discs $|w - w(s)| \leqq r(s, \Delta s)$ with $s \in E'$, so that $A(y_0 + e^{\pi K} \epsilon \Delta s) - A(y_0 - e^{\pi K} \epsilon \Delta s)$ is not less than the area of the part of Δ covered by these discs.

Set $m = \max \{mE'_u, mE'_v\}$, where E'_u and E'_v are the projections of E'_w on the u- and v-axes, so that, by Lemma 5, $m > 0$. Since $r(s, \Delta s) > \Delta s/2$, the area of the part of Δ covered by the discs $|w - w(s)| \leqq r(s, \Delta s)$ with $s \in E'$ is not less than $m\Delta s$. Hence,

$$A(y_0 + e^{\pi K} \epsilon \Delta s) - A(y_0 - e^{\pi K} \epsilon \Delta s) \geqq m\Delta s.$$

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(15) Since $s = s(x)$ is topological, this is equivalent to the Lusin's condition (N).

If Δs is sufficiently small, the left hand side does not exceed $2e^{\pi K}\epsilon\Delta s\{A'(y_0)+1\}$, so that

$$2e^{\pi K}\{A'(y_0)+1\}\epsilon \geqq m > 0.$$

Since $\epsilon>0$ is arbitrary, this is a contradiction. Thus, $T(x, y_0)$ is absolutely continuous on $x_1 \leqq x \leqq x_2$.

Theorem 1 is now completely proved.

THEOREM 2. *Let $w=T(z)$ be a topological mapping of some plane region D onto another such region. If T is K-QC in a neighborhood of each point of D, it is K-QC in D*(16).

COROLLARY. *A function $w=f(z)$, which is K-PA in a neighborhood of each point of D, is K-PA in D.*

Since the method is typical, we give a detailed proof.

Proof. Let $\Omega(z_1, z_2, z_3, z_4)$ be a quadrilateral contained in D together with its boundary. We have to show that $\bmod \Omega \leqq K \bmod T(\Omega)$. Making auxiliary conformal mappings, we may assume that Ω is the rectangle $0<x<1$, $0<y<h=\bmod \Omega$ with the four vertices $z_1=0$, $z_2=1$, $z_3=1+ih$, $z_4=ih$, and that $T(\Omega)$ is $0<u<1$, $0<v<H=\bmod T(\Omega)$ with the vertices $T(z_1)=0$, $T(z_2)=1$, $T(z_3)=1+iH$, $T(z_4)=iH$.

Since the region Ω can be covered by a denumerable infinity of neighborhoods, in each of which T is K-QC, we see, by part (iii) of Theorem 1, that for almost all $0<y_0<h$, the length of the T-image of the segment $0<x<1$, $y=y_0$ is represented by the integral $\int_0^1|dT(x, y_0)/dx|dx \leqq +\infty$. Since it connects a point on $u=0$ to a point on $u=1$, we have

$$1 \leqq \int_0^1 \left|\frac{dT(x, y_0)}{dx}\right| dx \tag{6}$$

for almost all $0<y_0<h$, so that, by Schwarz' inequality,

$$1 \leqq \int_0^1 \left|\frac{dT(x, y_0)}{dx}\right|^2 dx.$$

Since, as is easily seen, the partial derivatives u_x and v_x are measurable functions of (x, y) so is $|dT(x, y)/dx|^2=u_x^2+v_x^2$. Hence, by Fubini's theorem,

$$h \leqq \int_0^h \left[\int_0^1 (u_x^2+v_x^2)dx\right] dy = \iint_\Omega (u_x^2+v_x^2)dxdy.$$

By part (ii) of Theorem 1 the integrand is $\leqq K\cdot J(z)$ almost everywhere in Ω. Further, it is easily seen that $J(z)$ is, at each point where T is totally differentiable, equal to the derivative $DS(z)$ of the set function $S(E)$ defined in the proof of (i) of Theorem 1. Hence,

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(16) Cf. Ahlfors [1] and Hersch-Pfluger [9]. L.B.

$$h < K \iint_\Omega DS(z)dxdy.$$

Finally, since $S(E)$ is non-negative, the last integral does not exceed $S(\Omega) = H$, so that

(7) $$\text{mod}\, \Omega = h \leqq K \cdot H = K\, \text{mod}\, T(\Omega),$$ q.e.d.

As is easily seen from the above proof, we can state Theorem 2 in the following more general form, which will be of use later:

THEOREM 2′. *Let D be a plane region, and E be a point set in D, which is closed with respect to D and consists of a denumerable infinity of sets of finite 1-dimensional outer measure. Then, a topological mapping $w = T(z)$ of D, which is K-QC in some neighborhood of each point of $D-E$, is K-QC in D*(17). *Similarly, an interior mapping $w=f(z)$ of D into the w-sphere, which is K-PA in some neighborhood of each point of $D-E$, is K-PA in D.*

In fact, by a theorem due to Gross [6] (also, cf. [15, p. 279]), any point set of finite 1-dimensional outer measure on the $x+iy$-plane meets almost every straight-line $y=y_0$ at most at a finite number of points, so that, for almost all y_0, the intersection of $y=y_0$ and E is a denumerable set closed with respect to the intersection of $y=y_0$ and D. Since this property of E is invariant under conformal mappings, (6) holds for almost all y_0, so that the whole proof is valid also for this case.

THEOREM 3. *Suppose that a rectangle $0<x<1$, $0<y<h$ in the $z=x+iy$-plane is mapped, by a K-QC mapping $w=T(z)$, onto another rectangle $0<u<1$, $0<v<H$ in the $w=u+iv$-plane in such a manner that the vertices $z=0, 1, 1+ih, ih$ correspond*(18) *to $w=0, 1, 1+iH, iH$*(19). *Then, $H=K\cdot h$ if and only if $T(z) \equiv x+iKy$; and, $H=K^{-1}\cdot h$ if and only if $T(z)\equiv x+iK^{-1}y$.*

For $K=1$, we have the

COROLLARY 1. *A 1-QC mapping is a conformal mapping; a 1-PA function is an analytic function.*

On the other hand, it is easy to deduce the following

COROLLARY 2. *$w=T(z)\equiv |z|^K e^{i\arg z}$ is the unique K-QC mapping of $0<q<|z|<1$ onto $q^K<|w|<1$ with $T(1)=1$; similarly, $w=T(z)\equiv|z|^{K^{-1}}e^{i\arg z}$ is the unique K-QC mapping onto $q^{K^{-1}}<|w|<1$.*

Proof of Theorem 3. Only the necessity must be proved. Since equality is valid in (7), it must hold in every step of the proof of (7).

(17) In Ahlfors [1], this is proved under the assumption that E is an analytic arc.

(18) As for the boundary correspondence, cf. Theorem 4, §5.

(19) Naturally, $K^{-1}\cdot h \leqq H \leqq K\cdot h$.

First, for almost all $y=y_0$, the T-image of the segment $0<x<1$, $y=y_0$ must be of length 1, and hence this image is a segment $0<u<1$, $v=\text{const.}$, so that $dv(x, y_0)/dx=0$ for every $0<x<1$. Thus $v_x=0$ almost everywhere. Next, from the equality in Schwarz' inequality for almost all y_0, we obtain $|dT(x, y_0)/dx| = |du(x, y_0)/dx| = 1$ for almost all $0<x<1$. Since $u(x, y_0)$ must be increasing, $du(x, y_0)/dx=1$ for almost all x. By the absolute continuity, we have $u(x, y_0) = \int_0^x (du/dx)dx = x$ for almost all y_0 and every x. Since $u(x, y)$ is continuous, $u(x, y) \equiv x$.

Next, since $u_x^2+v_x^2=K\cdot J(z)$ almost everywhere, and since $u_x=1$, $v_x=0$, and $J(z)=v_y$ almost everywhere, we have $v_y=K^{-1}$ almost everywhere. Since $v(x_0, y)$ is absolutely continuous in y for almost all $0<x_0<1$, we have $v(x_0, y) = \int_0^y (dv/dy)dy = K^{-1}y$ for every $0<y<h$. Hence $v(x, y)\equiv K^{-1}y$.

5. ***K-QC* mappings of the unit disc onto itself.** In the present section, we study the properties of a K-QC mapping $w=T(z)$ of $|z|<1$ onto $|w|<1$. By virtue of the theorems to be proved, various results on conformal mappings and analytic functions can be extended to quasi-conformal mappings and pseudo-analytic functions. Such applications will be stated in §6.

THEOREM 4. *Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$. Then, T can be extended to a topological mapping of the closed disc $|z|\leqq 1$ onto $|w|\leqq 1$*(20).

Hence, by reflections in $|z|=1$ and $|w|=1$, and by the first part of Theorem 2′, we can extend T to a K-QC mapping of the whole z-sphere onto the whole w-sphere(21). Thus, the points on $|z|=1$ are not *boundary* points, but *interior* points of the region, where $w=T(z)$ is K-QC.

Proof. First, we shall prove that $T(z)$ has a definite boundary value at each point on $|z|=1$. Suppose that there exist in $|z|<1$ two sequences of points $\{z_n\}$, $\{z_n'\}$, $n=1, 2, \cdots$, converging to one and the same point z_0 on $|z|=1$, such that their T-images $\{w_n\}$, $\{w_n'\}$ in $|w|<1$ converge to two distinct points w_0, w_0' on $|w|=1$.

For a small $\rho>0$, let D_ρ denote the intersection of $|z|<1$ and $|z-z_0|<\rho$, and C_ρ the part of the boundary of D_ρ lying in $|z|<1$. We choose $R>0$ so small, that z_1 and z_1' lie outside D_R (we assume $z_1\neq z_1'$). For any given r $(0<r<R)$, choose n so large, that z_n and z_n' both belong to D_r. Let L_n, L_n' be two curves in $|w|<1$ connecting w_1 to w_n and w_1' to w_n', and let d_n be the distance between L_n and L_n'. Since $w_0\neq w_0'$, we can choose L_n and L_n' in such a manner that $d_n\geqq d>0$ for some d independent of n.

Now, for any $r<\rho<R$, the arc C_ρ meets the inverse images of L_n and L_n', so that the length of the T-image of C_ρ is $\geqq d$. Hence, by (iii) of Theorem 1, we have, for almost all ρ,

(20) Cf. Ahlfors [1], and also Grötzsch [7], Yûjôbô [18; 19].

(21) As to more general "pseudo-analytic continuation," cf. §7.

$$d \leqq \int_{C_\rho} \left| \frac{dw}{\rho d\theta} \right| \rho d\theta \qquad (z - z_0 = \rho e^{i\theta}),$$

so that, by Schwarz' inequality,

$$d^2 \leqq \pi\rho \int_{C_\rho} \left| \frac{dw}{\rho d\theta} \right|^2 \rho d\theta.$$

Dividing by ρ and integrating with respect to ρ from r to R, we obtain, by Fubini's theorem and (ii) of Theorem 1,

$$d^2 \log \frac{R}{r} \leqq \pi \iint_{D_R - D_r} \left| \frac{dw}{\rho d\theta} \right|^2 \rho d\rho d\theta \leqq \pi K \iint_{|z|<1} J(z) \rho d\rho d\theta \leqq \pi^2 K.$$

Since $r>0$ is arbitrary, this is a contradiction.

Thus, T can be extended to a mapping of the closed disc $|z| \leqq 1$, which is easily seen to be continuous. Since the inverse mapping T^{-1} also permits such an extension, we see that the extended T is a topological mapping of $|z| \leqq 1$ onto $|w| \leqq 1$, q.e.d.

Next, we shall prove the following Hersch-Pfluger's extension [9] (also, cf. §6) of Grötzsch's theorem [8].

Following Teichmüller [17], let $\log \Phi(P)$ denote for any $P>1$, the modulus of Grötzsch's extremal region G_P: $|z| >1$ slit along the half straight-line $P \leqq x \leqq +\infty$ on the real axis.

THEOREM 5. *Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$ with $T(0)=0$. Then, for any $0<|z|<1$,*

$$\left\{\Phi\left(\frac{1}{|z|}\right)\right\}^{K^{-1}} \leqq \Phi\left(\frac{1}{|w|}\right) \leqq \left\{\Phi\left(\frac{1}{|z|}\right)\right\}^{K}. \tag{8}$$

Further, for each z in $0<|z|<1$, there exist two T's unique save for rotations, each rendering one equality in (8).

Hence, by the well known inequality $P<\Phi(P)<4P$, we obtain(22)

$$4^{-K} \cdot |z|^K \leqq |T(z)| \leqq 4 \cdot |z|^{K^{-1}}. \tag{9}$$

Proof. Let A_z denote the unit disc in the z-plane slit along the segment from the origin to the point z, so that mod $A_z = \log \Phi(1/|z|)$. Similarly, define the annulus A_w in the w-plane: mod $A_w = \log \Phi(1/|w|)$. Since the image $T(A_z)$ is the unit disc slit along a curve from the origin to the point $w=T(z)$, we have, by a well known theorem of Grötzsch, mod $T(A_z) \leqq$ mod A_w. Hence, by Lemma 3, K^{-1} mod $A_z \leqq$ mod A_w, whence follows the first half of (8). Applying this to T^{-1}, we obtain the second half.

(22) As is seen from the known relation: $\log \Phi(P) = \log P + \log 4 - 1/4P^2 + O(1/P^4)$, the constant factor in the leftmost side of the following necessarily depends on K.

By a conformal mapping of G_P onto $1<|z|<\Phi(P)$ leaving the point $z=1$ fixed, any two boundary points of G_P lying on the same point of $P<x<+\infty$, $y=0$, are mapped into two points on $|z|=\Phi(P)$ having the same abscissa, and vice versa. From this fact and from Corollary 2 of Theorem 3, we obtain easily the existence and uniqueness of extremal mappings rendering equalities in (8).

THEOREM 6([23]). *Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$ with $T(1)=1$. If the point z tends to 1 inside a Stolz region, $|\arg(1-z)|<\phi<\pi/2$, its image $w=T(z)$ tends to 1 also inside a Stolz region, $|\arg(1-w)|<\psi<\pi/2$, where ψ depends only on ϕ and K([24]).*

Proof. It suffices to show the following fact: Let $w=T(z)$ be a K-QC mapping of the strip $-\infty<x<+\infty$, $0<y<1$ in the $z=x+iy$-plane onto the strip $-\infty<u<+\infty$, $0<v<1$ in the $w=u+iv$-plane, the real axis being mapped onto the real axis. Then, for any $0<y_0<1$, the distance between the T-image of the straight line $y=y_0$ and the real axis $v=0$ has a positive lower bound depending only on y_0 and K.

By reflection with respect to the real axes we extend T to a K-QC mapping of $-\infty<x<+\infty$, $-1<y<1$ onto $-\infty<u<+\infty$, $-1<v<1$. Let u_0+iv_0 be a point on the T-image of $y=y_0$, and let $z=x_0+iy_0$ and $z=x_1$ be the inverse images of $w=u_0+iv_0$ and $w=u_0$.

By a function $\tilde{w}=\tilde{w}(w)$, we map the strip $-\infty<u<+\infty$, $-1<v<1$ conformally onto $|\tilde{w}|<1$, in such a manner that $\tilde{w}(u_0)=0$. Then, as is easily seen, $|\tilde{w}(u_0+iv_0)|=\tan(\pi v_0/4)$. Similarly, by $\tilde{z}=\tilde{z}(z)$, we map $-\infty<x<+\infty$, $-1<y<1$ conformally onto $|\tilde{z}|<1$, in such a manner that $\tilde{z}(x_1)=0$. This time, we have $|\tilde{z}(x_0+iy_0)|\geqq\tan(\pi y_0/4)$.

Applying (9) to the composed K-QC mapping of $|\tilde{z}|<1$ onto $|\tilde{w}|<1$, we obtain $4^{-K}|\tilde{z}(x_0+iy_0)|^K\leqq|\tilde{w}(u_0+iv_0)|$. Hence,

$$0<\frac{4}{\pi}\tan^{-1}\left[4^{-K}\left\{\tan\left(\frac{\pi}{4}y_0\right)\right\}^K\right]\leqq v_0, \qquad \text{q.e.d.}$$

THEOREM 7. *Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$ with $T(0)=0$. Then, for any two points z_1, z_2 on the closed disc $|z|\leqq 1$,*

$$(48)^{-K}\cdot|z_2-z_1|^K\leqq|T(z_2)-T(z_1)|\leqq 48\cdot|z_2-z_1|^{K^{-1}}. \tag{10}$$

Yûjôbô [18] proved, under his definition of quasi-conformality, the second half of (10) with the constant factor $2e^{8\pi}$ replacing 48. In Ahlfors [1] this is proved with a constant factor depending on K. As will be shown in [11], the best possible value of this numerical constant is 16.

Proof. We use the following inequality due to Teichmüller [17]. For any $0<P<+\infty$, let $\log\Psi(P)$ denote the modulus of the $x+iy$-plane slit along

([23]) This result has been proved independently by J. Jenkins [22]. L.B.

([24]) As to some related results, cf. Yûjôbô [19].

the segment $-1 \leqq x \leqq 0$ and the half straight line $P \leqq x < +\infty$ on the real axis. Then,

$$\Psi(P) < 16P + 8.$$

Since T^{-1} is also K-QC, it suffices to prove the second half of the inequality. By reflections in the unit circumferences, we extend T to a K-QC mapping of $|z| < +\infty$ onto $|w| < +\infty$. We may assume that $0 < |z_2 - z_1| < 2$ and $|z_2| \leqq |z_1| \leqq 1$. Let A denote the annulus $|z_2 - z_1| < |z - z_1| < 1 + |z_1|$, so that

$$\text{mod}\, A = \log \frac{1 + |z_1|}{|z_2 - z_1|} \geqq \log \frac{1}{|z_2 - z_1|}.$$

One of the complementary continua of the image annulus $T(A)$ contains the points $w_1 = T(z_1)$ and $w_2 = T(z_2)$, the other contains the point at infinity and a point on $|w| = 1$, so that it contains a point on $|w - w_1| = 2$. Hence, by Teichmüller's theorem [17], the modulus of $T(A)$ does not exceed that of the $u+iv$-plane slit along the segment $-|w_2 - w_1| \leqq u \leqq 0$ and the ray $2 \leqq u < +\infty$ on the real axis, i.e.

$$\text{mod}\, T(A) \leqq \log \Psi \frac{2}{|w_2 - w_1|}.$$

Since $K^{-1} \text{mod}\, A \leqq \text{mod}\, T(A)$ and $|w_2 - w_1| \leqq 2$, we have

$$\left(\frac{1}{|z_2 - z_1|}\right)^{K^{-1}} \leqq \Psi\left(\frac{2}{|w_2 - w_1|}\right) < \frac{32}{|w_2 - w_1|} + 8 \leqq \frac{48}{|w_2 - w_1|},$$

whence the result follows.

For any bounded plane set E we denote by $C(E)$ the (inner) logarithmic capacity of E, and by $\Lambda_0(E)$ the (outer) logarithmic measure of E. Similarly, for any $0 < \alpha \leqq 2$, we denote by $C^{(\alpha)}(E)$ the (inner) capacity of order α of E, and by $\Lambda_\alpha(E)$ the (outer) α-dimensional measure of E. Theorem 7 implies

THEOREM 8. *Let $w = T(z)$ be a K-QC mapping of $|z| < 1$ onto $|w| < 1$ with $T(0) = 0$. Let E_z be a point set on the closed disc $|z| \leqq 1$, and E_w its image on $|w| \leqq 1$. Then,*

$$C(E_w) \leqq 48\{C(E_z)\}^{K^{-1}}, \qquad \Lambda_0(E_w) \leqq K \cdot \Lambda_0(E_z);$$

and(25) *for any $0 < \alpha \leqq 2$,*

$$C^{(\alpha)}(E_w) \leqq 48\{C^{(K^{-1}\alpha)}(E_z)\}^{K^{-1}}, \qquad \Lambda_\alpha(E_w) \leqq (48)^\alpha \Lambda_{K^{-1}\alpha}(E_z).$$

Naturally, these inequalities hold also for linear sets on the circumferences. On the other hand, the inequalities concerning capacities hold also for

(25) By the remark to Theorem 7, the constant 48 in the following equation may be replaced by 16. This value, however, seems not to be the best possible.

outer capacities, since, by definition, the outer capacity of a set is the infimum of the inner capacities of open sets containing it(26).

Hence we have the

COROLLARY. *By a K-QC mapping of a plane region any set of inner or outer logarithmic capacity zero is transformed into a set with the same property.*

Proof of Theorem 8. The inequalities concerning measures are direct consequences of the definitions and of the second half of (10). Since the inner capacity of a set is the supremum of capacities of closed sets contained in it, we may assume that E_z and E_w are closed. For a bounded closed set E, $C(E)$ and $C^{(\alpha)}(E)$ are, as is well known, equal to the transfinite diameter of E and to the transfinite diameter of order α of E, respectively (cf. [5; 13]). Thus the desired inequalities follow from the definitions and from the second half of (10).

Theorems 4–8 proved above are, of course, valid also for quasi-conformal mappings in the usual sense. Theorem 7 expresses a compactness property peculiar to our definition (cf. Ahlfors [1]).

THEOREM 9. *Let* $w=T_n(z)$, $n=1, 2, \cdots$, *be a sequence of K-QC mappings of* $|z|<1$ *onto* $|w|<1$ *satisfying* $T_n(0)=0$. *Then* $\{T_n\}$ *contains a subsequence, which converges uniformly on the closed disc* $|z|\leqq 1$, *to a K-QC mapping* $w=T(z)$ *of* $|z|<1$ *onto* $|w|<1$.

Proof. By the second half of (10) the complex valued functions $w=T_n(z)$ are equi-continuous on $|z|\leqq 1$. Hence, by Ascoli's theorem, we can find a subsequence of $\{T_n\}$, which converges uniformly on $|z|\leqq 1$ to a function $w=T(z)$. We denote this subsequence again by $\{T_n\}$. By the remark of Lemma 1 in §3, we have only to show that $w=T(z)$ is a topological mapping of $|z|\leqq 1$ onto $|w|\leqq 1$(27). First, applying (10) to T_n and letting $n\rightarrow+\infty$, we see that $T(z)$ is univalent and bicontinuous on $|z|\leqq 1$. Secondly, for any fixed w in $|w|\leqq 1$, the inverse image $T_n^{-1}(w)$ must have an accumulation point in $|z|\leqq 1$, where $T(z)$ has to assume the value w. Since obviously $|T(z)|\leqq 1$, $w=T(z)$ is a topological mapping of $|z|\leqq 1$ onto $|w|\leqq 1$, q.e.d.

We close this section with a list of three unsolved problems, which seem to be of fundamental importance in the theory of quasi-conformal mappings and pseudo-analytic functions(28).

(1) Is a K-QC mapping necessarily *measurable*? In other words, does it

(26) It makes no difference to take, instead of open sets, covering sets "open with respect to the closed disc." Further, if the set in question lies on the circumference, we may take as covering sets linear open sets on that circumference.

(27) Then, e.g. by Brouwer's theorem (cf. [2, p. 396]), $|z|<1$ is mapped onto $|w|<1$.

(28) The answer to questions (1) and (2) is affirmative. This follows from the equivalence theorem mentioned in footnote 11 and from the results of C. B. Morrey [24]. Cf. also Boyarskiĭ [21]. L.B.

transform a (closed) set of two-dimensional measure zero into a set with the same property?

(2) Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$. Is the extended T necessarily *absolutely continuous on* $|z|=1$? In other words, does it transform a (closed) set of linear measure zero on $|z|=1$ into a set on $|w|=1$ with the same property(29)?

(3) Consider the family of continuously differentiable K-QC mappings of a region D. Is this family *dense*(30) in the family of all K-QC mappings of D? In other words, can a K-QC mapping of D be approximated (uniformly on compact subsets) by continuously differentiable ones?

6. **Applications.** We state first some simple applications of Theorems 4–8 of §5, which hold also for quasi-conformal mappings and pseudo-analytic functions in the usual sense.

Let $w=w(z)$ be a K-QC mapping of $|z|<1$ onto a plane region Δ in the w-plane. We map Δ conformally(31) onto $|\zeta|<1$ by $\zeta=\zeta(w)$, in such a manner that the point $w=w(0)$ in Δ corresponds to $\zeta=0$. The composite mapping $\zeta=\zeta(w(z))=T(z)$ is a K-QC mapping of $|z|<1$ onto $|\zeta|<1$ with $T(0)=0$.

For the conformal mapping between Δ and $|\zeta|<1$, we have, e.g., the theorems of Carathéodory and Lindelöf on the correspondence between boundary elements of Δ and points on $|\zeta|=1$. Hence, by Theorems 4 and 6, we see that these theorems hold also for the K-QC mapping $w=w(z)$. Similarly, various theorems on conformal mappings can be extended to the class of K-QC mappings(32).

Next, let $w=f(z)$ be a K-PA function defined in $|z|<1$. As was remarked in §2, $w=f(z)$ is represented in the form $w=\phi(T(z))$, where $\zeta=T(z)$ is a plane K-QC mapping and $w=\phi(\zeta)$ is an analytic function. Without loss of generality, we may assume that $\zeta=T(z)$ maps $|z|<1$ onto $|\zeta|<1$ with $T(0)=0$.

Since, by Theorem 4, the T-image of an arc on $|z|=1$ is an arc on $|\zeta|=1$, we have immediately the following proposition, which will be of use in the next section.

Let $w=f(z)$ be continuous on $|z|\leqq 1$, and K-PA in $|z|<1$. If $f(z)$ is equal to a constant on an arc of $|z|=1$, it is identically equal to that constant.

More generally, we can state, by Theorem 4, an extension of Koebe's theorem on bounded analytic functions. Further, from Theorem 5 and Schwarz' lemma, we have the Hersch-Pfluger extension [9] of that lemma.

(29) Problem (2) is meaningful also for quasi-conformal mappings in the usual sense. Further, we can easily show that an affirmative answer to (2) leads to that of (1).

(30) With respect to the topology defined by uniform convergence on compact subsets.

(31) By the remark to Lemma 3 in §3, Δ cannot be conformally equivalent with $|\zeta|<+\infty$.

(32) Especially, theorems concerning logarithmic capacities of sets on the unit circumference. In order to settle whether or not we can extend theorems concerning linear measures, e.g. F. and M. Riesz's theorem on conformal mapping of a region bounded by a rectifiable Jordan curve, we have to solve the problem (2) stated in §5. A similar remark holds also for K-PA functions.

Also, an extension of Schottky's theorem follows from Theorem 5. As an example of such applications, we shall state and prove the following extension of Beurling's theorem [3](33).

Let $w=f(z)$ be K-PA in $|z|<1$, and suppose that the Riemannian image of $|z|<1$ by $f(z)$ has, as a covering surface of the w-sphere, a finite area. Then, except for a set of outer logarithmic capacity zero on $|z|=1$, $\lim f(z)$ exists when z tends to a point on $|z|=1$ inside a Stolz region. Further, if $w=a$ is an ordinary value for $f(z)$ in Beurling's sense, the set of points on $|z|=1$, where $\lim f(z)=a$, is of outer logarithmic capacity zero.

Proof. Let $w=f(z)$ be represented by $w=\phi(\zeta)$, $\zeta=T(z)$, as mentioned above. For the analytic function $w=\phi(\zeta)$ in $|\zeta|<1$, Beurling's theorem holds. Let E_ζ be the exceptional set on $|\zeta|=1$, where $\lim \phi(\zeta)$ does not exist, and let E_z be the image of E_ζ by $z=T^{-1}(\zeta)$ extended by Theorem 4. Since E_ζ is of outer logarithmic capacity zero, so is E_z by Theorem 8. Now, by Theorem 6, if z tends to a point on $|z|=1$ not belonging to E_z inside a Stolz region, its T-image tends to a point $\notin E_\zeta$ on $|\zeta|=1$ inside a Stolz region. Hence, $\lim f(z)=\lim \phi(T(z))$ exists. The remaining part is proved in the same way.

Next, we turn to applications of Theorem 9 which are peculiar to our definition.

THEOREM 10. *If a sequence $f_n(z)$, $n=1, 2, \cdots$, of K-PA functions defined in a region D converges(34) to a function $f(z)$ uniformly on compact subsets of D, then $f(z)$ is K-PA in D.*

Proof. Without loss of generality, we may assume D to be the unit disc $|z|<1$. Let $f_n(z)=\phi_n(\zeta)$, $\zeta=T_n(z)$ be the representation of $f_n(z)$, where $\zeta=T_n(z)$ is a K-QC mapping of $|z|<1$ onto $|\zeta|<1$ with $T_n(0)=0$, and $\phi_n(\zeta)$ is analytic in $|\zeta|<1$. From the sequence $\{T_n^{-1}(\zeta)\}$ we choose, by Theorem 9, a subsequence uniformly convergent on $|\zeta|\leqq 1$ to a K-QC mapping of $|\zeta|<1$ onto $|z|<1$, which we denote by $z=T^{-1}(\zeta)$. Let the subsequence be again denoted by $\{T_n^{-1}\}$. Since $\{f_n(z)\}$ converges uniformly on compact subsets of $|z|<1$, so does $\{\phi_n(\zeta)\}=\{f_n(T_n^{-1}(\zeta))\}$ in $|\zeta|<1$. Hence, the limit function $\phi(\zeta)=f(T^{-1}(\zeta))$ is analytic. Thus, $f(z)=\phi(T(z))$ is a K-PA function in $|z|<1$, q.e.d.

If, in the above proof, all the ϕ_n's are univalent, the limit function $\phi(\zeta)$ is either univalent or identically equal to a constant. Hence we have the following

COROLLARY 1. *If a sequence of K-QC mappings of D converges uniformly on compact subsets of D, the limit mapping is either a K-QC mapping of D or transforms D into a single point.*

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(33) The extension of Beurling's theorem to quasi-conformal mapping has been given also by A. Lohwater [23] and J. Jenkins [22]. L.B.

(34) With respect to the spherical distance.

As to *normal families* of *K-PA* functions, we can state the following theorem (also, cf. Yûjôbô [18]).

COROLLARY 2. *Let* $w=f_n(z)$, $n=1, 2, \cdots$, *be a sequence of uniformly bounded*(35) *K-PA functions defined in D. Then* $\{f_n\}$ *contains a subsequence which converges to a K-PA function uniformly on compact subsets of D.*

Proof. Since the analytic functions $\phi_n(\zeta)$ in the proof of Theorem 10 form a normal family in $|\zeta|<1$, we have only to choose a subsequence from $n=1, 2, \cdots$, for which both $\{\phi_n(\zeta)\}$ and $\{T_n(z)\}$ converge.

7. **Pseudo-analytic continuation.** Theorem 2′ of §4 can be regarded as a theorem on removal of singularities, if the topological character of $w=T(z)$ or the interiority of $w=f(z)$ is assured by some other conditions. We state here the following

COROLLARY OF THEOREM 2′. *Let D and E have the same meanings as in Theorem* 2′, *and let* $w=f(z)$ *be continuous*(36) *in D and K-PA in* $D-E$. *If E and its image* $f(E)$ *on the w-sphere are both totally disconnected, then* $w=f(z)$ *is K-PA in D.*

In fact, by the conditions on E and $f(E)$, and by a theorem on the prolongation of an interior mapping due to Stoïlow [16, p. 122], we can assert that $w=f(z)$ is an interior mapping of D.

By a similar method, we shall prove the following analogue of Painlevé's theorem on analytic continuation.

THEOREM 11. *Suppose that a simply connected region D in the z-plane is divided into two regions* D_1 *and* D_2 *by a rectifiable Jordan arc C in D connecting two boundary points of D. Let* $w=f(z)$ *be a function defined and continuous in D. If* $f(z)$ *is K-PA in* D_1 *and in* D_2 *respectively, and if the image* $f(C)$ *of C on the w-sphere has no interior points*(37), *then,* $w=f(z)$ *is K-PA in D.*

COROLLARY(38). *Suppose that* $w=f(z)$ *is continuous on the half disc* $|z|<1$, $y\geqq 0$ *in the* $z=x+iy$*-plane, K-PA in its interior, and real on the segment* $-1<x<1$, $y=0$. *Then, defining* $f(z)$ *in the lower half disc by*(39) $[f(\bar{z})]^*$, *we can extend* $f(z)$ *to a K-PA function in* $|z|<1$.

Proof. In view of the second half of Theorem 2′ it suffices to show that $w=f(z)$ is either identically equal to a constant or an interior mapping of D into the w-sphere. As was remarked in §6, if $f(z)$ is equal to a constant on an

(35) This may be replaced by any other condition, which assures the normality of a family of analytic functions.

(36) With respect to the spherical distance.

(37) Whether this condition can be removed or not, I cannot say.

(38) This can be more easily proved by the method in §6, i.e. by reduction to the case of analytic functions.

(39) A bracketed expression followed by a * denotes complex conjugation.

arc of C, it is identically equal to that constant in D. We assume that $f(z)$ is nonconstant, and show that it is an interior mapping. Since the discussion is purely topological, we may assume that D is the unit disc $|z|<1$, C the segment $-1<x<1$, $y=0$, and D_1, D_2 the upper and lower halves of D.

Suppose first that a continuum Γ in D is mapped by $w=f(z)$ into a single point. Since, by Baire's theorem (cf. [2, p. 108], or [15, p. 54]), a continuum is of the second category, at least one of the intersections $\Gamma\cap D_1$, $\Gamma\cap D_2$, and $\Gamma\cap C$ contains a subcontinuum of Γ. In any of these three cases, this leads to a contradiction to the hypothesis of nonconstancy of $f(z)$. Hence, $w=f(z)$ does not map any continuum into a single point.

Next, we shall prove that $w=f(z)$ is an open mapping. Suppose that the image $f(G)$ of an open set G in D contains a boundary point w_0 of itself. The inverse images of w_0 in G form a point set E_0 closed with respect to G. Since $w=f(z)$ is an interior mapping of D_1 and D_2 respectively, E_0 must lie on C.

Let z_0 be a point of E_0. Since, as remarked above, E_0 is totally disconnected, we can find a Jordan region d satisfying the following conditions:

(i) d contains z_0, and is contained in G together with its boundary curve γ; and

(ii) γ intersects C exactly twice, and has no points in common with E_0.

The region d is divided by the arc $C\cap d$ into two subregions d^+ and d^-, which lie respectively in the upper and lower halves of D.

Since the image $f(\gamma)$ of γ is a closed set and does not contain w_0, we can find a simply connected neighborhood U of w_0, which has no points in common with $f(\gamma)$. Now, the f-image of d^+ is an open set having w_0 on its boundary. Since the f-image of $C\cap d$ has no interior points, we can find a point w_1 in U, which is contained in $f(d^+)$ but not in $f(C\cap d)$. On the other hand, since the f-image of $d\cup\gamma$ is a closed set containing w_0 as a boundary point, we can find another point w_2 in U, which is an exterior point of $f(d\cup\gamma)$. We may assume that w_1 and w_2 are both different from $w=\infty$.

Note that in d^+ and in d^-, $w=f(z)$ is an orientation preserving topological mapping followed by a mapping by an analytic function. Let n^+ be the number of inverse images of w_1 in d^+, counted with their multiplicities. By the above choice of w_1, we have $n^+>0$. Denote by γ^+ the boundary curve of d^+ in its positive orientation with respect to d^+. Then, since $f(z)\neq w_1, w_2$ on γ^+, and $\neq w_2$ in d^+, we have[40]

$$0 < n^+ = \frac{1}{2\pi}\underset{\gamma^+}{\text{var. arg.}}\frac{f(z) - w_1}{f(z) - w_2}\,.$$

Similarly, let n^- be the number of inverse images of w_1 in d^-, and let γ^- denote the boundary curve of d^- in its positive orientation with respect to d^-. As before, we have

[40] By the following notation, we mean the variation along γ^+ of a branch of the argument, which is continuously extended along γ^+ from a fixed point on γ^+.

$$0 \leqq n^- \frac{1}{2\pi} \underset{\gamma^-}{\text{var.}} \arg \frac{f(z) - w_1}{f(z) - w_2} \cdot$$

Summing these two inequalities, we obtain

$$(11) \qquad 0 < \frac{1}{2\pi} \underset{\gamma}{\text{var.}} \arg \frac{f(z) - w_1}{f(z) - w_2},$$

where γ is taken in its positive orientation with respect to d. On the other hand, since $f(\gamma)$ lies outside U, the right hand side of (11) must vanish, which is a contradiction. Hence, $w=f(z)$ is an open mapping of D, q.e.d.

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