ON A THEOREM OF MORI AND THE DEFINITION OF QUASICONFORMALITY(1)

BY

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A posthumous paper by the late Professor Akira Mori [15] contains implicitly a solution of an important problem in the theory of quasiconformal mappings. More precisely, using Mori's Theorem I we can show, in a few lines, that two generally accepted "natural" definitions of quasiconformality are equivalent. In order to make this note readable, however, we shall need more than a few lines for a restatement of these definitions.

1. **Grötzsch's inequality.** The concept of quasiconformality is due to Grötzsch [10; 11; 12] who considered primarily homeomorphisms

$$w(z) = u(x, y) + iv(x, y) \qquad (z = x + iy)$$

of class C^1 with a positive Jacobian

$$J = u_x v_y - u_y v_x.$$

Such a mapping takes infinitesimal circles into infinitesimal ellipses; it is called quasiconformal if the eccentricity of these ellipses is uniformly bounded. This condition can be expressed analytically by either of the three equivalent differential inequalities:

$$\max_{0 \leqq \theta \leqq 2\pi} |w_x \cos\theta + w_y \sin\theta|^2 \leqq QJ, \tag{1'}$$

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 \leqq \left(Q + \frac{1}{Q}\right) J, \tag{1''}$$

$$|w_x + iw_y| \leqq \frac{Q-1}{Q+1} |w_x - iw_y|, \tag{1'''}$$

for some $Q \geqq 1$. This property is conformally invariant: if $w = w(z)$ has it, so does the function $W(\zeta) = F\{w[f(\zeta)]\}$ where F and f are conformal mappings.

Consider, in particular, a quasiconformal mapping, with constant Q, of the closed rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$ onto another closed rectangle $0 \leqq u \leqq a'$, $0 \leqq v \leqq b'$ assuming that the vertices $(0, 0)$, $(0, a)$, (a, b), $(0, b)$ are taken into $(0, 0)$, $(0, a')$, (a', b'), $(0, b')$ respectively. A simple application of the Schwartz inequality (cf. Ahlfors [1]) yields the Grötzsch inequality

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(2) $$\frac{a'}{b'} \leqq Q\,\frac{a}{b}\,.$$

More generally, let w be a quasiconformal homeomorphism of a domain D onto a domain Δ and let $R\subset D$ be a topological rectangle (topological image of the closed square with vertices $\alpha_1=(0, 0)$, $\alpha_2=(1, 0)$, $\alpha_3=(1,1)$, $\alpha_4=(0, 1)$. R can be mapped conformally onto the rectangle $0\leqq x\leqq a$, $0\leqq y\leqq b$ in such a way that the points α_1, α_2, α_3, α_4 go into the points $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$, respectively, and the uniquely determined number (a/b) is called the *modulus* of R and is denoted by mod R. In view of (2) and the conformal invariance of inequality (1) it follows that

(3) $$\text{mod } R \leqq Q \text{ mod } w(R).$$

Note that topological rectangles and their moduli may be considered on arbitrary Riemann surfaces.

2. **Geometric definition.** Quasiconformal mappings have proved to be a powerful tool in the theory of functions (cf. Ahlfors [1], Pfluger [18], Volkoviskii [23], Cacciopoli [6] and the references given there), especially in connection with Teichmüller's extremal quasiconformal mappings [22], and in the theory of partial differential equations (cf., in particular, Morrey [16], Lavrent'ev [13; 14], Bers [2; 3], Nirenberg [17], Finn [8], Bers and Nirenberg [4]). But in applying this tool it became necessary to extend the original definition. Of the proposed generalizations two are, in a certain sense, most general.

The geometric definition (Ahlfors, Pfluger, Mori) dispenses with all differentiability requirements and uses directly inequality (3). According to this definition a homeomorphism w of a plane domain D onto another such domain or, more generally, of a Riemann surface D onto another such surface, is *Q-quasiconformal* if (3) holds for every topological rectangle $R\subset D$.

Consider now an *interior* function $w(z)$ defined in a domain D. This means that $w(z)$ is continuous and either constant or has the following three properties. (i) The mapping w is light and open([2]). (ii) In the neighborhood of every point of D, save perhaps for a discrete set, the mapping w is a local homeomorphism. (iii) There exists a homeomorphism $\chi(z)$ of D onto a plane domain and an analytic function $f(\zeta)$ defined in $\chi(D)$ such that

(4) $$w(z) = f[\chi(z)].$$

The three properties (i), (ii), (iii) are *equivalent*. The implications (iii)→(i) and (iii)→(ii) are trivial. The implication (i)→(iii) is the well known result of Stoïlow [20]. The implication (ii)→(iii) is an easy consequence of the general uniformization theorem([3]). A nonconstant interior function w may be

([2]) Cf. Whyburn [24].

([3]) Cf. [3, p. 454].

considered as a homeomorphism of D onto a *Riemann covering surface* of a plane domain. Mori calls w pseudoanalytic if this homeomorphism is quasiconformal.

We shall say "quasiconformal" rather than "pseudoanalytic"(4) and shall call a function w, Q-quasiconformal according to the geometric definition if it is of the form (4) where χ is a Q-quasiconformal homeomorphism and f an analytic function. Mori himself noted that this definition is equivalent to his.

3. **Analytic definition.** We recall the concept of L_2 *derivatives* due to Sobolev [19] and Friedrichs [9]. Let f, g, h be measurable, locally square integrable, complex or real valued functions defined in a plane domain D. The relations

$$g = f_x, \qquad h = f_y \qquad \text{in the } L_2 \text{ sense}$$

mean that the following conditions are satisfied. (a) The identities

$$\iint_D f\omega_x dxdy = -\iint_D g\omega dxdy, \qquad \iint_D f\omega_y dxdy = -\iint_D h\omega dxdy$$

hold for every function ω of class C^1 with compact support $S \subset D$. (b) In every compact set $S \subset D$ there exist functions $f^{(n)}$ of class C^1 such that

$$\iint_S \{ |f^{(n)} - f|^2 + |f_x^{(n)} - g|^2 + |f_y^{(n)} - h|^2 \} dxdy \to 0$$

as $n \to \infty$. (c) The function $f(x, y)$ is absolutely continuous in x for almost all values of y and in y for almost all values of x, and $f_x = g$, $f_y = h$ almost everywhere in D.

It is known that each of the properties (a), (b), (c) *implies the other two.*

According to the analytic definition (Morrey, Cacciopoli, Bers and Nirenberg) a continuous function $w(z)$ in a domain D is Q-quasiconformal if it has L_2 derivatives satisfying inequality (1) almost everywhere.

4. **Beltrami equations.** Let

$$g_{11}(x, y)dx^2 + 2g_{12}(x, y)dxdy + g_{22}(x, y)dy^2 \tag{5}$$

be a Riemann metric defined in a domain D. A function $w = u + iv$ in D is said to be *conformal* with respect to this metric if u and v satisfy the Beltrami equations

$$gu_x = -g_{12}v_x + g_{11}v_y, \qquad gu_y = -g_{22}v_x + g_{12}v_y \tag{6'}$$

where

$$g^2 = g_{11}g_{22} - g_{12}^2,$$

(4) In the writer's opinion the term "pseudoanalytic" should be reserved for solutions of generalized Cauchy-Riemann equations.

which can be also written in the form

$$w_x + iw_y = \mu(w_x - iw_y), \tag{6''}$$

the complex valued function μ being given by

$$\mu = \frac{g_{11} - g_{22} + 2ig_{12}}{g_{11} + g_{22} + 2g}.$$

In what follows we assume that the functions g_{ik} are measurable and that the eccentricity of the metric is bounded by $Q \geqq 1$, that is, that

$$g_{11} + g_{22} \leqq 2Qg \tag{7'}$$

or, which is the same,

$$|\mu| < (Q-1)/(Q+1). \tag{7''}$$

A solution of (6) will be required to be continuous and to have L_2 derivatives satisfying the equation almost everywhere. Thus every solution of a Beltrami system is Q quasiconformal (by the analytic definition) if the eccentricity of the metric is bounded by Q, and conversely, every quasiconformal function (according to the analytic definition) is a solution of an appropriately chosen Beltrami system.

We note now some properties of a Beltrami system (6) satisfying (7).

(α) If w_1 and w_2 are two solutions of (6) in the same domain and w_1 is a homeomorphism, w_2 is an analytic function of w_2.

(β) If w is a solution of (6) so is $f(w)$, f being any analytic function.

(γ) In every domain there exists a homeomorphism satisfying (6).

(δ) If $w=u+iv$ is a univalent solution of (6) in a domain D, the Jacobian $J=u_xv_y-u_yv_x$ is positive almost everywhere, and for every measurable set $e \subset D$, $w(e)$ is measurable and has measure

$$\iint_D Jdxdy.$$

These results are due to Morrey [16]. For different proofs cf. Bers and Nirenberg [4] and Boyarskiĭ [5]. Boyarskiĭ's proof is based on the Calderón-Zygmund inequality [7]; it implies that the moduli of the derivatives of a solution of (6) are locally integrable to a power $p>2$, depending only on Q.

It can be shown that every measurable locally square integrable function having L_2 derivatives satisfying (6) is continuous. Hence the continuity hypothesis may be omitted from the analytic definition of quasiconformality.

We shall show next that the two definitions are equivalent. Hence statement (δ) answers in the affirmative a question raised by Mori [15, §5].

5. **Equivalence proof.** A homeomorphism which is Q-quasiconformal according to the analytic definition is so also according to the geometric definition. In fact, noting statement (δ) the usual proof of Grötzsch's inequality

can be repeated almost verbatim. Statement (α) now leads to the (known) observation that the analytic definition implies the geometric.

Now let $w(z)=u+iv$ be a homeomorphism of a domain D which is Q-quasiconformal according to the geometric definition. Mori proved (Theorem I) that $w(x+iy)$ is absolutely continuous on almost all lines $x=$const. and $y=$const.[5], and that there exists a null set $E\subset D$ such that at every point of $D-E$, w has a differential and its derivatives satisfy (1). In order to verify that w is a solution of a Beltrami system we must only show that its derivatives are locally square integrable.

For every Borel set $e\subset D$ let $s(e)$ denote the measure of the Borel set $w(e)$; $s(e)$ is a countably additive non-negative set function. Let $\Delta(z_0, r)$ denote the disc $|z-z_0|<r$. It is well known that the Lebesgue derivative

$$h(z_0) = \lim_{r\to 0} \frac{s[\Delta(z_0, r)]}{\pi r^2}$$

exists almost everywhere in D, and

$$\iint_e h\,dxdy \leqq s(e). \tag{8}$$

Mori observed, and it is easy to verify, that

$$h = J = u_x v_y - u_y v_x$$

at all points of $D-E$. Hence, for every compact set $S\subset D$ we have by (1) and (8) that

$$\iint_S (u_x^2 + u_y^2 + v_x^2 + v_y^2)\,dxdy \leqq (Q + 1/Q)\iint_S J\,dxdy$$
$$\leqq (Q + 1/Q)s(S) < +\infty.$$

Noting statement (β) of §4 we conclude that the geometric definition of Q-quasiconformality implies the analytic definition.

6. **An application.** Mori asked [15, §5] whether Q-quasiconformal functions of class C^1 are dense in the set of all Q-quasiconformal functions, in the sense of normal convergence (uniform convergence on compact subsets). The equivalence theorem yields the affirmative answer.

Let $w(z)$ be a Q-quasiconformal function; for the sake of simplicity we assume it to be defined in a subdomain D of the unit disc $|z|<1$. w is a solution of (6) with some μ satisfying $|\mu|<(Q-1)/(Q+1)$. We may assume that μ is defined for $|z|<1$.

By (γ) there exists a homeomorphic solution $W_0(z)$ of (6) defined in $|z|<1$. Since quasiconformal mappings are known to preserve conformal type the image Δ of $|z|<1$ under W_0 is not the whole plane. Let $F(\zeta)$ be the analytic

(5) This was also proved, independently, by Strebel [21].

function mapping Δ conformally onto the unit disc, $F[W_0(0)]=0$. The function $W(z)=F[W_0(z)]$ is a solution of (6) which maps $|z|<1$ homeomorphically onto itself and leaves the origin fixed.

By a fundamental property of quasiconformal mapping (discovered by Morrey, Ahfors and Lavrent'ev and, in its sharp form, by Mori)

$$(9)\qquad (|z_1 - z_2|/16)^Q \leqq |W(z_1) - W(z_2)| \leqq 16|z_1 - z_2|^{1/Q}.$$

W is a homeomorphism of $|z|\leqq 1$ and we may assume that $W(1)=1$, since this can be achieved by a rotation.

Now let $\{\mu^{(n)}\}$ be a sequence of complex valued real-analytic functions such that

$$|\mu^{(n)}| \leqq (Q-1)/(Q+1),$$
$$\mu^{(n)} \to \mu \text{ a.e. in } |z| < 1.$$

For every n there exists a solution $W^{(n)}(z)$ of the equation

$$W_x^{(n)} + iW_y^{(n)} = \mu^{(n)}(W_x^{(n)} - iW_y^{(n)})$$

which is a homeomorphism of $|z|\leqq 1$ onto itself satisfying the conditions $W^{(n)}(0)=0$, $W^{(n)}(1)=1$. These functions are real analytic and each satisfies inequalities (9). Hence we may assume, selecting if need be a subsequence, that $W^{(n)}(z)$ converges uniformly to a homeomorphism $W^{(\infty)}(z)$ of $|z|\leqq 1$ onto itself. Since, by (1),

$$\iint_D \{|W_x^{(n)}|^2 + |W_y^{(n)}|^2\}dxdy \leqq (Q + 1/Q)\pi$$

we may assume, selecting if need be a subsequence, that the functions $W_x^{(n)}$, $W_y^{(n)}$ converge weakly to certain L_2 functions which are easily seen to be the L_2 derivatives of $W^{(\infty)}$. It follows that $\mu^{(n)}\{W_x^{(n)}-iW_y^{(n)}\}$ converges weakly to $\mu(W_x^{(\infty)}-iW_y^{(\infty)})$, so that $W^{(\infty)}$ is a solution of (6). By §4, (α) we have that $W^{(\infty)}(z)=F[W(z)]$, F being analytic. But $F=W^{(n)}\circ W^{-1}$ is a homeomorphism of $|z|\leqq 1$ onto itself leaving 0 and 1 fixed. Hence $W^{(\infty)}=W$ and $W^{(n)}\to W$ uniformly.

Next, $w(z)$ must be of the form $w(z)=f[W(z)]$, f being analytic in $W(D)$. Let S be any subdomain of D with compact closure $\overline{S}\subset D$. If n is sufficiently large, $W^{(n)}(\overline{S})\subset W(D)$. The function $w^{(n)}(z)=f[W^{(n)}(z)]$ is Q-quasiconformal in S and is of class C^1, in fact real analytic there. Clearly $w^{(n)}\to w$ uniformly in S.

There are several obvious modifications of the preceding argument. These may be left to the reader.

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