GRAPHS AND SUBGRAPHS

BY

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The present paper contains a study of the properties of finite, undirected graphs. It is based primarily on the concept of *deficiency* which has been used by the author in previous papers to obtain certain basic results on directed graphs. For undirected graphs the situation proves to be more involved and it is necessary to replace the concept of *simple deficiency* by the *effective deficiency* which takes into account certain restrictions which appear in the undirected case.

These concepts are analyzed in Chapter 1. In Chapter 2 follows the proof of the general theorem about the existence of subgraphs with prescribed local degrees. The criterion obtained is related to a criterion established by *Tutte* but it is considerably simpler in its application, through the fact that it refers only to the properties of single subsets of the vertex set, while the criterion of *Tutte* involves the choice of pairs of subsets.

The proof is based on the alternating path method originally introduced by *Petersen* in graph theory, and subsequently used by most writers studying the existence of subgraphs, let us mention only the more recent papers by *Baebler*, *Belck*, *Gallai* and *Tutte*. Due to the applications our presentation of the alternating path theory differs in certain respects from the previous ones, but to save space the proofs have been based, as far as possible, upon those given by the preceding authors. The references are made to the paper by Tutte [12] which should be readily available to most readers.

In Chapter 3 one finds various new explicit factorizations for nonregular graphs of certain types. It is pointed out how all the known results on the factorization of regular graphs, in particular those by Baebler, Gallai and Tutte, follow as special cases. To conclude one finds an example to show that a certain limit for the factorization of odd regular graphs given by Baebler is actually the best possible.

CHAPTER 1. DEFICIENCY FUNCTIONS

1.1. Notations. Let S be some set and G an *undirected graph* with the *vertex set* S. An *edge* of G with the *endpoints* a and b shall be denoted by $E=(a, b)$. To simplify the following considerations we shall make two assumptions; we observe in passing that they are not absolutely essential since they may be obviated in part by a more detailed analysis:

1. S is a *finite set.*
2. There are *no loops* (a, a) in G.

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The *associated set* $R(A)$ of a subset A is the set of all vertices x such that there is at least one edge (a, x) with $a \in A$. For the *void set* $\emptyset$ we write $R(\emptyset)=0$. The *complement* of A in S is $\tilde{A}$.

We shall permit G to have *multiple edges*, that is, a finite number $\rho(a, b)$ of different edges,

$$E_i = (a, b)_i \qquad (i = 1, 2, \cdots, \rho(a, b))$$

joining the same pair of vertices. For any set B we write

$$\rho(a, B) = \rho(B, a) = \sum_{b \in B} \rho(a, b). \tag{1.1.1}$$

The *local degree* $\rho(a)$ of a vertex a is the number of edges in G issuing from a, consequently

$$\rho(a) = \rho(a, S) = \rho(a, R(a)). \tag{1.1.2}$$

For any subset A we shall write

$$\rho(A) = \sum \rho(a), \qquad a \in A. \tag{1.1.3}$$

This as one sees represents the total number of edges issuing from the vertices in A, counting twice those which have both their endpoints in A. For a given set A it is convenient to classify the edges in G as follows:

1. The edge $E=(e, f)$ *belongs to* A or *lies in* A when both ends are in A.
2. E *touches* A if exactly one end lies in A.
3. E is *outside of* A or *disjoint from* A if it lies in $\overline{A}$.

Let us denote by $\nu(A, B)$ the number of edges having one end in A and the other in B. The number of edges in each category just indicated is then

$$\nu(A, A), \qquad \nu(A, \overline{A}), \qquad \nu(\overline{A}, \overline{A})$$

so that

$$\nu(S, S) = \nu(A, A) + \nu(A, \overline{A}) + \nu(\overline{A}, \overline{A})$$

represents the total number of edges in G.

We extend the notation (1.1.1) by writing

$$\rho(A, B) = \sum_{a,b} \rho(a, b), \qquad a \in A, b \in B. \tag{1.1.4}$$

One readily verifies that

$$\nu(A, B) = \rho(A, B) - \nu(D, D), \qquad D = A \cdot B,$$

so that for disjoint sets

$$\nu(A, B) = \rho(A, B), \qquad A \cdot B = \emptyset.$$

The double number of edges in G is

$$\rho(S) = 2\cdot\nu(S, S) \equiv 0 \pmod 2. \tag{1.1.5}$$

More general, we have

$$\rho(A) = 2\cdot\nu(A, A) + \rho(A, \overline{A}) \tag{1.1.6}$$

so that

$$\rho(A) + \rho(A, \overline{A}) \equiv 0 \pmod 2. \tag{1.1.7}$$

It is sometimes convenient to say that a set A is a *peninsula of rank* $\nu(A, \overline{A})$. Thus a *peninsula A of rank* 1 or simply a *peninsula* is connected with the set $\overline{A}$ by a single edge, the *bridge*. According to (1.1.7) one has for any such peninsula

$$\rho(A) + 1 \equiv 0 \pmod 2. \tag{1.1.8}$$

Thus we see:

THEOREM 1.1.1. *A graph with even local degrees can have no peninsulas.*

The *section graph* $G(A)$, defined by the set A, is the subgraph of $G = G(S)$ consisting of the vertices in A and the edges of G lying in A. The *local degree* of a vertex $a \in A$ with respect to $G(A)$ is $\rho(a, A)$. The double number of edges in $G(A)$ is

$$\rho(A, A) \equiv 0 \pmod 2.$$

1.2. **The deficiency function.** To each vertex v in S we define a *multiplicity*

$$\kappa(v) \tag{1.2.1}$$

which is an integer satisfying

$$0 \leqq \kappa(v) \leqq \rho(v). \tag{1.2.2}$$

For any set A we put

$$\mu(A, v) = \min(\kappa(v), \rho(A, v)) \tag{1.2.3}$$

and

$$\mu(A, B) = \sum_v \mu(A, v), \qquad v \in B. \tag{1.2.4}$$

The definition (1.2.3) shows that $\mu(A, v)$ can take only one of the values $\kappa(v)$ or $\rho(A, v)$. We distinguish the cases:

1. When

$$\rho(A, v) < \kappa(v), \qquad \mu(A, v) = \rho(A, v)$$

then v is *unfilled from A*.

2. When

$$\rho(A, v) \geqq \kappa(v), \qquad \mu(A, v) = \kappa(v)$$

then v is *filled from* A.

The latter case we subdivide further:

2a. When

$$\rho(A, v) > \kappa(v), \qquad \mu(A, v) = \kappa(v)$$

then v *is overfilled from* A.

2b. When

$$\rho(A, v) = \kappa(v)$$

then v is *exactly filled from* A.

To (1.2.1) we shall also introduce a set of *complementary multiplicities*

$$\tilde{\kappa}(v) = \rho(v) - \kappa(v) \tag{1.2.5}$$

which also satisfy the conditions

$$0 \leqq \tilde{\kappa}(v) \leqq \rho(v). \tag{1.2.6}$$

As in (1.2.3) we write

$$\tilde{\mu}(A, v) = \min (\tilde{\kappa}(v), \rho(A, v)). \tag{1.2.7}$$

We shall use the notations

$$\kappa(A) = \sum_v \kappa(v), \qquad \tilde{\kappa}(A) = \sum_v \tilde{\kappa}(v), \qquad v \in A. \tag{1.2.8}$$

Next we define the basic concept of the *deficiency* of a set A by the formula

$$\delta(A) = \kappa(A) - \mu(A, R(A)) \tag{1.2.9}$$

with the special case

$$\delta(\emptyset) = 0. \tag{1.2.10}$$

Evidently

$$\mu(A, R(A)) = \mu(A, S)$$

so that (1.2.9) may also be written

$$\delta(A) = \kappa(A) - \mu(A, S). \tag{1.2.11}$$

In a previous paper on directed graphs we have derived a number of properties of the deficiency function (1.2.9). Most of these can be transferred directly to undirected graphs, so we shall mention only a few important facts. The deficiency (1.2.9) is an integer. If we denote its maximal value by δ_0 then the convention (1.2.10) gives $\delta_0 \geqq 0$. Sets A_0 for which

$$\delta(A_0) = \delta_0 \tag{1.2.12}$$

are called sets of *maximal deficiency* or *critical sets*. For such sets one finds:

THEOREM 1.2.1. *The critical sets form a ring of sets with a unique minimal such set contained in all others and a unique maximal critical set containing all others.*

Analogous to (1.2.9) and (1.2.11) there is a *complementary deficiency* defined by

$$\tilde{\delta}(A) = \tilde{\kappa}(A) - \tilde{\mu}(A, R(A)) = \tilde{\kappa}(A) - \tilde{\mu}(A, S). \tag{1.2.13}$$

As in the case of directed graphs one has:

THEOREM 1.2.2. *The deficiency of a set A and the complementary deficiency of its complement $\overline{A}$ are equal*

$$\delta(A) = \tilde{\delta}(\overline{A}). \tag{1.2.14}$$

This result is also a consequence of subsequent considerations. It shows that if $\tilde{\delta}_0$ is the maximal complementary deficiency then

$$\tilde{\delta}_0 = \delta_0 \tag{1.2.15}$$

and also that if A_0 is a critical set then its complement $\overline{A}_0$ is a complementary critical set.

To any subset A there exists a decomposition of the whole vertex set

$$S = O(A) + E(A) + U(A) = O + E + U \tag{1.2.16}$$

where O is the set of vertices which is *overfilled from* A, E the vertices *exactly filled from* A and U the vertices *unfilled from* A. We also use the notation

$$F = F(A) = O + E \tag{1.2.17}$$

for the set of vertices *filled from* A and

$$I = I(A) = E + U \tag{1.2.18}$$

for the set of vertices *not overfilled from* A. Correspondingly one has the following alternatives for a vertex v

$$\begin{aligned} &\mu(A, v) = \kappa(v), && v \in O,\\ &\mu(A, v) = \kappa(v) = \rho(A, v), && v \in E,\\ &\mu(A, v) = \rho(A, v), && v \in U. \end{aligned} \tag{1.2.19}$$

Each of the preceding sets may be decomposed further into an *inner* and *outer part*, for instance

$$\begin{aligned} &O = O_1 + O_2,\\ &O_1 = A \cdot O, \qquad O_2 = \overline{A} \cdot O \end{aligned} \tag{1.2.20}$$

and similarly for the others. One notices for instance that

$$A = O_1 + I_1, \qquad \overline{A} = O_2 + I_2. \tag{1.2.21}$$

Corresponding to (1.2.16) one also has the decomposition with respect to the complement $\overline{A}$ and the complementary multiplicities (1.2.6)

$$S = \tilde{O}(\overline{A}) + \tilde{E}(\overline{A}) + \tilde{U}(\overline{A}) = \tilde{O} + \tilde{E} + \tilde{U} \tag{1.2.22}$$

with the analogous definitions to (1.2.17) and (1.2.18). One verifies readily that

$$O = \tilde{U}, \quad E = \tilde{E}, \quad U = \tilde{O}, \quad I = \tilde{F}, \quad F = \tilde{I}. \tag{1.2.23}$$

For the outer and inner parts the reader may verify the relations

$$\begin{aligned} O_1 &= \tilde{U}_2, \quad U_1 = \tilde{O}_2, \quad I_1 = \tilde{F}_2, \quad F_1 = \tilde{I}_2, \quad E_1 = \tilde{E}_2, \\ O_2 &= \tilde{U}_1, \quad U_2 = \tilde{O}_1, \quad I_2 = \tilde{F}_1, \quad F_2 = \tilde{I}_1, \quad E_2 = \tilde{E}_1. \end{aligned} \tag{1.2.24}$$

We shall use these notations to transform the expression (1.2.11) for the deficiency. From (1.2.19) and (1.2.21) we find

$$\begin{aligned} \mu(A, S) &= \mu(A, A) + \mu(A, \overline{A}) \\ &= \mu(A, I_1) + \mu(A, O_1) + \mu(A, I_2) + \mu(A, O_2) \\ &= \rho(A, I_1) + \kappa(O_1) + \rho(A, I_2) + \kappa(O_2). \end{aligned}$$

By means of

$$\rho(A, \overline{A}) = \rho(A, I_2) + \rho(A, O_2)$$

one obtains further

$$\mu(A, S) = \rho(A, \overline{A}) + \rho(A, I_1) + \kappa(O_1) - \rho(A, O_2) + \kappa(O_2).$$

This we substitute in (1.2.11) and find after a simple reduction

$$\delta(A) + \rho(A, \overline{A}) = \kappa(I_1) - \rho(A, I_1) + \rho(A, O_2) - \kappa(O_2).$$

From (1.2.19) we find that this may also be written

$$\delta(A) + \rho(A, \overline{A}) = \kappa(I_1) - \rho(A, I_1) + \rho(A, F_2) - \kappa(F_2).$$

In the last difference we eliminate by means of the relations

$$\begin{aligned} \kappa(F_2) + \tilde{\kappa}(F_2) &= \rho(F_2) = \rho(\tilde{I}_1), \\ \rho(A, F_2) + \rho(\overline{A}, F_2) &= \rho(F_2) = \rho(\tilde{I}_1) \end{aligned}$$

and find the formula

$$\delta(A) + \rho(A, \overline{A}) = \kappa(I_1) - \rho(A, I_1) + \tilde{\kappa}(\tilde{I}_1) - \rho(\overline{A}, \tilde{I}_1). \tag{1.2.25}$$

This formula has the advantage of being self dual in the direct and complementary concepts so that it includes a proof of Theorem 1.2.2. A further self dual expression can be obtained by means of the formula

$$\rho(A, \bar{A}) = \rho(I_1, \tilde{I}_1) + \rho(I_1, \tilde{O}_1) + \rho(O_1, \tilde{I}_1) + \rho(O_1, \tilde{O}_1)$$

and the relations

$$\rho(A, I_1) = \rho(I_1, I_1) + \rho(I_1, O_1),$$
$$\rho(\bar{A}, \tilde{I}_1) = \rho(\tilde{I}_1, \tilde{I}_1) + \rho(\tilde{I}_1, \tilde{O}_1).$$

After some rearrangements and reductions one finds from (1.2.25)

$$(1.2.26) \qquad -\delta(A) = \rho(O_1, \tilde{O}_1) - \rho(I_1, \tilde{I}_1) + \tilde{\kappa}(I_1) + \kappa(\tilde{I}_1).$$

1.3. **Effective deficiency.** The set $O_1(A)$ consisted of those vertices in A which are overfilled from A. In general the corresponding section graph $G(O_1)$ will be disconnected and have the decomposition

$$(1.3.1) \qquad G(O_1) = G(C_1) + \cdots + G(C_k)$$

into maximal connected components where

$$(1.3.2) \qquad O_1 = C_1 + \cdots + C_k.$$

These C_i we call the *overfilled inner components* of A. They fall into two categories. Let

$$(1.3.3) \qquad E = (c, a), \qquad c \in C, a \in A - C,$$

be an edge in $G(A)$ touching C, where according to the definition of C the end a of E is not in O_1. For a given C the number of such edges (1.3.3) is

$$\nu(A - C, C).$$

We shall call C an even component when

$$(1.3.4) \qquad \kappa(C) + \nu(A - C, C) \equiv 0 \pmod 2$$

and an odd component when

$$(1.3.5) \qquad \kappa(C) + \nu(A - C, C) \equiv 1 \pmod 2.$$

The number $r^{(i)}$ of odd inner overfilled components we shall call the *inner restriction of* A.

The analogous concepts shall be introduced for the complementary multiplicities and applied to the complement $\bar{A}$. Corresponding to (1.3.1) we have

$$(1.3.6) \qquad G(\tilde{O}_1) = G(\tilde{C}_1) + \cdots + G(\tilde{C}_{\tilde{k}})$$

where

$$(1.3.7) \qquad \tilde{O}_1 = U_2 = \tilde{C}_1 + \cdots + \tilde{C}_{\tilde{k}}.$$

The $\tilde{C}_i$ are the maximal connected components of $\tilde{O}_1$, the set of vertices in $\bar{A}$ overfilled from $\bar{A}$ with respect to the complementary multiplicities, or equivalently, the connected components of the set U_2 of vertices outside of A

and unfilled from A. Such a component is odd when

$$\tilde{\kappa}(\tilde{C}) + \nu(\overline{A} - \tilde{C}, \tilde{C}) \equiv 1 \pmod{2}. \tag{1.3.8}$$

By means of (1.1.7) this condition can be reformulated into

$$\kappa(\tilde{C}) + \nu(A, \tilde{C}) \equiv 1 \pmod{2}. \tag{1.3.9}$$

The number $r^{(O)}$ of such *overfilled outer components* of A in (1.3.6) shall be called the *outer restriction* of A. The sum

$$r = r^{(i)} + r^{(O)} \tag{1.3.10}$$

is the *total restriction* of A.

Next we define: The *effective deficiency* of the set A is the quantity

$$\delta_e(A) = \delta(A) + r. \tag{1.3.11}$$

Thus one always has

$$\delta_e(A) \geqq \delta(A) \tag{1.3.12}$$

while Theorem 1.2.2 gives:

THEOREM 1.3.1. *The effective deficiency of a set A equals the effective deficiency of $\overline{A}$ with respect to the complementary multiplicities*

$$\delta_e(A) = \tilde{\delta}_e(\overline{A}). \tag{1.3.13}$$

Our first main problem is to establish necessary and sufficient conditions for a graph G to have a subgraph H such that the local degrees of H have the prescribed values

$$\rho_H(v) = \kappa(v). \tag{1.3.14}$$

We notice that if such a subgraph exists then its complementary graph

$$\overline{H} = G - H \tag{1.3.15}$$

must have the complementary degrees

$$\rho_{\overline{H}}(v) = \tilde{\kappa}(v). \tag{1.3.16}$$

We shall suppose for the moment that a subgraph H satisfying (1.3.14) exists. By means of H we shall give another formula for the deficiency of a set A. Within A, that is, in $G(A)$, there will be contained certain edges of H and we denote by $\rho_{H,A}(v)$ the number of such edges at each $v \in A$. For these quantities we have the upper bounds

$$\rho_{H,A}(v) \leqq \rho(A, v) = \mu(A, v), \qquad v \in I_1, \tag{1.3.17}$$

$$\rho_{H,A}(v) \leqq \kappa(v) = \mu(A, v), \qquad v \in O_1. \tag{1.3.18}$$

Thus the total number of edges of H in A is

$$(1.3.19) \qquad \rho_H(A, A) = \mu(A, A) - \sum_{v \in A} \lambda_{H,A}(v)$$

where each edge is counted twice, and we put

$$(1.3.20) \qquad \lambda_{H,A}(v) = \mu(A, v) - \rho_{H,A}(v).$$

On the other hand the total number of edges of H issuing from A is $\kappa(A)$, again counting double those within A. By subtracting (1.3.19) from this quantity we obtain the number of edges of H touching A

$$(1.3.21) \qquad \begin{aligned} \nu_H(A, \bar{A}) &= \kappa(A) - \mu(A, A) + \sum_{v \in A} \lambda_{H,A}(v) \\ &= \kappa(I_1) - \rho(A, I_1) + \sum_{v \in A} \lambda_{H,A}(v). \end{aligned}$$

The same argument may be applied to the set $\bar{A}$ and the graph $\bar{H}$ so that

$$(1.3.22) \qquad \nu_{\bar{H}}(A, \bar{A}) = \tilde{\kappa}(\tilde{I}_1) - \rho(\bar{A}, \tilde{I}_1) + \sum_{v \in \bar{A}} \tilde{\lambda}_{\bar{H},\bar{A}}(v).$$

By adding (1.3.21) and (1.3.22) we have

$$\begin{aligned} \rho(A, \bar{A}) = \kappa(I_1) + \tilde{\kappa}(\tilde{I}_1) &- \rho(A, I_1) - \rho(\bar{A}, \tilde{I}_1) \\ &+ \sum_{v \in A} \lambda_{H,A}(v) + \sum_{v \in \bar{A}} \tilde{\lambda}_{\bar{H},\bar{A}}(v) \end{aligned}$$

and by (1.2.25) this reduces to

$$(1.3.23) \qquad \delta(A) = - \sum_{v \in A} \lambda_{H,A}(v) - \sum_{v \in \bar{A}} \tilde{\lambda}_{\bar{H},\bar{A}}(v).$$

This result shows that the deficiency is nonpositive. However, we have stronger:

THEOREM 1.3.2. *When there exists a subgraph H with the local degrees* (1.3.14) *then for every subset A*

$$(1.3.24) \qquad \delta_e(A) \leqq 0.$$

Proof. It is evidently sufficient to show that for each inner overfilled component C of A there is at least one term (1.3.20) which is positive and similarly for the outer components. For an odd component C one has the congruence (1.3.5); on the other hand the analogue of the congruence (1.1.7)

$$\kappa(C) + \nu_H(C, \bar{C}) \equiv 0 \pmod 2$$

must hold for C and H. To fulfill them both one sees that at least one of the following two cases must occur:

1. There is some $\bar{H}$ edge (v, c) between C and $A - C$.
2. There is some H edge $(c_1, \bar{a})$ between C and $\bar{A}$.

In the first case the inequality holds in (1.3.17) for the vertex v, in the second case in (1.3.18) for $c_1 = v$. For the outer components the situation is analogous.

The expression (1.3.23) enables us to formulate in various ways the necessary and sufficient conditions for a set A to be of zero effective deficiency, provided a subgraph H satisfying (1.3.14) exists. We find first

$$(1.3.25) \qquad r^{(i)} = \sum_{v \in A} \lambda_{H,A}(v), \qquad r^{(O)} = \sum_{v \in \bar{A}} \lambda_{\bar{H},\bar{A}}(v).$$

When this is substituted in (1.3.21) and (1.3.22) we have the further criterion

$$(1.3.26) \qquad \begin{aligned} \nu_H(A, \bar{A}) &= \kappa(I_1) - \rho(A, I_1) + r^{(i)}, \\ \nu_{\bar{H}}(A, \bar{A}) &= \tilde{\kappa}(\tilde{I}_1) - \rho(\bar{A}, \tilde{I}_1) + r^{(O)} \end{aligned}$$

and this may also be rewritten

$$(1.3.27) \qquad \begin{aligned} \mu(A, A) &= \rho_H(A, A) + r^{(i)}, \\ \mu(A, \bar{A}) &= \rho_H(A, \bar{A}) + r^{(O)}. \end{aligned}$$

We leave the verification to the reader. The last criterion is, as one sees, not self dual in the complementary concepts. Expressed briefly it states that taking the restrictions into account, H must absorb all possibilities for placing its edges within or touching A.

Chapter 2. The method of alternating paths

2.1. **Definitions.** A path P in the graph G between a_0 and a_n is a sequence

$$(2.1.1) \qquad P = a_0, a_1, \cdots, a_n$$

of vertices such that

$$A_{i,i+1} = (a_i, a_{i+1}) \qquad (i = 0, 1, \cdots n - 1)$$

are edges in G. A *section* of (2.1.1) is any path of the form

$$P' = P(a_j, a_k) = a_j, a_{j+1}, \cdots, a_k.$$

A path is *cyclic* if $a_0 = a_n$.

Let H be a subgraph of G. The edges in G then fall into two categories:

α edges belonging to H;

β edges not belonging to H.

The symbols α and β we shall call the *characters* of the corresponding edges. In cases where the characters of certain edges may not be definitely known it is convenient to denote them by Greek letters $\gamma, \delta, \cdots$ where each is equal to α or β. The opposite of a character γ we denote by $\bar{\gamma}$. The symbols

$$\chi(a, b) = \gamma, \chi(c, d) = \bar{\gamma}$$

denote that the edge $(a, b) = a\gamma b$ has the character γ and the edge $(c, d) = c\bar{\gamma}d$ the opposite character $\bar{\gamma}$.

An *alternating path* is a path (2.1.1) with the two properties:

1. The characters of successive edges alternate

$$P = a_0\gamma a_1\bar{\gamma}a_2\gamma a_3 \cdots \delta a_n. \tag{2.1.2}$$

2. No edges appear twice in P.
An alternating path may contain cyclic sections and a single edge may also be considered such a path.

Since in the following we shall consider exclusively alternating paths they will for short only be denoted as *paths*. Next we select some fixed vertex c_0, the *center* and examine the family $F(P)$ of all paths P satisfying the conditions:

1. P will have the initial point c_0.

2. The first edge in P has the character β. To insure that $F(P)$ is not void we suppose that there exists at least one β edge having the end point c_0. A vertex a shall be called *accessible* if $a=c_0$ or a can be reached from c_0 by a path P. The other vertices are *inaccessible*.

The edges in the graph we classify as follows with respect to $F(P)$:

1. An edge E is *acursal* if there exists no path P in F containing E.

2. E is *cursal* if some P contains E. If $E=(a, b)$ appears in some P with the vertices in this order we shall say that E is *cursal from a to b*. The class of cursal edges shall be subdivided as follows:

1. The edge $E=(a, b)$ is *bicursal* if it is cursal in both directions and its end points a and b are called *bicursal neighbors*.

2. An edge is *unicursal* if it is cursal in one direction only. Its endpoints are then *unicursal neighbors*. The accessible vertices we classify as follows:

1. A vertex v is an *α vertex*, resp. a *β vertex* if every path in F to v ends in an α edge, resp. β edge.

2. v is a *bicursal vertex* if there are paths to v ending both in α and β edges.

To this we add by special definition:

1. The center c_0 is bicursal if there is a path returning to c_0 in a β edge.

2. If not c_0 is an α vertex.

Before we proceed to the main theorems in the theory of alternating paths we shall enumerate a few auxiliary facts.

THEOREM 2.1.1. *Let*

$$A = a\gamma b, \qquad B = b\bar{\gamma}c \tag{2.1.3}$$

be two adjoining edges of different character. If A is unicursal to b then B is cursal from b.

Proof. Tutte [12, Theorem V].

THEOREM 2.1.2. *A vertex v is bicursal if and only if it is the end of a bicursal edge.*

Proof. If $v\neq c_0$ is the end of a bicursal γ edge it can be entered by a path ending in this edge. Since it is possible to run through this edge also in the opposite direction there must be some $\bar{\gamma}$ edge entering v. Conversely, when v is bicursal there are two edges

$$A = b\alpha v, \qquad B = c\beta v$$

entering v as the end of paths. If A should be unicursal then Theorem 2.1.1 shows that B must be bicursal. When $v = c_0$ the entering β edge is evidently bicursal and if there is a bicursal edge at c_0 there must be some entering β edge.

THEOREM 2.1.3. *Every edge with one bicursal end v is cursal from v. No inaccessible vertex can be connected by an edge to a bicursal vertex.*

Proof. Since there are paths ending in α and β edges at $v \neq c_0$ any edge (v, d) must be cursal. For $v = c_0$ any β edge (c_0, d) is cursal and for an α edge the returning β edge can be continued through it.

THEOREM 2.1.4. *All entering unicursal edges at a bicursal vertex have the same character.*

Proof. If there are two entering edges (2.1.3) at a vertex $b \neq c_0$ and A is unicursal then B becomes bicursal according to Theorem 2.1.1. For $b = c_0$ all entering unicursal edges must be α edges.

2.2. **Bicursal equivalence.** For the present we shall restrict ourselves to the set $W(c_0)$ of accessible vertices. Two vertices a and b are *bicursally equivalent* if there exist vertices

$$(2.2.1) \qquad a = a_0, a_1, \cdots, a_n = b$$

such that any two successive ones are bicursal neighbors. This defines an equivalence relation and correspondingly we have a decomposition of the set W into disjoint blocks of equivalent vertices. Some of these blocks may be *singular*, that is, consist of a single vertex; others are *nonsingular* containing several vertices. Theorem 2.1.2 shows:

THEOREM 2.2.1. *The singular vertices are the α and β vertices. The nonsingular blocks consist of bicursal vertices.*

For the present we shall focus our attention upon the nonsingular blocks. We denote some fixed such block by $\mathfrak{B}$. Since every b in $\mathfrak{B}$ is accessible there exists a path from c_0 to b. We suppose first that c_0 is not in $\mathfrak{B}$. Let P be a path to some vertex in $\mathfrak{B}$ and e the first vertex in P belonging to $\mathfrak{B}$. We call e an *entrance* to $\mathfrak{B}$ and the edge

$$(2.2.2) \qquad E = d\gamma e$$

preceding e in P the *entering edge* of P while γ is the *entering character*. The section $P_0 = P(c_0, e)$ shall be called the *entering path*. Evidently every entering edge is unicursal and Theorem 2.1.4 shows that all entering edges at e must have the same character γ. In the case where $c_0 \in \mathfrak{B}$ we shall always denote the block by $\mathfrak{B}_0$ and call $c_0 = e$ the *entrance*; there are then no entering edges.

To each block $\mathfrak{B}$ of equivalent bicursal vertices we associate a subgraph $G_0(\mathfrak{B})$ of G having the set $\mathfrak{B}$ for its vertices and for its edges those bicursal edges which connect vertices in $\mathfrak{B}$. According to its definition $G_0(\mathfrak{B})$ must be connected and contain at least two vertices and an edge joining them.

In the following let us consider the entrance e of $G_0(\mathfrak{B})$ to be fixed. A path P which enters $\mathfrak{B}$ at e may continue through some of the edges of $G_0(\mathfrak{B})$. Let

$$e\bar{\gamma}b_1, \gamma b_2, \cdots, \delta b_k \tag{2.2.3}$$

be the corresponding section of P where all edges are bicursal while the next edge

$$b_k\bar{\delta}b_{k+1}$$

if it exists is unicursal. It should be noted that the vertex b_{k+1} may or may not belong to $\mathfrak{B}$. We call (2.2.3) the *section of P in $G_0(\mathfrak{B})$*. The totality of such sections we denote by $F(e)$.

THEOREM 2.2.2. *The family $F(e)$ must contain some nontrivial path, that is, it cannot reduce to e only.*

Proof. When c_0 is bicursal we have seen that there is a bicursal β edge at c_0. Thus we may suppose $\mathfrak{B} \neq \mathfrak{B}_0$. Let P be a path entering $\mathfrak{B}$ at e with the entering edge (2.2.2). If there exists some $\bar{\gamma}$ edge in $G_0(\mathfrak{B})$ having e as an end-point then $P(c_0, e)$ can be continued through this edge. However, not all edges in $G_0(\mathfrak{B})$ with end point e can be γ edges, because they are bicursal and there would have to be some unicursal $\bar{\gamma}$ edge entering e contradicting Theorem 2.1.4.

This proof also shows:

THEOREM 2.2.3. *If the entering character of $\mathfrak{B}$ at e is γ then not all edges in $G_0(\mathfrak{B})$ with the end point e can have the character γ.*

Let us now say that an edge B in $G_0(\mathfrak{B})$ is *cursal* or *bicursal* within $F(e)$ when it has this property in regard to this family of paths.

THEOREM 2.2.4. *Any edge in $G_0(\mathfrak{B})$ which is cursal within $F(e)$ is also bicursal within the same family.*

Proof. Tutte [12, Theorem VIII].

Next we define a vertex b in $G_0(\mathfrak{B})$ to be *accessible in $F(e)$* if it can be reached by a path in this family. Similarly b is *bicursal in $F(e)$* if it is the end point of paths in $F(e)$ terminating in α and β edges.

THEOREM 2.2.5. *Any vertex $b \neq e$ in $\mathfrak{B}$ which is accessible in $F(e)$ is also bicursal* (*Gallai, Auxiliary Theorem* I, *p.* 138).

Proof. Let $P(e, b)$ be a path in $F(e)$ ending in a γ edge A. By Theorem 2.2.4 A is bicursal in $F(e)$ so that there must be a path in $F(e)$ ending at b in a $\bar{\gamma}$ edge.

THEOREM 2.2.6. *Every vertex in $\mathfrak{B}$ is accessible in $F(e)$ by a nontrivial path.*

Proof. Tutte [10, Theorem X].

From Theorem 2.2.5 we conclude:

THEOREM 2.2.7. *Every vertex $b \neq e$ in $\mathfrak{B}$ is bicursal in $F(e)$, while e is bicursal if and only if there are both α and β edges in $G_0(\mathfrak{B})$ having e for an endpoint (Gallai, Auxiliary Theorem* II, *p.* 139).

Another consequence is:

THEOREM 2.2.8. *Each component $\mathfrak{B}$ contains at least* 3 *vertices and there are at least two $\bar{\gamma}$ edges in $G_0(\mathfrak{B})$ issuing from e.*

Proof. Theorem 2.2.6 shows that there are nontrivial returning paths to e in $F(e)$.

THEOREM 2.2.9. *The graph*

$$G_0(\mathfrak{B}) = G(\mathfrak{B})$$

is a section graph in G.

Proof. The theorem states that if a and b are vertices in $\mathfrak{B}$ connected by an edge $A = (a, b)$ in G then A belongs to $G_0(\mathfrak{B})$, that is, A is bicursal. When $a \neq e$, $b \neq e$ both vertices are bicursal in $F(e)$ and it follows readily from Theorem 2.2.4 that A is bicursal. Next let $A = (e, b)$ with $e \neq c_0$. Since b is bicursal in $F(e)$ we see that A must be cursal from b to e. When A has the character $\bar{\gamma}$ it is cursal from e to b by continuing the entry path. When $e\gamma b$ we make use of the fact that there is a returning path in $F(e)$ to e ending in a $\bar{\gamma}$ edge. Thus A must be bicursal. A similar argument applies when $e = c_0$.

Finally we have the important fact:

THEOREM 2.2.10. *Every graph $G(\mathfrak{B})$ has a single entrance e and when $e \neq c_0$ a single entering edge (d, e). All other edges (b, c) $b \in \mathfrak{B}$ touching $\mathfrak{B}$ are unicursal from b to c.*

Proof. Tutte [10, Theorem XI] (Gallai, Theorem I, p. 138).

We observed that the family of bicursal blocks defined a partition of the set W of accessible vertices into single α and β vertices and nonsingular blocks $\mathfrak{B}$. Theorem 2.2.10 makes it possible to assign a character also to the blocks $\mathfrak{B}$, namely the character of its entering edge. The block $\mathfrak{B}_0$ is defined to be an α block.

The unicursal edges from a bicursal block cannot lead to inaccessible vertices, hence if such an edge has the character γ it can only lead to γ vertices or γ blocks. For an α vertex a all cursal edges from a must be β edges going to β vertices or β blocks while all cursal edges from a β vertex must be α edges. Let us consider for a moment the family $\{\mathfrak{B}\}$ of all bicursal blocks, including the α and β vertices. One can then define a new graph, the *block graph* $\mathfrak{G}(\mathfrak{B})$ whose vertex set is $\{\mathfrak{B}\}$. Any pair of blocks $\mathfrak{B}_i$ and $\mathfrak{B}_j$, singular or nonsingular, can as we have shown be joined by at most a single unicursal edge in G. Correspondingly, when this happens we join the two blocks by a directed edge $(\mathfrak{B}_i, \mathfrak{B}_j)$ in $\mathfrak{G}(\mathfrak{B})$. From this definition we conclude:

THEOREM 2.2.11. *The block graph $\mathfrak{G}(\mathfrak{B})$ has unicursal edges, thus it is a directed graph in which every vertex can be reached by a directed path from $\mathfrak{B}_0$.*

2.3. **Bicursal point equivalence.** Next we introduce a larger partition of $W(c_0)$ containing the preceding bicursal equivalence. Two bicursal vertices a and b are *bicursally point equivalent* or simply *point equivalent* if there exists a sequence (2.2.1) of bicursal vertices such that each successive pair is joined by an edge in G, now not necessarily bicursal.

Under this definition the singular blocks remain the same as before while a nonsingular point equivalence block $\mathfrak{P}$ may include several of our previous bicursal blocks $\mathfrak{B}$. We shall prove:

THEOREM 2.3.1. *Every nonsingular point equivalence block $\mathfrak{P}$ has a single entrance e and when $e\neq c_0$ a single entering edge $E=(d, e)$ unicursal to e. All edges different from E touching $\mathfrak{P}$ are unicursal from $\mathfrak{P}$ to α and β vertices.*

Proof. (Compare Tutte [10, Theorem XII].) We write $\mathfrak{P}=\mathfrak{P}_0$ for the block containing c_0 if it exists. If $\mathfrak{P}\neq\mathfrak{P}_0$, let $Q(c_0, e)$ be a path from c_0 to $\mathfrak{P}$ and e the first vertex in $\mathfrak{P}$ belonging to Q. When $\mathfrak{P}=\mathfrak{P}_0$ we write $e=c_0$. The bicursal block $\mathfrak{B}_e$ to which e belongs we call the *leading block* in $\mathfrak{P}$.

Under the point equivalence some bicursal block $\mathfrak{B}_1$ may be connected to $\mathfrak{B}_e$ by an edge.

$$B = (b_0, b_1), \qquad b_0 \in \mathfrak{B}_e,\ b_1 \in \mathfrak{B}_1.$$

By Theorem 2.2.10 B must be unicursal from $\mathfrak{B}_e$ to $\mathfrak{B}_1$ and B becomes the single entering edge to $\mathfrak{B}_1$ with the entrance b_1. Evidently there is a path in $F(e)$ in $\mathfrak{B}_e$ which can be continued through B to b_1 and the other vertices in $\mathfrak{B}_1$. There may be several such neighboring blocks $\mathfrak{B}_1$ to $\mathfrak{B}_e$. In addition there may be unicursal edges from $\mathfrak{B}_e$ to α and β, vertices which by definition do not belong to $\mathfrak{P}$. To each block $\mathfrak{B}_1$ there may be successive bicursal blocks $\mathfrak{B}_2$ and so on. All of these are seen to be accessible through the preceding blocks by paths from e. Theorem 2.2.10 shows that all edges different from the entering edge E to $\mathfrak{B}_e$ and touching $\mathfrak{P}$ must be unicursal from $\mathfrak{P}$.

In the preceding we introduced the bicursal block graph $\mathfrak{G}(\mathfrak{B})$. Let us for a moment consider that part of $\mathfrak{G}(\mathfrak{B})$ whose vertices are the bicursal blocks $\mathfrak{B}_i$ contained in a point equivalence block $\mathfrak{P}$. Since each $\mathfrak{B}_i$ has but a single entering edge we conclude from our discussion:

THEOREM 2.3.2. *The section graph of the block graph $\mathfrak{G}(\mathfrak{B})$ defined by the bicursal blocks $\mathfrak{B}$ in a point block $\mathfrak{P}$ is topologically a tree which may be considered a partial ordering in the directed graph with the leading block $\mathfrak{B}_e$ as its maximal element.*

Theorem 2.3.2 enables us to assign a character also to each point block $\mathfrak{P}$, namely the character of the entering edge $E=(d, e)$ to its leading block $\mathfrak{B}_e$. We shall write $\mathfrak{P}_\alpha$ and $\mathfrak{P}_\beta$ respectively for the α and β blocks. For a block $\mathfrak{P}_\alpha$ the edge E is an α edge from a β vertex d and for a $\mathfrak{P}_\beta$ it is a β edge from an α vertex d. The number of such blocks $\mathfrak{P}_\alpha$ and $\mathfrak{P}_\beta$ we shall denote by k_α and k_β respectively.

2.4. **Properties of the accessible set.** For a given center c_0 let

$$W = W(c_0), \qquad \overline{W} = S - W \tag{2.4.1}$$

be the sets of accessible and of inaccessible vertices. We shall decompose the set W into four parts

$$W = W_\alpha + W_\beta + V_\alpha + V_\beta \tag{2.4.2}$$

where W_α and W_β are the sets of α and β vertices while

$$V_\alpha = \sum \mathfrak{P}_\alpha, \qquad V_\beta = \sum \mathfrak{P}_\beta \tag{2.4.3}$$

consist of the vertices belonging to α and to β point blocks $\mathfrak{P}_\alpha$ and $\mathfrak{P}_\beta$. As before we write $\mathfrak{P}_0$ for the α block containing c_0 and introduce the quantity

$$\begin{aligned} \epsilon_0 &= 1 \text{ if } \mathfrak{P}_0 \text{ exists,} \\ \epsilon_0 &= 0 \text{ if } \mathfrak{P}_0 \text{ does not exist.} \end{aligned} \tag{2.4.4}$$

Finally we write

$$A = W_\alpha + V_\alpha \tag{2.4.5}$$

and call this the *α component of $W(c_0)$*.

The preceding analysis of the set of accessible vertices was based upon some prescribed subgraph H of G. The local degrees of H

$$\kappa(v) = \rho_H(v), \qquad \bar{\kappa}(v) = \rho_{\overline{H}}(v) \tag{2.4.6}$$

may be considered to be a set of multiplicities for G in the sense of §1.2. In §1.4 we also defined a family of sets associated with a given set A. When these concepts are applied to the set A in (2.4.5) and the multiplicities (2.4.6) we can show:

THEOREM 2.4.1. *For the α component A in* (2.4.5) *one has*

$$I_1 = W_\alpha, \qquad O_1 = V_\alpha \tag{2.4.7}$$

and

$$F_2 \supset W_\beta, \qquad I_2 \supset \overline{W}, \qquad U_2 \supset V_\beta. \tag{2.4.8}$$

We divide the proof into a series of lemmas:

LEMMA 1. *No vertex $v \in W_\alpha$ is overfilled from A.*

Proof. This follows from the fact that any edge $E=(v, a)$, $a \in A$, must be an α edge, hence there are at most $\rho_H(v)$ edges to v from A. If namely E were a β edge and $a \in W_\alpha$ there would be entering β edges to the α vertices v and a; if $a \in \mathfrak{P}_\alpha$ then E cannot be the entering α edge since this would come from a β vertex, thus E is cursal from $\mathfrak{P}_\alpha$ to v, hence an α edge.

LEMMA 2. *Every vertex $v \in V_\alpha$ is overfilled from A.*

Proof. Let $v \in \mathfrak{P}_\alpha$ and suppose first $v \neq e$. Then all α edges at v are either bicursal edges within $\mathfrak{P}_\alpha$ or they are cursal to α vertices, hence there are $\rho_H(v)\alpha$ edges at v all lying in A. But there are also β edges at v within $\mathfrak{P}_\alpha$ since v is bicursal. When $v=c_0$ all α edges at c_0 are cursal to $\mathfrak{P}_0$ or to α vertices so that the lemma is true. Finally take $v=e\neq c_0$. Except for the entering edge all α edges at v must go to $\mathfrak{P}_\alpha$ or α vertices so that there are $\rho_H(v)-1$ such edges in A; in addition according to Theorem 2.2.8 there are at least two β edges within $\mathfrak{P}_\alpha$ at v.

The combination of Lemmas 1 and 2 gives the formulas (2.4.7). We have also the following consequence of these observations:

THEOREM 2.4.2. *The inner overfilled components of the set A are the blocks $\mathfrak{P}_\alpha$ and these are odd except when $\mathfrak{P}_\alpha = \mathfrak{P}_0$.*

Proof. We have seen that the $\mathfrak{P}_\alpha$ are the overfilled blocks of A with respect to the multiplicities (2.4.6). All α edges with an end in $\mathfrak{P}_0$ go to vertices in $\mathfrak{P}_0$ or W_α so that when (1.1.7) is applied to $\mathfrak{P}_0$ and H

$$\rho_H(\mathfrak{P}_0) + \rho(\mathfrak{P}_0, A - \mathfrak{P}_0) \equiv 0 \pmod 2$$

and $\mathfrak{P}_0$ is even. The same holds for a block $\mathfrak{P}_\alpha \neq \mathfrak{P}_0$, except for the single entering α edge coming from W_β, so that in this case

$$\rho_H(\mathfrak{P}_\alpha) + \rho(\mathfrak{P}_\alpha, A - \mathfrak{P}_\alpha) \equiv 1 \pmod 2.$$

The number of inner restrictions of A in the sense of §1.3 is therefore

$$r^{(i)} = k_\alpha - \epsilon_0 \qquad (2.4.9)$$

where ϵ_0 is defined by (2.4.4).

We proceed next to the proof of (2.4.8):

LEMMA 3. *Every vertex $v \in W_\beta$ is filled from A.*

Proof. All $\rho_H(v)\alpha$ edges at v must be cursal to A.

LEMMA 4. *No vertex $v \in \overline{W}$ can be overfilled from A.*

Proof. Theorem 2.1.3 shows that any edge to v from A must come from W_α, thus it is an α edge since v is inaccessible.

LEMMA 5. *No vertex $v \in V_\beta$ can be filled from A.*

Proof. Let $v \in \mathfrak{P}_\beta$ and take first $v \neq e$. Every edge between $\mathfrak{P}_\beta$ and A must be cursal from v, hence an α edge. Since v is bicursal in $\mathfrak{P}_\beta$ there must be incoming α edges to v also from $\mathfrak{P}_\beta$, thus the number of edges to v from A cannot reach $\rho_H(v)$. When $v = e$ one of the edges from A is the entering β edge, but then by Theorem 2.2.8 there are at least two α edges at e lying within $\mathfrak{P}_\beta$.

From this proof we deduce further:

THEOREM 2.4.3. *The outer overfilled odd components of A with respect to the complementary multiplicities* (2.4.6) *are the sets $\mathfrak{P}_\beta$.*

Proof. We have just seen that the $\mathfrak{P}_\beta$ are overfilled blocks for $\overline{A}$ with respect to the complementary multiplicities. Furthermore, all β edges from $\mathfrak{P}_\beta$ go to W_β except for the single entering edge from W_α. When (1.1.7) is applied to $\overline{H}$ and $\mathfrak{P}_\beta$ we find therefore

$$\rho_{\overline{H}}(\mathfrak{P}_\beta) + \rho(\mathfrak{P}_\beta, \overline{A} - \mathfrak{P}_\beta) \equiv 1 \pmod 2.$$

It remains to show that there are no other such odd blocks $\mathfrak{Q}$ in $\overline{A}$. The vertices in $\mathfrak{Q}$ would have to be inaccessible because they are disjoint from the $\mathfrak{P}_\beta$ and cannot belong to W_β according to (2.4.8). However, since $\mathfrak{Q}$ is odd, there must either be some β edge from $\mathfrak{Q}$ to A or some α edge to W_β. But both alternatives would make a vertex in $\mathfrak{Q}$ accessible as one readily sees. We conclude that the number of outer restrictions of A is

$$r^{(O)} = k_\beta. \tag{2.4.10}$$

Finally we have:

THEOREM 2.4.4. *The deficiencies of the set A with respect to the multiplicities* (2.4.6) *are*

$$\delta(A) = -k_\alpha - k_\beta - \epsilon_0, \qquad \delta_e(A) = 0. \tag{2.4.11}$$

Proof. To calculate the deficiency we notice that

$$\kappa(A) = \rho_H(A)$$

is the number of edges of H issuing from the vertices in A. The conditions of Theorem 2.4.1 indicate which vertices in S are filled from A and we conclude

$$\mu(A, S) = \rho(A, W_\alpha + \overline{W} + V_\beta) + \rho_H(W_\beta + V_\alpha).$$

This is also the number of edges of H entering from A except that in the terms

$$\rho(A, V_\beta), \qquad \rho_H(V_\alpha)$$

we have counted respectively the k_β entering β edges from A and the $k_\alpha - \epsilon_0$ entering α edges from W_β. Adjusting for this one finds

(2.4.12) $\rho_H(A) + k_\alpha + k_\beta - \epsilon_0 = \rho(A, W_\alpha + \overline{W} + V_\beta) + \rho_H(W_\beta + V_\alpha).$

When this is combined with the definition (1.2.11) of the deficiency and (1.3.11) of the effective deficiency we obtain (2.4.11).

2.5. **The main theorem.** We assume as previously that a family of multiplicities

(2.5.1) $$\kappa(v)$$

is prescribed for the vertices of the graph G. We proved in Theorem 1.3.2 that if there exists a subgraph H of G with these local degrees then

(2.5.2) $$\delta_e(A) \leqq 0$$

for every subset A of S, that is, no subset can have a positive effective deficiency with respect to these multiplicities. We shall now complete this result to the main theorem:

THEOREM 2.5.1. *In order that a graph have a subgraph H with the local degrees* (2.5.1) *it is necessary and sufficient that for every subset A of the vertex set S the condition* (2.5.2) *be fulfilled.*

To prove the remaining sufficiency part of this theorem we define a subgraph H of G to be *compatible with the multiplicities* (2.5.1) if one always has

(2.5.3) $$\rho_H(v) \leqq \kappa(v).$$

The quantity

(2.5.4) $$d_H(v) = \kappa(v) - \rho_H(v)$$

shall be called the *deficit of H at v*. The *deficit of H in the set A* is the sum

(2.5.5) $$d_H(A) = \sum_{v\in A} d_H(v) = \kappa(A) - \rho_H(A)$$

while the total deficit of H is

(2.5.6) $$d_H(S) = \kappa(S) - \rho_H(S).$$

If a and b are two vertices with positive deficits joined by an edge not in H, then the adjunction of (a, b) to H will reduce the total deficit by two units. More general, let there exist an alternating path P joining a and b, beginning and ending in a β edge. By transforming H such that all edges of H in P are changed into nonedges and all nonedges into edges, we obtain a new compatible subgraph H' of G. The local degrees of H and H' are the same at all vertices except at a and b where the deficits of H' are one unit smaller. We say that H' has been obtained from H by an *alternating augmentation*. In this process it is conceivable that the path be cyclic $a=b$. An augmentation is still possible if $d_H(a) \geqq 2$.

We define further: A subgraph H is a *maximal compatible subgraph* for the multiplicities (2.5.1) if it cannot be alternatingly augmented. We shall then arrive at our desired result by taking H as some such maximal graph and show that if the condition (2.5.2) is fulfilled for all subsets no vertex of H can have a positive deficit.

For this purpose let us suppose that c_0 is a vertex such that

$$d_H(c_0) \geqq 1. \tag{2.5.7}$$

With c_0 as center we construct the accessible set $W(c_0)$ with respect to H. The maximality of H shows that no vertex v in W can have a positive deficit if there is a path to v ending in a β edge. As a consequence we must have

$$\rho_H(v) \leqq \kappa(v), \qquad v \in W_\alpha + \overline{W} \tag{2.5.8}$$

and

$$\rho_H(v) = \kappa(v), \qquad v \in W_\beta + V_\alpha + V_\beta - c_0. \tag{2.5.9}$$

If $c_0 \in \mathfrak{P}_0$ there is by definition a returning path to c_0 ending in a β edge. Since no augmentation is possible we conclude

$$d_H(c_0) = 1, \qquad c_0 \in \mathfrak{P}_0. \tag{2.5.10}$$

We now consider the α component A of $W(c_0)$ defined in (2.4.5) and compute its deficiency

$$\delta(A) = \kappa(A) - \mu_\kappa(A, S) \tag{2.5.11}$$

with respect to the multiplicities (2.5.1). Here

$$\mu_\kappa(A, S) = \mu_\kappa(A, W_\alpha + V_\beta + \overline{W}) + \mu_\kappa(A, V_\alpha + W_\beta)$$

and according to (2.5.8) and Theorem 2.4.1

$$W_\alpha + V_\beta + \overline{W}$$

is a set not overfilled from A with respect to (2.5.1). Thus we have

$$\mu_\kappa(A, W_\alpha + V_\beta + \overline{W}) = \rho(A, W_\alpha + V_\beta + \overline{W}).$$

The set

$$V_\alpha + W_\beta$$

is filled from A with respect to the multiplicities (2.5.9) according to (2.4.7) and (2.4.8) except at $c_0 \in \mathfrak{P}_0$ where by (2.5.10)

$$\kappa(c_0) = \rho_H(c_0) + 1. \tag{2.5.12}$$

We have therefore

$$\mu_\kappa(A, V_\alpha + W_\beta) = \rho_H(V_\alpha + W_\beta) + \epsilon_0.$$

These expressions we substitute in (2.5.11) and after reduction by means of

(2.4.12) we find for the deficiency with respect to the multiplicities (2.5.1)

$$\delta(A) + k_\alpha + k_\beta = \kappa(A) - \rho_H(A) = d_H(A). \tag{2.5.13}$$

To determine the effective deficiency with respect to the $\kappa(v)$ we must find the odd overfilled components of A. According to (2.5.9) all components $\mathfrak{P}_\alpha \neq \mathfrak{P}_0$ and $\mathfrak{P}_\beta$ are odd overfilled components of A also for (2.5.1). But the exception (2.5.12) shows that $\mathfrak{P}_0$ which was even for the $\rho_H(v)$ multiplicities becomes odd for the $\kappa(v)$. Thus we have for the restrictions

$$r^{(i)} = k_\alpha, \qquad r^{(O)} \geqq k_\beta.$$

When this is used in (2.5.13) one finds for the effective deficiency with respect to (2.5.1)

$$\delta_e(A) = d_H(A) + (r^{(O)} - k_\beta). \tag{2.5.14}$$

This however contains a contradiction to our assumption that the condition (2.5.2) should be fulfilled for all subsets. The first term on the right in (2.5.14) is positive according to (2.5.7) and the second is nonnegative. This concludes the proof of Theorem 2.5.1.

The preceding proof shows that in applying the condition (2.5.2) it is not necessary that it be verified for all subsets A since if there is no subgraph corresponding to the given multiplicities then some α component A in (2.4.5) must have positive effective deficiency and these sets are special in several ways.

THEOREM 2.5.2. *An α component A of an accessible set with respect to a maximal compatible subgraph H with positive deficit has only odd inner overfilled components and*

$$\rho(O_1, \tilde{O}_1) = 0. \tag{2.5.15}$$

Proof. Only (2.5.15) requires justification. We saw that all edges from some $\mathfrak{P}_\alpha$ to $\overline{A}$ must go to β vertices and these were not overfilled with respect to the $\tilde{\kappa}(v)$.

Among the simplest consequences of the condition (2.5.2) is that one must have

$$\kappa(S_i) \equiv 0 \pmod 2 \tag{2.5.16}$$

for every connected component S_i of S.

CHAPTER 3. APPLICATIONS

3.1. **Expressions for the deficiencies.** We shall turn to the applications of the preceding theory and begin by deriving certain convenient formulas for the deficiencies. As our starting point we take the expression (1.2.26)

$$\delta(A) + \rho(O_1, \tilde{O}_1) = \rho(I_1, \tilde{I}_1) - \tilde{\kappa}(I_1) - \kappa(\tilde{I}_1). \tag{3.1.1}$$

Here we introduce for short the quantities λ and $\tilde{\lambda}$ defined by

$$\tilde{\kappa}(I_1) = \tilde{\lambda}\cdot\rho(I_1), \qquad \kappa(\tilde{I}_1) = \lambda\cdot\rho(\tilde{I}_1) \tag{3.1.2}$$

and find for the right hand side in (3.1.1)

$$\rho(I_1, \tilde{I}_1) - \tilde{\lambda}\cdot\rho(I_1) - \lambda\cdot\rho(\tilde{I}_1). \tag{3.1.3}$$

Next we make use of the expressions

$$\rho(I_1) = \rho(I_1, S), \qquad S = I_1 + O_1 + \tilde{I}_1 + \tilde{O}_1$$

so that

$$\rho(I_1) = \rho(I_1, I_1) + \rho(I_1, O_1) + \rho(I_1, \tilde{I}_1) + \rho(I_1, \tilde{O}_1)$$

and similarly for $\rho(\tilde{I}_1)$. In addition we write

$$\rho(I_1, \tilde{O}_1) = \rho(A, \tilde{O}_1) - \rho(O_1, \tilde{O}_1),$$
$$\rho(\tilde{I}_1, O_1) = \rho(\overline{A}, O_1) - \rho(O_1, \tilde{O}_1).$$

When all this is substituted in (3.1.3) and in turn in (3.1.1) we obtain after some rearrangements

$$\begin{aligned}\delta(A) = (1 - \lambda - \tilde{\lambda})[\rho(I_1, \tilde{I}_1) - \rho(O_1, \tilde{O}_1)] - [\tilde{\lambda}\cdot\rho(I_1, I_1) + \lambda\cdot\rho(\tilde{I}_1, \tilde{I}_1)]\\ - [\tilde{\lambda}\cdot\rho(I_1, O_1) + \lambda\cdot\rho(\overline{A}, O_1)] - [\tilde{\lambda}\cdot\rho(A, \tilde{O}_1) + \lambda\cdot\rho(\tilde{I}_1, \tilde{O}_1)].\end{aligned} \tag{3.1.4}$$

To derive an expression for the effective deficiency we first make use of the decomposition (1.3.2) for O_1 to rewrite the third bracket in (3.1.4) as follows:

$$-\sum_s [\tilde{\lambda}\cdot\rho(I_1, C_s) + \lambda\cdot\rho(\overline{A}, C_s)]. \tag{3.1.5}$$

Similarly we introduce in the fourth bracket of (3.1.4) the sum (1.3.7) of the outer overfilled components $\tilde{C}_t$ and obtain

$$-\sum_t [\tilde{\lambda}\cdot\rho(A, \tilde{C}_t) + \lambda\cdot\rho(\tilde{I}_1, \tilde{C}_t)]. \tag{3.1.6}$$

The effective deficiency $\delta_e(A)$ is derived from (3.1.4) by adding the restrictions

$$r = r^{(i)} + r^{(O)}$$

to it. Evidently we can write

$$r^{(i)} = \sum_s \epsilon_s, \qquad r^{(O)} = \sum_t \tilde{\epsilon}_t \tag{3.1.7}$$

when we define

$$\epsilon_s = 0 \text{ when } C_s \text{ is even,}$$
$$\epsilon_s = 1 \text{ when } C_s \text{ is odd}$$

and analogously for $\tilde{\epsilon}_t$. The sums (3.1.7) can then be included in the sums (3.1.5) and (3.1.6) and so we find

$$\begin{aligned}\delta_e(A) &= (1 - \lambda - \tilde{\lambda})\left[\rho(I_1, \tilde{I}_1) - \rho(O_1, \tilde{O}_1)\right] \\ &\quad - \tilde{\lambda}\cdot\rho(I_1, I_1) - \lambda\cdot\rho(\tilde{I}_1, \tilde{I}_1) - \sum_1 - \sum_2\end{aligned} \tag{3.1.8}$$

where we have put

$$\begin{aligned}\sum_1 &= \sum_s [\tilde{\lambda}\cdot\rho(I_1, C_s) + \lambda\cdot\rho(\overline{A}, C_s) - \epsilon_s], \\ \sum_2 &= \sum_t [\lambda\cdot\rho(A, \tilde{C}_t) + \lambda\cdot\rho(\tilde{I}_1, \tilde{C}_t) - \tilde{\epsilon}_t].\end{aligned} \tag{3.1.9}$$

We shall from now on suppose that the multiplicities for the graph has *constant proportions*, that is, the quotients

$$\lambda = \frac{\kappa(v)}{\rho(v)}, \qquad \tilde{\lambda} = \frac{\tilde{\kappa}(v)}{\rho(v)}, \qquad \lambda + \tilde{\lambda} = 1 \tag{3.1.10}$$

have fixed values independent of the vertex v. In this case one also has

$$\lambda = \frac{\kappa(A)}{\rho(A)}, \qquad \tilde{\lambda} = \frac{\tilde{\kappa}(A)}{\rho(A)}$$

independent of the set A. These constants then coincide with those defined in (3.1.2).

Under these conditions the expression (3.1.8) for the effective deficiency simplifies since the first term vanishes. Thus one sees that a set can only have a positive effective deficiency if one or more of the terms in the sums (3.1.9) become negative. This is only possible for terms corresponding to an odd inner overfilled component $C_s = C$ or an odd outer component $\tilde{C}_t = \tilde{C}$. Let us examine first when an inequality

$$\tilde{\lambda}\cdot\rho(I_1, C) + \lambda\cdot\rho(\overline{A}, C) < 1 \tag{3.1.11}$$

can hold. We select the notation such that

$$\lambda \leqq 1/2, \qquad \tilde{\lambda} \geqq 1/2. \tag{3.1.12}$$

THEOREM 3.1.1. *The inequality* (3.1.11) *can hold only under one of the two alternatives*

(3.1.13) (1) $$\rho(\overline{A}, C) < 1/\lambda$$

and C is an isolated component of $G(A)$ with

$$\kappa(C) \equiv 1 \pmod 2 \tag{3.1.14}$$

(2) *C is a peninsula whose bridge lies in A and*

(3.1.15) $$\kappa(C) \equiv 0 \pmod 2.$$

Proof. We notice that the sum

(3.1.16) $$\rho(C, \overline{C}) = \rho(I_1, C) + \rho(\overline{A}, C)$$

is the total number of edges connecting C with the rest of the graph. There must be at least one such edge since C is an odd overfilled component of A and (2.5.16) is supposed to hold. According to (3.1.12) there are only the two alternatives

$$\rho(I_1, C) = 0, \qquad \rho(I_1, C) = 1.$$

The first leads to case (1). The second yields

$$\rho(\overline{A}, C) = 0$$

from (3.1.11), hence C is a peninsula according to (3.1.16). The congruence (3.1.15) must hold since C is an odd overfilled component. We leave to the reader the proof of the analogous result for the outer components:

THEOREM 3.1.2. *The condition*

(3.1.17) $$\tilde{\lambda}\cdot\rho(A, \tilde{C}) + \lambda\cdot\rho(\tilde{I}_1, \tilde{C}) < 1$$

can hold only under one of the alternatives:

(1) *All edges from $\tilde{C}$ be in $\overline{A}$ and*

(3.1.18) $$\rho(A, \tilde{C}) = 0, \qquad \rho(\tilde{C}, \tilde{I}_1) < 1/\lambda,$$

(3.1.19) $$\kappa(\tilde{C}) \equiv 1 \pmod 2.$$

(2) *$\tilde{C}$ is a peninsula whose bridge does not lie in $\overline{A}$ and*

(3.1.20) $$\kappa(\tilde{C}) \equiv 0 \pmod 2.$$

3.2. **Decomposition theorems for graphs.** Let n be the greatest common divisor of the local degrees of the graph so that

(3.2.1) $$\rho(v) = n\cdot\rho_1(v).$$

We can then select a set of multiplicities

(3.2.2) $$\kappa(v) = m\cdot\rho_1(v), \qquad \tilde{\kappa}(v) = \tilde{m}\cdot\rho_1(v)$$

with the constant proportions

(3.2.3) $$\lambda = \frac{m}{n} \leqq \frac{1}{2}, \qquad \tilde{\lambda} = \frac{\tilde{m}}{n} \geqq \frac{1}{2}, \qquad \lambda + \tilde{\lambda} = 1.$$

We shall prove:

THEOREM 3.2.1. *Let G be a graph without peninsulas and with local degrees and multiplicities defined by* (3.2.1) *and* (3.2.2). *If the graph has the property that no set C with*

$$(3.2.4) \qquad \kappa(C) \equiv 1 \pmod 2$$

can be a peninsula of rank

$$(3.2.5) \qquad \rho(C, \overline{C}) < \frac{n}{m}$$

then there exists a subgraph corresponding to the given multiplicities.

Proof. Since there are no peninsulas of rank 1 the cases (2) are excluded in Theorems 3.1.1 and 3.1.2 while case (1) is eliminated by the special condition of the theorem. Thus no subset A of S has a positive effective deficiency. It should be noted that the conditions

$$(3.2.6) \qquad \kappa(S_i) \equiv 0 \pmod 2$$

for the maximal connected components S_i of S are also consequences of our condition if one agrees to call them *peninsulas of rank* 0.

Theorem 3.2.1 applies in particular to graphs whose local degrees are even since they have no peninsulas by Theorem 1.1.1. A special case of some interest occurs for the multiplicities

$$(3.2.7) \qquad \kappa(v) = \tilde{\kappa}(v) = \frac{1}{2}\cdot\rho(v), \qquad \lambda = \tilde{\lambda} = \frac{1}{2}\cdot$$

One verifies the special condition

$$\delta(A) = \delta(\overline{A}) = \bar{\delta}(A) = \bar{\delta}(\overline{A})$$

satisfied for any subset A and also for its effective deficiencies. Theorem **3.2.1** gives:

THEOREM 3.2.2. *A connected graph with even local degrees decomposes into two subgraphs with the same degrees if and only if*

$$(3.2.8) \qquad \rho(S) \equiv 0 \pmod 4.$$

Our next application is:

THEOREM 3.2.3. *Let G be a graph with the local degrees*

$$\rho(v) = 2n'\cdot\rho_1(v).$$

Then G decomposes into a direct sum of n' subgraphs H each with the degrees

$$\rho_H(v) = 2\cdot\rho_1(v).$$

Proof. It is sufficient to prove that there exists one such subgraph H. But G has no peninsulas and the condition (3.2.4) cannot be fulfilled for even multiplicities.

We show further:

THEOREM 3.2.4. *Let G be a graph with even degrees* (3.2.1). *If G has no* *peninsulas C of rank* 0 *or* 2 *satisfying* (3.2.4) *then a subgraph exists correspond ing to the multiplicities* (3.2.2) *whenever*

$$n/4 \leqq m \leqq 3n/4.$$

Proof. We notice that since the $\rho(v)$ are even also the number

$$\rho(C, \overline{C}) \equiv \rho(C) + \rho(C, \overline{C}) \equiv 0 \pmod 2$$

is even and so in (3.2.5) we can replace the upper bound by the greatest even integer below n/m. Under our restrictions we have

$$n/m \leqq 4, \qquad \rho(C, \overline{C}) \leqq 2$$

so the theorem follows from Theorem 3.2.1.

Finally, let us apply Theorem 3.2.1 to the case where both numbers n and m in (3.2.1) and (3.2.2) are odd. We see that for any set A one must have

$$\rho(A) \equiv \kappa(A) \pmod 2.$$

This excludes the cases (2) in Theorems 3.1.1 and 3.1.2, since the conditions (3.1.15) and (3.1.20) are not compatible with (1.1.8). From (3.2.4) one finds

$$\rho(C, \overline{C}) \equiv 1 \pmod 2$$

so that the bound in (3.2.5) may be replaced by greatest odd integer below n/m. For $n/m = 3$ this gives the bound $k = 1$ and so we may state:

THEOREM 3.2.5. *Let G be a graph without peninsulas having degrees* (3.3.1) *and multiplicities* (3.3.2) *where n is odd. Then G has a subgraph corresponding to each odd m satisfying the condition*

$$n/3 \leqq m \leqq n.$$

We notice that if G has a subgraph H for some odd m then it has one for every greater odd m, since Theorem 3.2.3 applies to $\overline{H}$.

Let us point out briefly that the preceding theory contains as applications all the known theorems about regular graphs and subgraphs. In these cases we have the constant values

$$\rho(v) = n, \qquad \kappa(v) = m, \qquad \tilde{\kappa}(v) = \tilde{m}, \qquad \rho_1(v) = 1.$$

Theorem 3.2.3 gives the result of *Petersen* that every regular graph of even degree $n = 2 \cdot n_1$ decomposes into n_1 graphs of degree 2. Theorem 3.2.2 is also known for regular graphs.

From Theorem 3.2.4 we obtain the result of *Gallai*: A regular connected graph of even degree n has a regular subgraph of odd degree m provided there are no peninsulas C with an odd number of vertices and rank

$$\rho(C, \overline{C}) \leqq k$$

where k is the greatest even integer below n/m. There exist regular subgraphs for all odd degrees m satisfying $m \geqq n/4$ provided there are no peninsulas of rank 2.

For graphs of odd degrees we have the important result of *Baebler*, which includes the well known result of *Petersen* about graphs of third degree: Let G be regular of odd degree and $m < n/2$ some odd integer. Then G has a regular subgraph of degree m provided there are no peninsulas C with an odd number of vertices and rank

$$\rho(C, \overline{C}) \leqq k$$

where k is the greatest odd integer below n/m. If G has no peninsulas there exists a regular subgraph of odd degree m whenever

$$m \geqq n/3.$$

This latter result can also be expressed that G contains the direct sum of at least $[n/3]$ subgraphs of degree 2. We shall verify through an example that in general this is the best bound possible. Some details in the following construction are left to the reader.

We select an odd number $n = 2n_1 - 1$ and construct first a graph C with $2n_1 + 1$ vertices

$$c_1;\ a_1,\ a_2 \cdots a_{n_1};\ b_1, \cdots, b_{n_1}.$$

We join these vertices by all possible edges excepting only

$$(c_1, a_1), \quad (c_1, b_1), \quad (a_i, b_i) \qquad (i = 1, 2, \cdots n_1).$$

One verifies that at all vertices

$$a_i, b_i \qquad (i = 2, \cdots n_1)$$

there are n edges, while there are only $n-1$ edges at the vertices a_1, b_1, c_1.

In the next step we construct n replicas $C^{(k)}$ of C. We connect these to a new graph by adding 3 outside vertices (a_0, b_0, c_0) and the edges

$$(a_0, a_1^{(k)}), \quad (b_0, b_1^{(k)}), \quad (c_0, c_1^{(k)}) \qquad (k = 1, 2, \cdots n).$$

The resulting graph is seen to be regular of degree n, connected and without peninsulas. For any multiplicity $m < n/2$ the deficiency of the set

$$A = \sum C^{(k)}, \qquad \overline{A} = \{a_0, b_0, c_0\}$$

is seen to be $-3m$. When m is odd the $C^{(k)}$ are the odd overfilled components of A while $\overline{A}$ has none. The restriction of A is therefore n so that

$$\delta_e(A) = n - 3m$$

is the effective deficiency. Since it is positive whenever $m < n/3$ there are no subgraphs of these odd degrees.

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