

THE INDEPENDENCE OF CERTAIN DISTRIBUTIVE LAWS IN BOOLEAN ALGEBRAS

BY
DANA SCOTT

Let α be a regular cardinal number. We shall prove the following:

THEOREM. *There is a complete Boolean algebra that is (β, γ) -distributive for every $\beta < \alpha$ and every cardinal γ , but is not (α, α) -distributive⁽¹⁾.*

The method of proof is to construct the desired algebra as the algebra of all regular open sets of a suitable topological space. To this end we note first

LEMMA 1. *There is a 0-dimensional Hausdorff space \mathfrak{X} such that*

(i) *the class of open sets of \mathfrak{X} is closed under the formation of β -termed intersections for every $\beta < \alpha$;*

(ii) *the class of nowhere-dense sets of \mathfrak{X} is closed under the formation of β -termed unions for every $\beta < \alpha$;*

(iii) *there is an $\alpha \times 2$ -termed sequence C of nonempty open-closed sets of \mathfrak{X} such that*

(iii₁) $C_{\xi 0} \cup C_{\xi 1} = \mathfrak{X}$ for $\xi < \alpha$; and

(iii₂) $\bigcap_{\xi < \alpha} C_{\xi f(\xi)}$ is nowhere-dense for $f \in 2^\alpha$.

Proof. Let the set of points of the space \mathfrak{X} be the set of all subsets of α . (Notice that α is considered as an ordinal number, and that each ordinal $\beta < \alpha$ is also a point of \mathfrak{X} .) If x and y are two subsets of α , denote by $[x, y]$ the interval of all sets z such that $x \subseteq z \subseteq y$. As a basis for the open sets of \mathfrak{X} take the collection of all intervals $[x, y]$ such that $x \cup (\alpha - y) \subseteq \beta$ for some $\beta < \alpha$. An empty interval is also included in the basis. Suppose that $\beta < \alpha$ and $\{[x_\xi, y_\xi] : \xi < \beta\}$ is a sequence of basic open sets where $x_\xi \cup (\alpha - y_\xi) \subseteq \gamma_\xi < \alpha$ for $\xi < \beta$. We have

$$\bigcap_{\xi < \beta} [x_\xi, y_\xi] = \left[\bigcup_{\xi < \beta} x_\xi, \bigcap_{\xi < \beta} y_\xi \right]$$

and

$$\bigcup_{\xi < \beta} x_\xi \cup \left(\alpha - \bigcap_{\xi < \beta} y_\xi \right) \subseteq \bigcup_{\xi < \beta} \gamma_\xi.$$

From the regularity of α it follows that $\bigcup_{\xi < \beta} \gamma_\xi < \alpha$; thus the intersection of the sequence of basic open sets is again a basic open set. (Notice that if the

Received by the editors April 4, 1956.

⁽¹⁾ For terminology see Smith-Tarski [3]. This theorem has also been proved in a weaker form in Smith [2].

sequence is a decreasing sequence of nonempty sets, then the intersection is nonempty.) An *open* set then is a union of basic open sets. It is obvious that there are no isolated points in the space \mathfrak{X} and that \mathfrak{X} is Hausdorff. Also clear is the proof that every basic open set is closed, showing that \mathfrak{X} is 0-dimensional. An easy computation using the set-theoretical distributive law and the fact just established about the intersections of basic open sets yields finally a proof of (i).

Let \emptyset be the collection of all nonempty basic open sets. Let $\beta < \alpha$ and N be a β -termed sequence of nowhere-dense sets. To show that $N^* = \bigcup_{\xi < \beta} N_\xi$ is nowhere dense, it suffices to show that for every $Y \in \emptyset$ there is a $Z \in \emptyset$ such that $Z \subseteq Y$ and $Z \cap N^* = \emptyset$. By the axiom of choice let \mathfrak{C} be a function that chooses a set from every nonempty family of subsets of our space. Let $Y \in \emptyset$ and define by recursion a β -termed sequence G such that for $\xi < \beta$

$$G_\xi = \mathfrak{C} \left\{ Z : Z \in \emptyset \text{ and } Z \subseteq Y \cap \bigcap_{\eta < \xi} G_\eta \text{ and } Z \cap N_\xi = \emptyset \right\}.$$

We proceed by induction to show that this sequence is well-defined. Thus suppose that G_η is well-defined for all $\eta < \xi$ where $\xi < \beta$. It is clear that the sequence is decreasing up to this point and hence $Y \cap \bigcap_{\eta < \xi} G_\eta \in \emptyset$. The fact that N_ξ is nowhere-dense implies that there is a $Z \in \emptyset$ such that $Z \subseteq Y \cap \bigcap_{\eta < \xi} G_\eta$ and $Z \cap N_\xi = \emptyset$. It follows at once that G_ξ is well-defined. It is obvious now that the whole sequence G is decreasing, and hence $G^* = \bigcap_{\xi < \beta} G_\xi \in \emptyset$ and $G^* \subseteq Y$ and $G^* \cap N^* = \emptyset$. This argument shows that N^* is nowhere-dense and establishes property (ii).

To prove (iii) we have only to let

$$C_{\xi 0} = [\{\xi\}, \alpha]$$

and

$$C_{\xi 1} = [0, \alpha - \{\xi\}] \qquad \text{for } \xi < \alpha.$$

Since these sets are basic open sets they are also closed. Formula (iii₁) is obvious and (iii₂) is a consequence of the simple fact that

$$\bigcap_{\xi < \alpha} C_{\xi f(\xi)} = \{f^{-1}(0)\} \qquad \text{for } f \in 2^\alpha.$$

This completes the proof of Lemma 1.

If $\alpha = \omega$ our space is nothing more than the Cantor Discontinuum. For larger α the space is compact only in the sense that every open cover can be reduced to one of power less than α . The proof of (ii) above could easily be modified to show that no nonempty open set is an α -termed union of nowhere-dense sets—the analogue of the Baire Category Theorem. A rather different construction of the space has been given by Sikorski in [1] (see especially p. 129 where the space is called \mathfrak{D}_μ where $\alpha = \omega_\mu$.) Our construction here

would seem neater since there is no need of any non-Archimedean metric; however, the particular form of the space \mathfrak{X} is of no importance for the present purpose.

Let \mathfrak{R} be the algebra of all regular open sets of the space \mathfrak{X} . That \mathfrak{R} is a complete Boolean algebra is well-known⁽²⁾. The Boolean operations of \mathfrak{R} will be denoted by the usual symbols $+$, \cdot , \sum , \prod . The unit element of \mathfrak{R} is \mathfrak{X} itself, while the zero element is just the empty set O . The next lemma, which we state without proof, relates the Boolean operations in \mathfrak{R} to the set-theoretical operations in \mathfrak{X} . We use the symbols in X and $\text{cl } X$ to denote the interior and closure of the set X .

LEMMA 2. *If β is any ordinal and X is a β -termed sequence of regular open sets (i.e. elements of \mathfrak{R}), then*

- (i) $\sum_{\xi < \beta} X_\xi = \text{in cl } \bigcup_{\xi < \beta} X_\xi$;
- (ii) $\prod_{\xi < \beta} X_\xi = \text{in cl } \bigcap_{\xi < \beta} X_\xi$;
- (iii) $\sum_{\xi < \beta} X_\xi - \bigcup_{\xi < \beta} X_\xi$ is nowhere-dense;
- (iv) $\bigcap_{\xi < \beta} X_\xi - \prod_{\xi < \beta} X_\xi$ is nowhere-dense.

LEMMA 3. *\mathfrak{R} is (β, γ) -distributive for every $\beta < \alpha$ and every γ .*

Proof. Let $\beta < \alpha$ and let γ be any ordinal. Given a $\beta \times \gamma$ -termed sequence X of regular open sets and an open set A satisfying the formula

$$(1) \quad \sum_{\eta < \gamma} X_{\xi\eta} = A \neq O \quad \text{for each } \xi < \beta,$$

then we must show that there is a function $f \in \gamma^\beta$ such that⁽³⁾

$$(2) \quad \prod_{\xi < \beta} X_{\xi f(\xi)} \neq O.$$

Thus, by way of contradiction, assume that for all functions $f \in \gamma^\beta$

$$(3) \quad \prod_{\xi < \beta} X_{\xi f(\xi)} = O.$$

By virtue of Lemma 2 (iv), formula (3) implies

$$(4) \quad \bigcap_{\xi < \beta} X_{\xi f(\xi)} \text{ is nowhere-dense.}$$

Since $\beta < \alpha$ and each set $X_{\xi\eta}$ is open, we have by Lemma 1 (i)

$$(5) \quad \bigcap_{\xi < \beta} X_{\xi f(\xi)} \text{ is open.}$$

Formulas (4) and (5) yield at once

⁽²⁾ See for example Tarski [4]. A subset of a topological space is called a *regular open set* if it is equal to the interior of its closure.

⁽³⁾ For the equivalence of this form of the distributive law to other forms see Smith-Tarski [3, Theorem 2.2].

$$(6) \quad \bigcap_{\xi < \beta} X_{\xi f(\xi)} = 0 \quad \text{for every } f \in \gamma^\beta.$$

Hence we can derive from (6) the formula

$$(7) \quad \bigcup_{f \in \gamma^\beta} \bigcap_{\xi < \beta} X_{\xi f(\xi)} = 0.$$

In view of the general set-theoretical distributive law, (7) implies

$$(8) \quad \bigcap_{\xi < \beta} \bigcup_{\eta < \gamma} X_{\xi \eta} = 0.$$

From formula (8) we derive

$$(9) \quad A = A - \bigcap_{\xi < \beta} \bigcup_{\eta < \gamma} X_{\xi \eta} = \bigcup_{\xi < \beta} \left(A - \bigcup_{\eta < \gamma} X_{\xi \eta} \right).$$

Now by Lemma 2 (iii) and formula (1) we have for each $\xi < \beta$

$$(10) \quad A - \bigcup_{\eta < \gamma} X_{\xi \eta} \text{ is nowhere-dense.}$$

By virtue of Lemma 1 (ii), formulas (9) and (10) imply that the set A is nowhere-dense, which contradicts the assumption that A is a nonempty open set. The proof of Lemma 3 is thus complete.

LEMMA 4. \mathfrak{R} is not (α, α) -distributive.

Proof. Clearly the terms of the sequence C of Lemma 1 (iii) are regular open sets. In terms of the Boolean operations of \mathfrak{R} conditions (iii₁) and (iii₂) may be written as

$$(iii'_1) \quad C_{\xi 0} + C_{\xi 1} = \mathfrak{X} \quad \text{for } \xi < \alpha;$$

$$(iii'_2) \quad \prod_{\xi < \alpha} C_{\xi f(\xi)} = 0 \quad \text{for } f \in 2^\alpha.$$

Whence we see that the sequence C itself offers a counterexample to the $(\alpha, 2)$ -distributive law.

Our theorem is now a direct consequence of Lemmas 3 and 4.

BIBLIOGRAPHY

1. R. Sikorski, *Remarks on some topological spaces of high power*, Fund. Math. vol. 37 (1950) pp. 123–136.
2. E. C. Smith, *A distributivity condition for Boolean algebras*, Ann. of Math. vol. 64 (1956) pp. 551–561.
3. E. C. Smith and A. Tarski, *Higher degrees of distributivity and completeness in Boolean algebras*, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 230–257.
4. A. Tarski, *Les fondements de la géométrie des corps*, Księga pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego Lwów vol. 9 (1927) pp. 29–33.

PRINCETON UNIVERSITY,
PRINCETON, N. J.