THE INDEPENDENCE OF CERTAIN DISTRIBUTIVE LAWS IN BOOLEAN ALGEBRAS

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Let α be a regular cardinal number. We shall prove the following:

THEOREM. *There is a complete Boolean algebra that is* (β, γ)*-distributive for every* $\beta < \alpha$ *and every cardinal* γ*, but is not* (α, α)*-distributive*(1).

The method of proof is to construct the desired algebra as the algebra of all regular open sets of a suitable topological space. To this end we note first

LEMMA 1. *There is a* 0*-dimensional Hausdorff space* $\mathfrak{X}$ *such that*

(i) *the class of open sets of* $\mathfrak{X}$ *is closed under the formation of* β*-termed intersections for every* $\beta < \alpha$;

(ii) *the class of nowhere-dense sets of* $\mathfrak{X}$ *is closed under the formation of* β*-termed unions for every* $\beta < \alpha$;

(iii) *there is an* $\alpha \times 2$*-termed sequence* C *of nonempty open-closed sets of* $\mathfrak{X}$ *such that*

(iii_1) $C_{\xi 0} \cup C_{\xi 1} = \mathfrak{X}$ *for* $\xi < \alpha$; *and*

(iii_2) $\bigcap_{\xi < \alpha} C_{\xi f(\xi)}$ *is nowhere-dense for* $f \in 2^\alpha$.

Proof. Let the set of points of the space $\mathfrak{X}$ be the set of all subsets of α. (Notice that α is considered as an ordinal number, and that each ordinal is the set of all smaller ordinals. Thus, for example, every ordinal $\beta < \alpha$ is also a point of $\mathfrak{X}$.) If x and y are two subsets of α, denote by $[x, y]$ the interval of all sets z such that $x \subseteq z \subseteq y$. As a basis for the open sets of $\mathfrak{X}$ take the collection of all intervals $[x, y]$ such that $x \cup (\alpha - y) \subseteq \beta$ for some $\beta < \alpha$. An empty interval is also included in the basis. Suppose that $\beta < \alpha$ and $\{[x_\xi, y_\xi] : \xi < \beta\}$ is a sequence of basic open sets where $x_\xi \cup (\alpha - y_\xi) \subseteq \gamma_\xi < \alpha$ for $\xi < \beta$. We have

$$\bigcap_{\xi<\beta} [x_\xi, y_\xi] = \left[\bigcup_{\xi<\beta} x_\xi, \bigcap_{\xi<\beta} y_\xi \right]$$

and

$$\bigcup_{\xi<\beta} x_\xi \cup \left(\alpha - \bigcap_{\xi<\beta} y_\xi \right) \subseteq \bigcup_{\xi<\beta} \gamma_\xi.$$

From the regularity of α it follows that $\bigcup_{\xi<\beta} \gamma_\xi < \alpha$; thus the intersection of the sequence of basic open sets is again a basic open set. (Notice that if the

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(1) For terminology see Smith-Tarski [3]. This theorem has also been proved in a weaker form in Smith [2].

sequence is a decreasing sequence of nonempty sets, then the intersection is nonempty.) An *open* set then is a union of basic open sets. It is obvious that there are no isolated points in the space $\mathfrak{X}$ and that $\mathfrak{X}$ is Hausdorff. Also clear is the proof that every basic open set is closed, showing that $\mathfrak{X}$ is 0-dimensional. An easy computation using the set-theoretical distributive law and the fact just established about the intersections of basic open sets yields finally a proof of (i).

Let $\mathfrak{O}$ be the collection of all nonempty basic open sets. Let $\beta<\alpha$ and N be a β-termed sequence of nowhere-dense sets. To show that $N^*=\bigcup_{\xi<\beta} N_\xi$ is nowhere dense, it suffices to show that for every $Y\in\mathfrak{O}$ there is a $Z\in\mathfrak{O}$ such that $Z\subseteq Y$ and $Z\cap N^*=O$. By the axiom of choice let $\mathfrak{C}$ be a function that chooses a set from every nonempty family of subsets of our space. Let $Y\in\mathfrak{O}$ and define by recursion a β-termed sequence G such that for $\xi<\beta$

$$G_\xi = \mathfrak{C}\left\{Z : Z \in \mathfrak{O} \text{ and } Z \subseteq Y \cap \bigcap_{\eta<\xi} G_\eta \text{ and } Z \cap N_\xi = O\right\}.$$

We proceed by induction to show that this sequence is well-defined. Thus suppose that G_η is well-defined for all $\eta<\xi$ where $\xi<\beta$. It is clear that the sequence is decreasing up to this point and hence $Y\cap\bigcap_{\eta<\xi} G_\eta\in\mathfrak{O}$. The fact that N_ξ is nowhere-dense implies that there is a $Z\in\mathfrak{O}$ such that $Z\subseteq Y\cap\bigcap_{\eta<\xi} G_\eta$ and $Z\cap N_\xi=O$. It follows at once that G_ξ is well-defined. It is obvious now that the whole sequence G is decreasing, and hence $G^*=\bigcap_{\xi<\beta} G_\xi\in\mathfrak{O}$ and $G^*\subseteq Y$ and $G^*\cap N^*=O$. This argument shows that N^* is nowhere-dense and establishes property (ii).

To prove (iii) we have only to let

$$C_{\xi 0} = [\{\xi\}, \alpha]$$

and

$$C_{\xi 1} = [O, \alpha - \{\xi\}] \qquad \text{for } \xi < \alpha.$$

Since these sets are basic open sets they are also closed. Formula (iii$_1$) is obvious and (iii$_2$) is a consequence of the simple fact that

$$\bigcap_{\xi<\alpha} C_{\xi f(\xi)} = \{f^{-1}(0)\} \qquad \text{for } f \in 2^\alpha.$$

This completes the proof of Lemma 1.

If $\alpha=\omega$ our space is nothing more than the Cantor Discontinuum. For larger α the space is compact only in the sense that every open cover can be reduced to one of power less than α. The proof of (ii) above could easily be modified to show that no nonempty open set is an α-termed union of nowhere-dense sets—the analogue of the Baire Category Theorem. A rather different construction of the space has been given by Sikorski in [1] (see especially p. 129 where the space is called $\mathfrak{D}_\mu$ where $\alpha=\omega_\mu$.) Our construction here would seem neater since there is no need of any non-Archimedean metric; however, the particular form of the space $\mathfrak{X}$ is of no importance for the present purpose.

Let $\mathfrak{R}$ be the algebra of all regular open sets of the space $\mathfrak{X}$. That $\mathfrak{R}$ is a complete Boolean algebra is well-known[2]. The Boolean operations of $\mathfrak{R}$ will be denoted by the usual symbols $+$, $\cdot$, $\sum$, $\prod$. The unit element of $\mathfrak{R}$ is $\mathfrak{X}$ itself, while the zero element is just the empty set O. The next lemma, which we state without proof, relates the Boolean operations in $\mathfrak{R}$ to the set-theoretical operations in $\mathfrak{X}$. We use the symbols in X and cl X to denote the interior and closure of the set X.

LEMMA 2. *If β is any ordinal and X is a β-termed sequence of regular open sets (i.e. elements of $\mathfrak{R}$), then*
(i) $\sum_{\xi<\beta} X_\xi = \mathrm{in\ cl} \bigcup_{\xi<\beta} X_\xi$;
(ii) $\prod_{\xi<\beta} X_\xi = \mathrm{in\ cl} \bigcap_{\xi<\beta} X_\xi$;
(iii) $\sum_{\xi<\beta} X_\xi - \bigcup_{\xi<\beta} X_\xi$ *is nowhere-dense*;
(iv) $\bigcap_{\xi<\beta} X_\xi - \prod_{\xi<\beta} X_\xi$ *is nowhere-dense.*

LEMMA 3. *$\mathfrak{R}$ is (β, γ)-distributive for every $\beta<\alpha$ and every γ.*

Proof. Let $\beta<\alpha$ and let γ be any ordinal. Given a $\beta\times\gamma$-termed sequence X of regular open sets and an open set A satisfying the formula

(1) $$\sum_{\eta<\gamma} X_{\xi\eta} = A \neq O \qquad \text{for each } \xi < \beta,$$

then we must show that there is a function $f\in\gamma^\beta$ such that[3]

(2) $$\prod_{\xi<\beta} X_{\xi f(\xi)} \neq O.$$

Thus, by way of contradiction, assume that for all functions $f\in\gamma^\beta$

(3) $$\prod_{\xi<\beta} X_{\xi f(\xi)} = O.$$

By virtue of Lemma 2 (iv), formula (3) implies

(4) $$\bigcap_{\xi<\beta} X_{\xi f(\xi)} \text{ is nowhere-dense.}$$

Since $\beta<\alpha$ and each set $X_{\xi\eta}$ is open, we have by Lemma 1 (i)

(5) $$\bigcap_{\xi<\beta} X_{\xi f(\xi)} \text{ is open.}$$

Formulas (4) and (5) yield at once

(2) See for example Tarski [4]. A subset of a topological space is called a *regular open set* if it is equal to the interior of its closure.

(3) For the equivalence of this form of the distributive law to other forms see Smith-Tarski [3, Theorem 2.2].

$$\bigcap_{\xi<\beta} X_{\xi f(\xi)} = O \qquad \text{for every } f \in \gamma^\beta. \tag{6}$$

Hence we can derive from (6) the formula

$$\bigcup_{f\in\gamma^\beta} \bigcap_{\xi<\beta} X_{\xi f(\xi)} = O. \tag{7}$$

In view of the general set-theoretical distributive law, (7) implies

$$\bigcap_{\xi<\beta} \bigcup_{\eta<\gamma} X_{\xi\eta} = O. \tag{8}$$

From formula (8) we derive

$$A = A - \bigcap_{\xi<\beta} \bigcup_{\eta<\gamma} X_{\xi\eta} = \bigcup_{\xi<\beta}\left(A - \bigcup_{\eta<\gamma} X_{\xi\eta}\right). \tag{9}$$

Now by Lemma 2 (iii) and formula (1) we have for each $\xi<\beta$

$$A - \bigcup_{\eta<\gamma} X_{\xi\eta} \text{ is nowhere-dense.} \tag{10}$$

By virtue of Lemma 1 (ii), formulas (9) and (10) imply that the set A is nowhere-dense, which contradicts the assumption that A is a nonempty open set. The proof of Lemma 3 is thus complete.

LEMMA 4. *$\mathfrak{R}$ is not (α, α)-distributive.*

Proof. Clearly the terms of the sequence C of Lemma 1 (iii) are regular open sets. In terms of the Boolean operations of $\mathfrak{R}$ conditions (iii_1) and (iii_2) may be written as

$$C_{\xi 0} + C_{\xi 1} = \mathfrak{X} \qquad \text{for } \xi < \alpha; \tag{iii_1'}$$

$$\prod_{\xi<\alpha} C_{\xi f(\xi)} = O \qquad \text{for } f \in 2^\alpha. \tag{iii_2'}$$

Whence we see that the sequence C itself offers a counterexample to the $(\alpha, 2)$-distributive law.

Our theorem is now a direct consequence of Lemmas 3 and 4.

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