ADDENDA TO THE PAPER *ON A PROBLEM OF MIHLIN*(1)

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The proof of the formula (2.8) in the paper quoted is incomplete. Though meanwhile we have proved and generalized Theorem 1 by a different method [1], the initial approach still seems to be indispensable in establishing Theorem 2. The needed addition is not difficult but perhaps not obvious and so we give it here.

In (2.1) the functions $Y_n(z')$ depend on x, that is $Y_n(z') = Y_n(x, z')$ but the proof of (2.8) as it stands in the paper applies only to the case when $Y_n(x, z')$ is independent of x. We indicate now how to complete the argument. We write $Y_n(x, z') = \sum_m \alpha_m(x) Y_{nm}(z')$ where $Y_{nm}(z')$ is a complete orthonormal system of (fixed) spherical harmonics of degree n and $\sum |\alpha_m(x)|^2 = 1$. Then $\tilde{f}_{n\epsilon}(x) = \sum_m \alpha_m(x)\tilde{f}_{nm\epsilon}(x)$ where $\tilde{f}_{nm\epsilon}$ is given by (2.5) with $Y_{nm}(z')$ in the integral instead of $Y_n(z')$, and by Schwarz's inequality it follows that $|\tilde{f}_{n\epsilon}(x)|^2 \leqq \sum_m |\tilde{f}_{nm\epsilon}(x)|^2$. Integrating and applying Plancherel's theorem to the righthand side we obtain

$$\|\tilde{f}_{n,\epsilon}\|^2 \leqq \int \sum_m |\hat{\tilde{f}}_{nm\epsilon}(x)|^2 dx, \tag{1}$$

where $\hat{\tilde{f}}_{nm\epsilon}$ is the Fourier transform of $\tilde{f}_{nm\epsilon}$. Now $\hat{\tilde{f}}_{nm\epsilon}$ is the product of the Fourier transform $\hat{f}(x)$ of $f(x)$ and the Fourier transform of the kernel equal to

$$(2\pi)^{-k/2} Y_{n,m}(x') \, |x|^{-k}$$

for $|x| > \epsilon$ and zero elsewhere. By Lemma 2 the latter transform is in absolute value less than $An^{-1-\lambda}|Y_{n,m}(x')|$, A being an absolute constant. Thus from (1) we obtain

$$\begin{aligned} \|\tilde{f}_{n,\epsilon}\|^2 &\leqq A^2 n^{-2-2\lambda} \int |\hat{f}(x)|^2 \sum_m Y_{nm}(x')^2 dx \\ &\leqq A^2 n^{-2-2\lambda} \|\hat{f}\|^2 \max_{x'} \Big[\sum_m Y_{nm}(x')^2 \Big]. \end{aligned} \tag{2}$$

Now

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(1) Transactions of the American Mathematical Society, vol. 78 (1955) pp. 209–224. The reexamination of our paper was prompted by a criticism by Mihlin (*On the theory of multidimensional singular equations*, Bull. of the Leningrad Univ., no. 1, (1956), Series on Math., Mechanics and Astronomy). Although the dependence of $Y_m(z')$ on x was implicitly assumed in the paper it was overlooked in the proof of (2.8).

$$\max_{x'}\left[\sum_m Y_{nm}(x')^2\right] = \max_{\beta,x'}\left[\sum_m \beta_m Y_{nm}(x')\right]^2,$$

where $\sum \beta_m^2 = 1$. Since $\sum_m \beta_m Y_{nm}(x')$ is a normalized spherical harmonic of degree n, according to lemma 1 its absolute value is dominated by Bn^λ, where B is again an absolute constant. Finally since $\|\hat{f}\| = \|f\|$ from (2) and the last estimate we obtain (2.8).

We conclude with one more remark. It is clear from the context of the paper that in the case of Theorem 2 the function $\tilde{f}_\epsilon$ was understood to be defined by (2.4). Actually one can prove that the integral in (1.5) is almost everywhere absolutely convergent and therefore can be used to define $\tilde{f}_\epsilon$ directly. We will return to this and related problems on another occasion.

Reference

1. A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. vol. 78 (1956) pp. 289–309.