

CLOSED COVERINGS IN ČECH HOMOLOGY THEORY

BY

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1. Introduction. We treat here such topics as homology local connectedness, regular convergence, and the Vietoris mapping theorem⁽¹⁾. In general, we are interested in those topics of topology which are based on the technique of chain-realizations. We present an alternative technique, based on one central theorem (Theorem 2.3). One justification for our technique is that it allows the use of compact coefficient groups as well as fields. Moreover, we hope that with further study it will provide a certain amount of unification of the topics treated.

The paper is divided into two parts. The first part consists of §§2, 3, and 4. In §2 we state the basic theorem and some of its corollaries. In §§3 and 4 we give a few applications of our theorem to the topics already noted. In the second part, which is not dependent on the first part, we prove the basic theorem. In §§5 and 6 we develop in detail a Kelley-Pitcher theory of finite closed coverings of compact spaces. This theory was partially developed in the well-known Kelley-Pitcher paper on exact sequences [3, pp. 703–706]. In §7 we use this development to prove the basic theorem. We would be interested in knowing whether or not the basic theorem can be proved by more elementary means.

2. The basic theorems. In this paper, a space will always be a Hausdorff space. Suppose that $\alpha = (A^1, \dots, A^r)$ is an ordered, finite covering of a space X . In r -space, let \bar{A}^i be the point whose j th coordinate is δ_j^i . Then the nerve of α will be the collection of all simplices $(\bar{A}^{i_0}, \dots, \bar{A}^{i_q})$ with $A^{i_0} \cap \dots \cap A^{i_q} \neq \emptyset$. If X is compact, $H_n(X)$ will denote the n -dimensional Čech homology group with coefficients in a fixed group \mathcal{G} , which may be either a field or a compact abelian group; $H_0(X)$ will denote the reduced 0-dimensional group. For a covering α , $H_n(\alpha)$ denotes the homology group of the nerve of α .

If $u = (U^1, \dots, U^r)$ is an open covering of X , there is the projection homomorphism $\pi_u: H_n(X) \rightarrow H_n(u)$ which assigns to each element of $H_n(X)$ its u -coordinate.

By a *closed covering* α of a compact space X , we will always mean an ordered, finite covering $\alpha = (A^1, \dots, A^r)$ by closed sets in X such that every point of X is in the interior of some A^i . Given such an α , there exists an open covering $u = (U^1, \dots, U^r)$ of X such that $U^i \supset A^i$ and $U^{i_0} \cap \dots \cap U^{i_q} \neq \emptyset$

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(1) For bibliographical notes concerning homology local connectedness, see Wilder [6]; for a bibliography on regular convergence, see White [4]; for a modern treatment of the Vietoris mapping theorem, see Begle [1].

if and only if $A^{i_0} \cap \dots \cap A^{i_q} \neq \emptyset$ (this is a special case of (6.1)). Then u and α have the same nerve and $H_n(\alpha) = H_n(u)$. Define a projection homomorphism $\pi_\alpha: H_n(X) \rightarrow H_n(\alpha)$ by $\pi_\alpha = \pi_u$. It may be checked that π_α is independent of u .

If α and β are finite collections of closed subsets of X , then β *refines* α , or $\beta > \alpha$, if and only if given an element B of β there exists an element A of α with $B \subset A$. A projection $\pi_{\beta\alpha}$ assigns to each B in β such an A . The induced homomorphism of $H_n(\beta)$ into $H_n(\alpha)$ will also be denoted by $\pi_{\beta\alpha}$ (or occasionally by just π). If α and β are closed coverings of X with $\beta > \alpha$, it may be checked that $\pi_{\beta\alpha}\pi_\beta = \pi_\alpha$.

If A and B are closed subsets of X with $A \subset B$, we denote by $I_{AB}: H_n(A) \rightarrow H_n(B)$, or occasionally by I , the injection homomorphism which is induced by inclusion. We denote by $H_n(A; B)$ the image of I_{AB} .

(2.1) DEFINITION. If α and β are finite collections of closed subsets of X and n is a non-negative integer, we write $\beta^n > \alpha$ (β *n-refines* α) if and only if given an element B of β there exists an element A of α with $B \subset A$ and $H_j(B; A) = 0$ for all $j \leq n$.

We write $\beta^n \gg \alpha$ (β *strongly n-refines* α) if and only if $\beta > \alpha$ and there exists a projection $\pi_{\beta\alpha}: \beta \rightarrow \alpha$ such that $H_j(B^{i_0} \cap \dots \cap B^{i_q}; \pi B^{i_0} \cap \dots \cap \pi B^{i_q}) = 0$ for all B^{i_0}, \dots, B^{i_q} in β and all $j \leq n$.

(2.2) If $\gamma^n > \beta$ and β *star-refines* α , then $\gamma^n \gg \alpha$.

Proof. Since $\gamma^n > \beta$, there exists a projection $\pi': \gamma \rightarrow \beta$ such that $H_j(C; \pi' C) = 0$ for each C in γ and each $j \leq n$. Since β *star-refines* α , there is a projection $\pi'': \beta \rightarrow \alpha$ such that if B is in β then every element of β which intersects B is in $\pi'' B$. Let $\pi = \pi'' \pi'$. Suppose $C^{i_0} \cap \dots \cap C^{i_q} \neq \emptyset$. Then

$$C^{i_0} \cap \dots \cap C^{i_q} \subset C^{i_0} \subset \pi' C^{i_0} \subset \pi C^{i_0} \cap \dots \cap \pi C^{i_q}.$$

Then $H_j(C^{i_0} \cap \dots \cap C^{i_q}; \pi C^{i_0} \cap \dots \cap \pi C^{i_q}) = 0$ since $H_j(C^{i_0}; \pi' C^{i_0}) = 0$, $j \leq n$.

The following is the basic theorem of the paper; its proof is deferred to §7.

(2.3) THEOREM. Suppose that $A_{-1} \supset A_0 \supset \dots \supset A_n$ is a sequence of closed subsets of a compact space X , and that $\alpha_{-1} \ll^n \alpha_0 \ll^n \dots \ll^n \alpha_n$, where α_i is a closed covering of A_i . Then

(i) the kernel of $\pi_{\alpha_n}: H_j(A_n) \rightarrow H_j(\alpha_n)$ is contained in the kernel of the injection $I: H_j(A_n) \rightarrow H_j(A_{-1})$ for all $j \leq n$, and

(ii) the image of $\pi_{\alpha_n, \alpha_0}: H_j(\alpha_n) \rightarrow H_j(\alpha_0)$ is contained in the image of the projection $\pi_{\alpha_0}: H_j(A_0) \rightarrow H_j(\alpha_0)$ for all $j \leq n+1$.

We now study the implications of the theorem in case $A_i = X$, all i .

(2.4) DEFINITION. Let α, β be closed coverings of the compact space X . We say that α, β *determine* $H_n(X)$ if and only if $\alpha < \beta$ and $\pi_\alpha: H_n(X) \rightarrow H_n(\alpha)$ maps $H_n(X)$ isomorphically onto the image of the projection $\pi_{\beta\alpha}: H_n(\beta) \rightarrow H_n(\alpha)$. We say that α, β *partially determine* $H_n(X)$ if and only if image

$\pi_\alpha = \text{image } \pi_{\beta\alpha}$ (this latter is equivalent to the classical notion that β is a normal refinement [7, p. 140] of α).

(2.5) *If α, β (partially) determine $H_n(X)$ and $\gamma > \beta$, then α, γ (partially) determine $H_n(X)$.*

Proof. It is always true that $\text{image } \pi_\alpha \subset \text{image } \pi_{\gamma\alpha}$. But $\text{image } \pi_{\gamma\alpha} \subset \text{image } \pi_{\beta\alpha} = \text{image } \pi_\alpha$, so $\text{image } \pi_\alpha = \text{image } \pi_{\gamma\alpha}$.

(2.6) *If β, γ partially determine $H_n(X)$ and if $\beta > \alpha$ then α, γ partially determine $H_n(X)$.*

Proof. We have $\text{image } \pi_{\gamma\alpha} = \pi_{\beta\alpha} (\text{image } \pi_{\gamma\beta}) = \pi_{\beta\alpha} (\text{image } \pi_\beta) = \text{image } \pi_\alpha$.

The following consequence of (2.3) is used as the basis for the rest of the paper.

(2.7) **THEOREM.** *If X is a compact space and if $\alpha_{-1} \ll^n \alpha_0 \ll^n \dots \ll^n \alpha_{2n}$, where α_i is a closed covering of X , then α_n, α_{2n} determine $H_j(X)$, for all $j \leq n$, and partially determine $H_{n+1}(X)$.*

Proof. In (2.3 i), let $A_i = X$. Hence according to (2.3 i) the kernel of π_{α_n} is contained in the kernel of the identity map of $H_j(X)$, $j \leq n$, and hence π_{α_n} is an isomorphism into for $j \leq n$. In (2.3 ii), set $A_i = X$ and consider the coverings $\alpha_n, \dots, \alpha_{2n}$. In (2.3 ii), the image of $\pi_{\alpha_{2n}, \alpha_n} : H_j(\alpha_{2n}) \rightarrow H_j(\alpha_n)$ is contained in the image of π_{α_n} for $j \leq n+1$. Since the opposite inclusion always holds, the two images are equal. Hence α_n, α_{2n} partially determine $H_j(X)$ for $j \leq n+1$. The theorem follows.

3. Locally connected space; the Vietoris mapping theorem. We consider here a few properties of locally connected spaces; these are known for the case when the coefficient group is a field [7, Chap. 6]. On the basis of these, we give a new proof of the Vietoris mapping theorem as given by Begle [1].

(3.1) **DEFINITION.** A compact space X is said to be lc^n , n a non-negative integer, if and only if given $x \in X$ and a closed neighborhood U of x , there exists a closed neighborhood V of x with $H_j(V; U) = 0$, all $j \leq n$.

(3.2) *If X is an lc^n , compact space, then given a closed covering α of X , there exists a closed covering β of X with $\beta^n \gg \alpha$.*

Proof. Suppose that γ is a closed covering which star-refines α . For each $x \in X$, there is an element C of γ which contains x in its interior. Hence there is a closed neighborhood V_x of x with $H_j(V_x; C) = 0$ for $j \leq n$. Let β be a finite collection of the V_x whose interiors cover X . Then $\beta^n > \gamma$ and γ star-refines α . By (2.2), $\beta^n \gg \alpha$.

The following is equivalent to a classical theorem [7, p. 180] when the coefficient group is a field.

(3.3) **THEOREM.** *Suppose that X is a compact lc^n space. For each sufficiently small covering α of X , there exists a closed covering $\beta > \alpha$ such that α, β determine $H_j(X)$, all $j \leq n$, and partially determine $H_{n+1}(X)$.*

Proof. Let α_{-1} be a closed covering of X . By (3.2), there are closed cover-

ings $\alpha_0, \dots, \alpha_{n-1}$ with $\alpha_{-1} \ll^n \dots \ll^n \alpha_{n-1}$. By (3.2), for all sufficiently small α_n we have $\alpha_n \gg \alpha_{n-1}$. Given such an α_n , we get a sequence $\alpha_{-1} \ll^n \dots \ll^n \alpha_{2n}$. By (2.7), α_n, α_{2n} determine $H_j(X)$, $j \leq n$, and partially determine $H_{n+1}(X)$.

(3.4) DEFINITION. We shall say that the coefficient group G is *elementary* if it is either a field or an elementary compact group⁽²⁾. In case G is elementary, we say that $H_n(X)$, or a subgroup thereof, is elementary if it is a finite dimensional vector space when G is a field, or an elementary compact group when G is an elementary compact group.

Condition (b) of the following theorem is similar to property $(P, Q)_n$ of Wilder [7, p. 193].

(3.5) THEOREM. *If the coefficient group is elementary, then the following are equivalent for a compact space X :*

- (a) X is lc^n ;
- (b) if A and B are closed subsets of X with A in the interior of B , then $H_j(A; B)$ is elementary for $j \leq n$;
- (c) if A is a closed subset of X and U is a closed neighborhood of A , then there is a closed neighborhood V of A with $H_j(V; U) = H_j(A; U)$ for all $j \leq n$.

Proof. To show that (a) implies (b), suppose (a) holds and that A is contained in the interior of B (written $A \subset \subset B$). Let $A_{-1} = B$. We may find a sequence $B = A_{-1}, A_0, \dots, A_n = A$ such that $A_{i+1} \subset \subset A_i$. Let α_{-1} be a closed covering of A_{-1} . There exists a closed covering β of A_{-1} which star-refines α_{-1} . Since A_0 is in the interior of A_{-1} and X is lc^n , there is a closed covering α_0 of A_0 with $\alpha_0 \gg \beta$. By (2.2), $\alpha_0 \gg \alpha_{-1}$. Similarly there exist closed coverings α_i of A_i with $\alpha_{-1} \ll^n \alpha_0 \ll^n \dots \ll^n \alpha_n$. According to (2.3 i), the kernel K of $\pi_{\alpha_n}: H_j(A) \rightarrow H_j(\alpha_n)$ is contained in the kernel K' of $I_{AB}: H_j(A) \rightarrow H_j(B)$ for $j \leq n$. Now $H_j(\alpha_n)$ is elementary, together with its subgroups and factor groups. Hence $H_j(A)/K \approx \text{image } \pi_{\alpha_n}$ is elementary. Then

$$H_j(A)/K' \approx (H_j(A)/K)/(K'/K)$$

is elementary, being a factor group of an elementary group. But $H_j(A; B) \approx H_j(A)/K'$; hence (a) implies (b).

Suppose now that (b) holds and that $A \subset \subset U$. For each closed V with $A \subset \subset V \subset U$ let $K(V) = \bigcap H_j(V'; V)$ where the intersection is taken over all V' with $A \subset \subset V' \subset \subset V$. Since each $H_j(V'; V)$ is elementary, there is a V' with $K(V) = H_j(V'; V)$. We note that I_{VU} maps $K(V)$ onto $K(U)$. For if $x \in K(U)$, then $x \in H_j(V'; U)$ and there is a $y \in H_j(V')$ with $I_{V'U}(y) = x$. Then $I_{VU}(I_{V'V}(y)) = x$, and $I_{V'V}(y) \in K(V)$. Hence the $K(V)$, together with the $I_{V'V}$, constitute an inverse mapping system of elementary groups, and I_{VU} maps $K(V)$ onto $K(U)$. Hence, given $x \in K(U)$, there is a function assigning to each V an $x(V) \in K(V)$ with $x(U) = x$ and $I_{V'V}(x(V')) = x(V)$. Hence there

(2) An elementary compact group is defined to the direct sum of a finite number of groups, each of which is the reals mod 1 or a finite cyclic group.

is a $y \in H_j(\cap V) = H_j(A)$ with $I_{AV}(y) = x(V)$. In particular, $I_{AV}(y) = x$. Then $H_j(A; U) = \cap H_j(V; U)$. But each $H_j(V; U)$ is elementary, so that for some V , $H_j(A; U) = H_j(V; U)$. It follows that (b) implies (c).

It is easy to see that (c) implies (a), by taking for A an arbitrary one-point set.

(3.6) VIETORIS MAPPING THEOREM. *Suppose that f is a continuous map of a compact space X onto a compact space Y such that $H_j(f^{-1}(y)) = 0$ for all $j \leq n$ and $y \in Y$, where the coefficient group is elementary. Then $f_*: H_j(X) \rightarrow H_j(Y)$ is an isomorphism onto for $j \leq n$, and is onto for $j = n+1$.*

Proof. Case I: X is lc^{n+1} . Let β_{-1} be an arbitrary closed covering of Y . Suppose that γ is a star-refinement of β_{-1} ; then $f^{-1}(\gamma)$ is a star-refinement of $f^{-1}(\beta_{-1})$. Suppose $y \in Y$. Then y is interior to some element C of γ , and $f^{-1}(y)$ is interior to $f^{-1}(C)$. Since X is lc^n and $H_j(f^{-1}(y)) = 0$ for $j \leq n$, there exists by (3.5) a V with $f^{-1}(y) \subset \subset V \subset f^{-1}(C)$ and $H_j(V; f^{-1}(C)) = 0$, $j \leq n$. There exists a closed neighborhood B_y of y with $f^{-1}(B_y) \subset V$. Let β_0 be a finite subcollection of $\{B_y\}$ whose interiors cover Y . Then $f^{-1}(\beta_0)^n \supset f^{-1}(\gamma)$ and $f^{-1}(\gamma)$ star-refines $f^{-1}(\beta_{-1})$; by (2.2), $f^{-1}(\beta_0)^n \gg f^{-1}(\beta_{-1})$. In a similar manner we obtain a sequence $\beta_{-1} < \beta_0 < \dots < \beta_{2n}$ of closed coverings of Y such that if $\alpha_i = f^{-1}(\beta_i)$ then $\alpha_{-1} \ll^n \dots \ll^n \alpha_{2n}$. Then by (2.7) α_n, α_{2n} determine $H_j(X)$, $j \leq n$, and partially determine $H_{n+1}(X)$. Since β_n can be made arbitrarily small, for every sufficiently small covering β of Y there exists a refinement β' of β such that the coverings $\alpha = f^{-1}(\beta)$, $\alpha' = f^{-1}(\beta')$ determine $H_j(X)$, $j \leq n$, and partially determine $H_{n+1}(X)$.

Consider the diagram

$$\begin{array}{ccccc}
 H_j(X) & & & & \\
 \downarrow f_* & \nearrow \pi_{\alpha'} & & \searrow \pi_{\alpha} & \\
 & H_j(\alpha) & & H_j(\alpha') & \\
 & \parallel & \longleftarrow \pi_{\beta', \beta} & \parallel & \\
 & H_j(\beta) & & H_j(\beta') & \\
 & \nearrow \pi_{\beta} & & \searrow \pi_{\beta'} & \\
 & H_j(Y) & & &
 \end{array}$$

where $\alpha, \beta, \alpha', \beta'$ are as above. There is commutativity: $\pi_{\beta'} f_* = \pi_{\beta}$, $\pi_{\beta} f_* = \pi_{\alpha'}$, etc.

Note first that f_* is an isomorphism into for $j \leq n$. For $\pi_{\alpha} = \pi_{\beta'} f_*$ and π_{α} is an isomorphism into; hence f_* is an isomorphism into.

By (3.3), $H_j(X)$ is elementary for $j \leq n+1$; hence $K = \text{image } f_*$ is elementary. Moreover, $\text{image } \pi_{\beta'} f_* = \text{image } \pi_{\beta}$ for $j \leq n+1$. For if $y \in H_j(Y)$ then

$$\pi_\beta(y) = \pi_{\beta'\beta}(\pi_{\beta'}(y)) = \pi_{\alpha'\alpha}(\pi_{\beta'}(y)).$$

Since α, α' partially determine $H_j(X)$, there is an $x \in H_j(X)$ with $\pi_\alpha(x) = \pi_\beta(y)$. Then $\pi_{\beta f_*}(x) = \pi_\beta(y)$. Hence

$$\pi_\beta(K) = \pi_\beta(H_j(Y)) \quad \text{for all sufficiently small } \beta.$$

Suppose now that $y \in H_j(Y)$. $K - y$ denotes the set $\{k - y : k \in K\}$. For each β there is a $k \in K$ with $\pi_\beta(k) = \pi_\beta(y)$, so that $k - y \in L_\beta = (\ker \pi_\beta) \cap (K - y)$. Now $K - y$ is a translate of K , which is elementary. The L_β are nonempty and decrease with β . Hence $\bigcap_\beta L_\beta \neq \emptyset$. If $z \in \bigcap_\beta L_\beta$, then $\pi_\beta(z) = 0$ for all β , so that $z = 0$. Hence $0 \in K - y$ and $y \in K$. Hence f_* maps $H_j(X)$ onto $H_j(Y)$, $j \leq n + 1$.

Case II: X a compact space. We may consider X as embedded in an l^{n+1} compact space X' (for example, X' a product of intervals). The map f generates a decomposition of X whose elements are the sets $f^{-1}(y)$, $y \in Y$. Extend this decomposition to a decomposition of X' by admitting as elements the one-point sets of $X' - X$. This upper semi-continuous decomposition of X' has a decomposition space Y' and a decomposition map $F: X' \rightarrow Y'$. We may identify Y with $F(X)$ and f with $F|X$. We have the diagram

$$\begin{array}{ccccccccc} H_{j+1}(X') & \rightarrow & H_{j+1}(X', X) & \rightarrow & H_j(X) & \rightarrow & H_j(X') & \rightarrow & H_j(X', X) \\ \downarrow F_{*1} & & \downarrow F'_* & & \downarrow f_* & & \downarrow F_{*2} & & \downarrow F'_* \\ H_{j+1}(Y') & \rightarrow & H_{j+1}(Y', Y) & \rightarrow & H_j(Y) & \rightarrow & H_j(Y') & \rightarrow & H_j(Y', Y) \end{array}$$

where the downward homomorphisms are induced by F . Now both the F'_* are isomorphisms onto, since F maps $X' - X$ homeomorphically onto $Y' - Y$. For $j \leq n$, F_{*2} is an isomorphism onto by Case I, and F_{*1} is onto. By the five-lemma of Eilenberg-Steenrod [2, p. 16], f_* is an isomorphism onto. For $j = n + 1$, F_{*2} is onto; by the five-lemma, f_* is then onto. The theorem follows.

4. Regular convergence. In this section, we prove some theorems concerning regular convergence. For the known facts where the coefficient group is a field, see [5].

(4.1) **DEFINITION.** A sequence (A_i) of closed subsets of a compact space X is said to converge *n-regularly* to the subset A of X if and only if (A_i) converges to A , and given $x \in A$ and a closed neighborhood U of x there exist a closed neighborhood V of x and a positive integer N such that $H_j(V \cap A_i; U \cap A_i) = 0$ for $j \leq n$ and $i \geq N$.

If $\beta = (B^1, \dots, B^r)$ is a covering of X and $A \subset X$ then $\beta \cap A$ denotes the covering $(B^1 \cap A, \dots, B^r \cap A)$ of A .

(4.2) *If the sequence (A_i) converges n-regularly to A , then given a closed covering α of X there exist a closed covering β of X and a positive integer N such that $\alpha \cap A_i \ll^n \beta \cap A_i$ for all $i \geq N$.*

Proof. Let γ be a closed covering of X such that γ star-refines α . For each

$x \in X$ there is a C in γ with x in its interior. Hence there is a closed neighborhood B_x of x and a positive integer N_x such that $H_j(B_x \cap A_i; C \cap A_i) = 0$ for $j \leq n$ and $i \geq N_x$. Let β be a finite subcollection of B_x whose interiors cover X . Then $\beta \cap A_i \gg \gamma \cap A_i$ for i sufficiently large. Since $\gamma \cap A_i$ star-refines $\alpha \cap A_i$, then $\beta \cap A_i \gg \alpha \cap A_i$ for i sufficiently large.

(4.3) *If (A_i) converges n -regularly to A , then for each sufficiently small closed covering α of X , there exists a closed covering $\beta \gg \alpha$ and a positive integer N such that $\alpha \cap A_i, \beta \cap A_i$ determine $H_j(A_i)$ for $j \leq n$ and $i \geq N$ and partially determine $H_{n+1}(A_i)$ for $i \geq N$.*

Proof. The proof follows easily from (4.2) and (2.7).

(4.4) **DEFINITION.** Suppose that A is a closed subset of the compact space X , and that α is a closed covering of X . We say that α is in *general position relative to A* if and only if whenever A^{i_0}, \dots, A^{i_q} are elements of α with $A^{i_0} \cap \dots \cap A^{i_q} \cap A \neq \emptyset$ then $\text{int } A^{i_0} \cap \dots \cap \text{int } A^{i_q} \cap A \neq \emptyset$.

(4.5) *If A is a closed subset of the compact space X and α is a closed covering of X , there is a refinement β of α in general position relative to A .*

Proof. Suppose $\alpha = (A^1, \dots, A^r)$. For each i_0, \dots, i_q , select a point x_{i_0, \dots, i_q} of $\text{int } A^{i_0} \cap \dots \cap \text{int } A^{i_q} \cap A$, if such a point exists. Let $\beta = (B^1, \dots, B^r)$ be a closed covering of X such that B^i is in the interior of A^i and such that B^i contains (in its interior) all the x_{i_0, \dots, i_q} which belong to $\text{int } A^i$. Then β is the desired covering.

It will be noted that if (A^i) converges to A , and α is general position relative to A , then the nerves of $\alpha \cap A$ and $\alpha \cap A_i$ coincide for i sufficiently large.

(4.6) **THEOREM.** *Suppose that the sequence (A_i) of closed subsets of the compact space X converges to A , and that for each sufficiently small closed covering α of X there exists a closed covering β of X and a positive integer N such that $\alpha \cap A_i, \beta \cap A_i$ determine $H_n(A_i)$ for $i \geq N$. Then for each α , there is a β such that $\alpha \cap A, \beta \cap A$ determine $H_n(A)$.*

Proof. We consider closed coverings $\alpha < \beta < \gamma$ which are in general position relative to A . We use the diagram

$$\begin{array}{ccccc}
 H_n(A) & \xrightarrow{\pi_3} & H_n(\gamma \cap A) = H_n(\gamma \cap A_i) & \xleftarrow{\pi'_3} & H_n(A_i) \\
 & \searrow \pi_2 & \downarrow \pi''_2 & \swarrow \pi'_2 & \\
 & & H_n(\beta \cap A) = H_n(\beta \cap A_i) & & \\
 & \searrow \pi_1 & \downarrow \pi''_1 & \swarrow \pi'_1 & \\
 & & H_n(\alpha \cap A) = H_n(\alpha \cap A_i) & &
 \end{array}$$

where all the π 's are projections and where i is always large enough so that the indicated equalities hold.

Suppose that $x \in H_n(A)$, $x \neq 0$, and $\pi_1(x) = 0$. Suppose $\alpha \cap A_i, \beta \cap A_i$ deter-

mine $H_n(A_i)$, $i \geq N$, that $\pi_2(x) \neq 0$, and that $\beta \cap A_i$, $\gamma \cap A_i$ determine $H_n(A_i)$, $i \geq N$. Since $\pi_3(x) = \pi_2' \pi_2(x)$, there is a $y \in H_n(A_i)$ with $\pi_2'(y) = \pi_2(x)$. Then $y \neq 0$ since $\pi_2'(y) \neq 0$. Then also $\pi_1'(y) \neq 0$, since $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$. But $\pi_1'(y) = \pi_1(x)$. This is a contradiction. Hence π_1 is an isomorphism into.

Suppose that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$, $i \geq N$. We show that image $\pi_1 = \text{image } \pi_1''$; we know that image $\pi_1 \subset \text{image } \pi_1''$. Let $y \in H_n(\beta \cap A)$. There is an $x_i \in H_n(A_i)$ with $\pi_1'(x_i) = \pi_1''(y)$. For each closed covering γ of X , γ in general position relative to A , with $\gamma > \beta$, $\pi_1' \pi_2''(\pi_3'(x_i)) = \pi_1'(x_i) = \pi_1''(y)$. Hence for each such γ there is a $z_\gamma (= \pi_3'(x_i))$ in $H_n(\gamma \cap A)$ with $\pi_1'' \pi_2''(z_\gamma) = \pi_1''(y)$. Hence there is an $x \in H_n(A)$ whose $(\alpha \cap A)$ -coordinate is $\pi_1'(y)$. Then $\pi_1(x) = \pi_1''(y)$. Hence if $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$, $i \geq N$, and α , β are in general position relative to A then $\alpha \cap A$, $\beta \cap A$ determine $H_n(A)$.

(4.7) THEOREM. Suppose that the sequence (A_i) of closed subsets of the compact space X converges n -regularly to the subset A . Then $H_n(A) \approx H_n(A_i)$ for i sufficiently large.

Proof. There exist, by (4.5) and (4.3), closed coverings α , β of X , in general position relative to A , such that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$. By (4.6), $\alpha \cap A$, $\beta \cap A$ determine $H_n(A)$. Hence $H_n(A_i)$ and $H_n(A)$, for i sufficiently large, are isomorphic to the image of the projection $\pi: H_n(\beta \cap A) \rightarrow H_n(\alpha \cap A)$, and are isomorphic to each other.

5. The Kelley-Pitcher theory for simplicial pairs. In §§5 and 6 we continue an investigation of Kelley and Pitcher [3, pp. 703-706] seeking relationships between the groups of a space, the groups of the nerve of a covering of the space, and the groups of intersections of elements of the covering. The treatment is self-contained; the portion of this section through (5.5) is due to Kelley-Pitcher.

By a *simplicial pair* we mean a pair (X, α) consisting of a finite simplicial complex X and a covering $\alpha = (A^1, \dots, A^r)$ by subcomplexes. X_α will denote the nerve of α . However, $C_p(\alpha)$ and $H_p(\alpha)$ will indicate the chain group and homology group of the nerve X_α . If a simplex of X_α is of the form $T_q = (\tilde{A}^{i_0} \cdot \dots \cdot \tilde{A}^{i_q})$ then define $\cap T_q = A^{i_0} \cap \dots \cap A^{i_q}$. Consider X_α as having a (-1) -dimensional simplex T_{-1} with $\cap T_{-1} = X$.

If S_p is a p -simplex of X let ΛS_p denote the subcomplex of X_α consisting of all T_q with $S_p \subset \cap T_q$. If A^{i_0}, \dots, A^{i_q} are all the elements of α which contain S_p , $p \geq 0$, then ΛS_p consists of all faces of the simplex $(\tilde{A}^{i_0} \cdot \dots \cdot \tilde{A}^{i_q})$. If S_{-1} is the (-1) -dimensional simplex of X , then $\Lambda S_{-1} = X_\alpha$. The boundary operator will be denoted by ∂ .

Denote by $C_{p,q}(\alpha)$ the set of all linear forms $\sum g S_p \cdot T_q$ where $g \in \mathfrak{G}$ and S_p , T_q are oriented simplices of X , X_α respectively with $S_p \subset \cap T_q$. Agree that $g S_p \cdot (-T_q) = g(-S_p) \cdot T_q = (-g) S_p \cdot T_q$ and that forms are to be added as

usual. If $A_p = \sum g_i S_p^i$ is a p -chain of $\bigcap T_q$, define $A_p \cdot T_q = \sum g_i S_p^i \cdot T_q$. If $B_q = \sum g_i T_q^i$ is a q -chain of $\bigwedge S_p$, define $S_p \cdot B_q = \sum g_i S_p \cdot T_q^i$.

The group $C_{p,-1}(\alpha)$ will be identified with the chain group $C_p(X)$ under the identification $A_p \cdot T_{-1} \leftrightarrow A_p$. The group $C_{-1,q}(\alpha)$ will be identified with the chain group $C_q(\alpha)$ under the identification $S_{-1} \cdot B_q \leftrightarrow B_q$.

A homomorphism $\Delta: C_{p,q} \rightarrow C_{p-1,q}$ is defined by $\Delta(A_p \cdot S_q) = (\partial A_p) \cdot S_q$ and linearity. A homomorphism $D: C_{p,q} \rightarrow C_{p,q-1}$ is defined by $D(S_p \cdot B_q) = S_p \cdot (\partial B_q)$ and linearity.

(5.1) If $p \geq 0$ the sequence

$$\cdots \rightarrow C_{p,q+1} \xrightarrow{D} C_{p,q} \xrightarrow{D} C_{p,q-1} \rightarrow \cdots$$

is exact. If $p = -1$, the sequence coincides with

$$\cdots \rightarrow C_{q+1}(\alpha) \xrightarrow{\partial} C_q(\alpha) \xrightarrow{\partial} C_{q-1}(\alpha) \rightarrow \cdots$$

Proof. It is clear that $DD=0$ since $\partial\partial=0$. Suppose $p \geq 0$ and that $D(\sum S_p^i \cdot B_q^i) = \sum S_p^i \cdot (\partial B_q^i) = 0$, where the S_p^i form a basis for $C_p(X)$. Then $\partial B_q^i = 0$. Since B_q^i is on the simplex $\bigwedge S_p^i$, then $B_q^i = \partial C_q^i$ for some C_q^i in $\bigwedge S_p^i$. Then $D(\sum S_p^i \cdot C_q^i) = \sum S_p^i \cdot B_q^i$, and the sequence is exact for $p \geq 0$. The statement for $p = -1$ can be easily seen.

(5.2) The sequence

$$\cdots \rightarrow C_{p+1,q} \xrightarrow{\Delta} C_{p,q} \xrightarrow{\Delta} \cdots$$

has $\Delta\Delta=0$. The homology group $H_{p,q}(\alpha)$ of the sequence corresponding to the indices p, q is isomorphic to the direct sum $\sum H_p(\bigcap T_q)$, where the summation is extended over all q -simplices $T_q = (\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ with $i_0 < \cdots < i_q$.

Proof. It is clear that $\Delta\Delta=0$. If $q = -1$, $C_{p,-1} = C_p(X)$ and $\Delta = \partial$ on $C_p(X)$. Hence the homology group, if $q = -1$, is $H_p(X) = H_{p,-1}(\alpha)$. Suppose $q \geq 0$. Elements z of $C_{p,q}$ are uniquely represented as $z = \sum A_p^i \cdot T_q^i$, where the T_q^i are the oriented q -simplices of the form $(\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ with $i_0 < \cdots < i_q$ and where $A_p^i \in C_p(\bigcap T_q^i)$. This representation sets up an isomorphism of $C_{p,q}$ with $\sum C_p(\bigcap S_q)$ and the statement follows easily.

Define $E_{p,q}$ to be the kernel of $D: C_{p,q} \rightarrow C_{p,q-1}$. For $p \geq 0$, it follows from (5.1) that $E_{p,q}$ is the image of $D: C_{p,q+1} \rightarrow C_{p,q}$. We have the sequence

$$\cdots \rightarrow E_{p+1,q} \xrightarrow{\Delta} E_{p,q} \xrightarrow{\Delta} \cdots$$

Define $K_{p,q}$ to be the homology group of this sequence corresponding to indices p, q .

(5.3) We have $K_{p,-1} = H_p(X)$.

Proof. $K_{p,-1}$ is the homology group of

$$\cdots \rightarrow E_{p+1,-1} \xrightarrow{\Delta} E_{p,-1} \xrightarrow{\Delta} E_{p-1,-1} \rightarrow \cdots$$

Now $D:C_{p,-1} \rightarrow C_{p,-2} = 0$ so that $E_{p,-1} = C_{p,-1} = C_p(X)$. Moreover, $\Delta = \partial$ on $C_p(X)$. The result follows.

(5.4) We have $K_{-1,q} = H_q(\alpha)$.

Proof. We have $C_{-1,q} = C_q(\alpha)$. Also $D:C_{-1,q} \rightarrow C_{-1,q-1}$ coincides with $\partial:C_q(\alpha) \rightarrow C_{q-1}(\alpha)$. Hence $E_{-1,q}$ is the cycle group $Z_q(\alpha)$. Now $E_{0,q}$ is the image of $D:C_{0,q+1} \rightarrow C_{0,q}$. Hence the image of $\Delta:E_{0,q} \rightarrow E_{-1,q}$ is also the image of $\Delta D:C_{0,q+1} \rightarrow C_{-1,q}$. But $\Delta D(\sum A_0^i \cdot T_{q+1}^i) = \sum (\partial A_0^i) \cdot (\partial T_{q+1}^i)$ where ∂A_0^i is of the form $g_i S_{-1}$ with g_i the coefficient sum of A_0^i . It follows that the image of $\Delta:E_{0,q} \rightarrow E_{-1,q}$ is the bounding cycle group $B_q(\alpha)$. The result follows.

For each $p \geq 0$, the sequence

$$0 \rightarrow E_{p,q} \xrightarrow{l'} C_{p,q} \xrightarrow{D} E_{p,q-1} \rightarrow 0$$

is exact, where l' is inclusion. If $p = -1$ the sequence is exact except that D may not be onto. Taking homology groups and using a well-known method of constructing exact sequences [3, p. 687] we get the exact sequence

$$\cdots \rightarrow K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q} \rightarrow \cdots$$

$\rightarrow K_{0,q-1} \rightarrow K_{-1,q} \rightarrow H_{-1,q}$, where l is induced by inclusion l' , m by D , and $\eta^{(p)}$ by the boundary operation. Since $H_{-1,q} = 0$, we get

(5.5) The sequence

$$\cdots \rightarrow K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q} \rightarrow \cdots \rightarrow K_{0,q-1} \rightarrow K_{-1,p} \rightarrow 0$$

is exact.

The map $\eta^{(p)}:K_{p,q-1} \rightarrow K_{p-1,q}$, $p \geq 0$, is of particular interest. Suppose z represents an element x of $K_{p,q-1}$. Then $z = D(z')$ where $z' \in C_{p,q}$. Then $\Delta z'$ represents $\eta^{(p)}(x)$.

$$\begin{aligned} \text{In } H_p(X) = K_{p,-1} &\xrightarrow{\eta^{(p)}} K_{p-1,0} \xrightarrow{\eta^{(p-1)}} K_{p-1,2} \rightarrow \cdots \rightarrow K_{0,p-1} \xrightarrow{\eta^{(0)}} K_{-1,p} \\ &= H_p(\alpha), \text{ define } \eta = \eta^{(0)} \cdots \eta^{(p-1)} \eta^{(p)}. \end{aligned}$$

(5.6) THEOREM. Suppose (X, α) is a simplicial pair such that if S is a simplex of X there is a vertex v of S such that if $v \in A^i$ then $S \subset A^i$. Any function f assigning to each vertex v of X a vertex \bar{A}^i of X_α with $v \in A^i$ is a simplicial map of X into X_α . The homomorphism $f_*:H_p(X) \rightarrow H_p(\alpha)$ is given by $f_* = \epsilon \eta$, where $\epsilon = (-1)^{p(p+1)/2}$.

Proof. For a vertex v of X , Λv denotes the simplex of X_α whose vertices are those \bar{A}^i with $v \in A^i$. Suppose $v \sim v'$ if and only if $\Lambda v = \Lambda v'$. This is an equivalence relation. Suppose the elements of each equivalence class are simply ordered by an ordering $>$. Suppose also that $v > v'$ whenever Λv contains $\Lambda v'$ as a proper face. We have then a partial ordering $>$ on the vertices of X

such that if $v > v'$ then $\Lambda v \supset \Lambda v'$, if $\Lambda v = \Lambda v'$ either $v > v'$ or $v' > v$, if Λv contains $\Lambda v'$ properly then $v > v'$.

Suppose S is a simplex of X . There is a vertex v such that if v' is any other vertex of S then $\Lambda v \subset \Lambda v'$. It follows that there is a vertex v_0 of S (with $\Lambda v_0 = \Lambda v$) such that $v_0 < v'$ for all vertices v' of S . It then follows that the vertices of S may be ordered uniquely as $v_0 < v_1 < \dots < v_p$. In this proof, all oriented simplices $S = (v_0 \dots v_p)$ will have $v_0 < \dots < v_p$. If $v_0 \in A^i$ then $S \subset A^i$. Also if $v_i \in A^i$ then $v_p \in A^i$.

Consider f as in the statement of the theorem. If $S = (v_0 \dots v_p)$ then $f(v_j)$ is a vertex \tilde{A}^{i_j} with $v_j \in A^{i_0}$. Then $v_p \in A^{i_0} \cap \dots \cap A^{i_p}$ and f is simplicial.

A vertex of X_α will be denoted by a symbol of the form w_j . Define a homomorphism $\mathfrak{J}: C_{p,q} \rightarrow C_{p,q+1}$, $p \geq 0$, by

$$g(v_0 \dots v_p) \cdot (w_0 \dots w_q) \rightarrow g(v_0 \dots v_p) \cdot (f(v_0)w_0 \dots w_q)$$

and linearity. If $S = (v_0 \dots v_p)$ then $(w_0 \dots w_q) \subset \Lambda S$ and $f(v_0) \in \Lambda S$. Since ΛS is a simplex, the definition has meaning.

It may be verified that $D\mathfrak{J} + \mathfrak{J}D = \text{identity}$. Hence if $D(z) = 0$ then $D(T(z)) = z$. Hence (*) if z represents an element x of $K_{p,q}$, $p \geq 0$, then $\Delta T(z)$ represents $\eta^{(p)}(x)$ in $K_{p-1,q+1}$.

Let $z = \sum g_i(v_{0i} \dots v_{pi})$ represent an element x of $K_{p,-1} = H_p(X)$. By (*), $\eta^{(p)}(x)$ in $K_{p-1,0}$ is represented by

$$\begin{aligned} \sum g_i \partial(v_{0i} \dots v_{pi}) \cdot f(v_{0i}) \\ = \sum g_i(v_{1i} \dots v_{pi}) \cdot f(v_{0i}) + \sum_{j>0} (-1)^j g_i(v_{0i} \dots \hat{v}_{ji} \dots v_{pi}) \cdot f(v_{0i}). \end{aligned}$$

By (*) again, $\eta^{(p-1)}\eta^{(p)}(x)$ in $K_{p-1,2}$ is represented by

$$\begin{aligned} \sum g_i \partial(v_{1i} \dots v_{pi}) \cdot (f(v_{1i})f(v_{0i})) \\ = \sum g_i(v_{2i} \dots v_{pi}) \cdot (f(v_{1i})f(v_{0i})) \\ + \sum_{j>1} (-1)^{j-1} g_i(v_{1i} \dots \hat{v}_{ji} \dots v_{pi}) \cdot (f(v_{1i})f(v_{0i})). \end{aligned}$$

Continuing, the element $\eta^{(1)} \dots \eta^{(p)}(x)$ in $K_{0,p-1}$ is represented by

$$\begin{aligned} \sum g_i \partial(v_{p-1i} v_{pi}) (f(v_{p-1i}) \dots f(v_{0i})) \\ = \sum g_i v_{pi} \cdot (f(v_{p-1i}) \dots f(v_{0i})) - \sum g_i v_{p-1i} \cdot (f(v_{p-1i}) \dots f(v_{0i})). \end{aligned}$$

Hence $\eta(x)$ is represented by

$$\sum g_i S_{-1} \cdot (f(v_{pi}) \dots f(v_{0i})) = (-1)^{p(p+1)/2} f(z).$$

The theorem follows for $p \geq 0$. For $p = -1$ the theorem is trivial.

If (X, α) and (Y, β) are simplicial pairs then a map $F: (X, \alpha) \rightarrow (Y, \beta)$ is a pair $F = (f, g)$ consisting of a simplicial map $f: X \rightarrow Y$ and a function $g: \alpha \rightarrow \beta$ which assigns to each A^i in α an element $gA^i = B^i$ of β with $fA^i \subset gA^i$. The cor-

responding simplicial map from X_α to Y_β will also be denoted by g . Suppose $S_p \subset \cap T_q$ where T_q is a simplex of X_α . Then $f(S_p) \subset \cap gT_q$. If A is a chain of X then fA will denote its image under f . *There is the function*

$$\sum A^i \cdot T_q^i \rightarrow \sum fA^i \cdot gT_q^i$$

of $C_{p,q}(\alpha)$ into $C_{p,q}(\beta)$, which we denote by F_c . It may be seen that $DF_c = F_c D$. Hence F_c maps $E_{p,q}(\alpha)$ into $E_{p,q}(\beta)$. Moreover, since $\Delta F_c = F_c \Delta$, there is induced a map $F_k: K_{p,q}(\alpha) \rightarrow K_{p,q}(\beta)$. Similarly F_c induces a map $F_h: H_{p,q}(\alpha) \rightarrow H_{p,q}(\beta)$. The following may be seen.

(5.7) *There is commutativity in each rectangle of*

$$\begin{array}{ccccccc} \cdots & \rightarrow & K_{p,q}(\alpha) & \rightarrow & H_{p,q}(\alpha) & \rightarrow & K_{p,q-1}(\alpha) \rightarrow \cdots \\ & & \downarrow F_k & & \downarrow F_h & & \downarrow F_k \\ \cdots & \rightarrow & K_{p,q}(\beta) & \rightarrow & H_{p,q}(\beta) & \rightarrow & K_{p,q-1}(\beta) \rightarrow \cdots \end{array}$$

Moreover, $F_k: K_{p,-1}(\alpha) \rightarrow K_{p,-1}(\beta)$ coincides with $f_*: H_p(X) \rightarrow H_p(Y)$ and $F_k: K_{-1,q}(\alpha) \rightarrow K_{-1,q}(\beta)$ with $g_*: H_q(\alpha) \rightarrow H_q(\beta)$.

(5.8) *Suppose that $F = (f, g)$ and $F' = (f', g)$ are two maps of (X, α) into (Y, β) such that given a simplex S of $A^i \in \alpha$ then there is a simplex S' of $gA^i \in \beta$ such that $f(S) \cup f'(S) \subset S'$. Then $F_h = F'_h$ and $F_k = F'_k$.*

Proof. For a simplex S of X suppose that S' is the smallest simplex of Y containing $f(S) \cup f'(S)$. If $S \in \cap T_q$ then $S' \in \cap gT_q$. Define $\mathfrak{J}: C_{p,q}(\alpha) \rightarrow C_{p+1,q}(\beta)$, $p \geq 0$, by

$$g(v_0 \cdots v_p) \cdot T_q \rightarrow \left[\sum (-1)^i g(f(v_0) \cdots f(v_i) f'(v_i) \cdots f'(v_p)) \right] \cdot gT_q$$

and linearity. Then $\Delta \mathfrak{J} + \mathfrak{J} \Delta = F'_c - F_c$ for $p \geq 1$, $\Delta \mathfrak{J} = F'_c - F_c$ for $p = 0$. Moreover, $D\mathfrak{J} = \mathfrak{J}D$. Hence $\mathfrak{J}: E_{p,q}(\alpha) \rightarrow E_{p+1,q}(\beta)$, $p \geq 0$. It follows that $F_h = F'_h$ and $F_k = F'_k$ for $p \geq 0$.

The case $p = -1$ remains. On $C_{-1,p}$, $F_c = F'_c$. Hence the conclusion follows for $p = -1$.

The following theorem may be easily proved.

(5.9) *If*

$$(X, \alpha) \xrightarrow{F} (Y, \beta) \xrightarrow{F'} (Z, \gamma)$$

then $(F'F)_h = F'_h F_h$ and $(F'F)_k = F'_k F_k$.

6. The Kelley-Pitcher theory of compact pairs. In this section we extend the Kelley-Pitcher theory to pairs (X, α) , where X is a compact space and $\alpha = (A^1, \cdots, A^r)$ a collection of closed subsets of X which covers X . We begin the section by summarizing the results.

For such a compact pair (X, α) , define $H_{p,q}(\alpha) = \sum H_p(\cap T_q)$, where the summation is over all q -simplices $T_q = (\bar{A}^{i_0} \cdots \bar{A}^{i_q})$ of X_α with $i_0 < \cdots < i_q$. If T_q is a simplex as above and if $A_p \in H_p(\cap T_q)$ then denote by $A_p \cdot T_q$ the

element of $H_{p,q}$ whose T_q -coordinate is A_p and all of whose other coordinates are 0. Agree also that $A_p \cdot (-T_q) = (-A_p) \cdot T_q$. Then $H_{p,q}$ is the set of linear forms $\sum A_p^i \cdot T_q^i$ where $A_p^i \in H_p(\cap T_q^i)$.

A map F of a pair (X, α) into a pair (Y, β) will be a pair $F = (f, g)$ where $f: X \rightarrow Y$ is continuous and where $g: \alpha \rightarrow \beta$ assigns to each A^i in α an element $gA^i = B^i$ in β such that $fA^i \subset gA^i$. The simplicial map of X_α into Y_β will also be denoted by g . If T_q is an unoriented simplex of X_α then $f(\cap T_q) \subset \cap gT_q$. The map f of $\cap T_q$ into $\cap gT_q$ is denoted by $f| \cap T_q$. There is the induced homomorphism $(f| \cap T_q)_*: H_p(\cap T_q) \rightarrow H_p(\cap gT_q)$. A homomorphism $F_h: H_{p,q}(\alpha) \rightarrow H_{p,q}(\beta)$ is defined by $A_p \cdot T_q \rightarrow [(f| \cap T_q)_* A_p] \cdot gT_q$ and linearity.

If

$$(X, \alpha) \xrightarrow{F} (Y, \beta) \xrightarrow{F'} (Z, \gamma)$$

where $F = (f, g)$ and $F' = (f', g')$ then the composition $F'F: (X, \alpha) \rightarrow (Z, \gamma)$ is defined to be $(f'f, g'g)$.

We define for each (X, α) a group $K_{p,q}(\alpha)$; a map $F: (X, \alpha) \rightarrow (Y, \beta)$ will induce a homomorphism $F_k: K_{p,q}(\alpha) \rightarrow K_{p,q}(\beta)$. There will also be homomorphisms

$$K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q}.$$

The following theorems constitute the theory for compact pairs, and are proved later in the section.

THEOREM I. For each (X, α) and each q the sequence

$$\cdots \rightarrow K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q} \xrightarrow{l} \cdots \rightarrow K_{0,q-1} \xrightarrow{\eta^{(0)}} K_{-1,q} \rightarrow 0$$

is exact.

THEOREM II. For each (X, α) we have $K_{p,-1} = H_p(X)$ and $K_{-1,q} = H_p(\alpha)$.

THEOREM III. The homomorphism $\eta = \eta^{(0)} \cdots \eta^{(p)}$ of $H_p(X)$ into $H_p(\alpha)$ given by

$$H_p(X) = K_{p,-1} \xrightarrow{\eta^{(p)}} K_{p-1,0} \rightarrow \cdots \rightarrow K_{0,p-1} \xrightarrow{\eta^{(0)}} K_{-1,p} = H_p(\alpha)$$

is such that $\eta = (-1)^{p(p+1)/2} \pi_\alpha$, where π_α is the projection homomorphism of $H_p(X)$ into $H_p(\alpha)$.

THEOREM IV. If

$$(X, \alpha) \xrightarrow{F} (Y, \beta) \xrightarrow{F'} (Z, \gamma)$$

then $(F'F)_h = F'_h F_h$ and $(F'F)_k = F'_k F_k$.

THEOREM V. If $F = (f, g)$ maps (X, α) into (Y, β) then $F_k: K_{p,-1}(\alpha) \rightarrow K_{p,-1}(\beta)$

coincides with $f_*: H_p(X) \rightarrow H_p(Y)$; also $F_k: K_{-1,q}(\alpha) \rightarrow K_{-1,q}(\beta)$ coincides with $g_*: H_q(\alpha) \rightarrow H_q(\beta)$.

THEOREM VI. If $F: (X, \alpha) \rightarrow (Y, \beta)$, there is commutativity in each rectangle of

$$\begin{array}{ccccccc} \cdots & \rightarrow & K_{p,q}(\alpha) & \xrightarrow{l} & H_{p,q}(\alpha) & \xrightarrow{m} & K_{p,q-1}(\alpha) \xrightarrow{\eta^{(p)}} \cdots \\ & & \downarrow F_k & & \downarrow F_k & & \downarrow F_k \\ \cdots & \rightarrow & K_{p,q}(\beta) & \xrightarrow{l} & H_{p,q}(\beta) & \xrightarrow{m} & K_{p,q-1}(\beta) \longrightarrow \cdots \end{array}$$

We now begin the proofs.

(6.1) LEMMA. Suppose that X is a compact Hausdorff space, and that $\alpha = (A^1, \dots, A^r)$ is a closed covering of X . If $\beta = (B^1, \dots, B^s)$ is a collection of closed subsets of X , there is a collection $\gamma = (C^1, \dots, C^s)$ of closed subsets of X with C^i containing B^i in its interior, $i=1, \dots, s$, and with a set of the type $C^{i_0} \cap \dots \cap C^{i_p}$ intersecting some A^i if and only if $B^{i_0} \cap \dots \cap B^{i_p}$ intersects A^i .

Proof. It is sufficient to prove that if $1 \leq k \leq s$ then there is a $\gamma = (C^1, \dots, C^s)$ with $C^i = B^i$ for $i \neq k$, C^k containing B^k in its interior, and $C^{i_0} \cap \dots \cap C^{i_p} \cap A^i \neq \emptyset$ if and only if $B^{i_0} \cap \dots \cap B^{i_p} \cap A^i \neq \emptyset$. Suppose that C^k contains B^k in its interior and that C^k intersects any set of the type $B^{i_0} \cap \dots \cap B^{i_p} \cap A^i$ if and only if B^k intersects $B^{i_0} \cap \dots \cap B^{i_p} \cap A^i$. Let $C^i = B^i$ for $i \neq k$. Then $C^{i_0} \cap \dots \cap C^{i_p} \cap A^i \neq \emptyset$ if and only if $B^{i_0} \cap \dots \cap B^{i_p} \cap A^i \neq \emptyset$. The assertion follows.

(6.2) DEFINITION. Suppose that (X, α) is a compact pair. Then a finite open covering u of X is a *special* covering of (X, α) if and only if

(i) whenever U^{i_0}, \dots, U^{i_p} are elements of u with $U^{i_0} \cap \dots \cap U^{i_p}$ intersecting each of A^{i_0}, \dots, A^{i_p} then $U^{i_0} \cap \dots \cap U^{i_p}$ intersects $A^{i_0} \cap \dots \cap A^{i_p}$;

(ii) whenever U^{i_0}, \dots, U^{i_p} are elements of u with $U^{i_0} \cap \dots \cap U^{i_p} \neq \emptyset$, there is an element U^{i_k} such that if $U^{i_k} \cap A^i \neq \emptyset$ then $U^{i_0} \cap \dots \cap U^{i_p} \cap A^i \neq \emptyset$. The set of all such special coverings will be denoted by $U(\alpha)$.

(6.3) THEOREM. Suppose that (X, α) is a compact pair. If v is an open covering of X , there is a special open covering u of (X, α) which refines v .

Proof. Let Y^m consist of all $x \in X$ with $x \in A^i$ for at least m values of i . Then Y^m is closed, $Y^{m+1} \subset Y^m$ and $Y^1 = X$. We prove the following by induction for $r \geq m \geq 1$.

L_m : there exists a covering $\beta_m = (B^1, \dots, B^s)$ of Y^m by closed subsets of Y^m such that β_m refines v and such that β_m satisfies (i) and (ii) of the definition of a special covering with the U 's replaced by B 's.

L_r may be seen to be true. Suppose L_m is true and that $\beta_m = (B^1, \dots, B^s)$ is the desired covering of Y^m . By Lemma 6.1, there is a collection $\gamma = (C^1, \dots, C^s)$ of closed subsets of Y^{m-1} with C^i containing B^i in its interior

relative to Y^{m-1} and with $C^{i_0} \cap \cdots \cap C^{i_p}$ intersecting A^j if and only if $B^{i_0} \cap \cdots \cap B^{i_p}$ intersects A^j . It may be seen that γ satisfies (i) and (ii) of Definition 6.2, since β_m does. We may also suppose γ sufficiently small so that γ refines v .

Now suppose $x \in Y^{m-1} - Y^m$. Then x is contained in exactly $m-1$ A^j 's. There is also a neighborhood of x relative to Y^{m-1} every point of which is contained in exactly those A^j 's which contain x . Hence we may expand γ to a covering

$$\beta_{m-1} = (C^1, \dots, C^s, C^{s+1}, \dots, C^{s+t})$$

of Y^{m-1} by closed sets, where all points of C^{s+i} are contained in exactly the same A^j 's, and where β_{m-1} refines v . Suppose $C^{i_0} \cap \cdots \cap C^{i_p}$ intersects each of A^{j_0}, \dots, A^{j_q} . If $i_0, \dots, i_p \leq s$, then $C^{i_0} \cap \cdots \cap C^{i_p}$ intersects $A^{j_0} \cap \cdots \cap A^{j_q}$ since γ satisfied (i). If some $i_k > s$ then A^{j_0}, \dots, A^{j_q} must each be one of the A^j 's which contains C^{i_k} . Then

$$A^{j_0} \cap \cdots \cap A^{j_q} \supset C^{i_k} \supset C^{i_0} \cap \cdots \cap C^{i_p}.$$

Hence β_{m-1} satisfies (i).

Suppose now that $C^{i_0} \cap \cdots \cap C^{i_p} \neq \emptyset$. If $i_0, \dots, i_p \leq s$, there is a C^{i_k} such that if $C^{i_k} \cap A^j \neq \emptyset$ then $C^{i_0} \cap \cdots \cap C^{i_p} \cap A^j \neq \emptyset$, since γ satisfies (ii). Suppose $i_k > s$. If $C^{i_k} \cap A^j \neq \emptyset$ then $A^j \supset C^{i_k}$. Then $C^{i_0} \cap \cdots \cap C^{i_p} \cap A^j = C^{i_0} \cap \cdots \cap C^{i_p} \neq \emptyset$. Hence β_{m-1} satisfies (ii). The induction is complete.

Now L_1 is true, with $\beta_1 = (B^1, \dots, B^s)$. Let $u = (U^1, \dots, U^s)$ be such that u refines v , $U^i \supset B^i$, and such that $U^{i_0} \cap \cdots \cap U^{i_p}$ intersects A^j if and only if $B^{i_0} \cap \cdots \cap B^{i_p}$ intersects A^j . Then u is a special covering, and the theorem follows.

Suppose that (X, α) is a compact pair. For each special covering $u \in U(\alpha)$ there is the nerve X_u . There is also a covering $\alpha_u = (A_u^1, \dots, A_u^r)$ of X_u , where A_u^i consists of all simplexes $\tau^p = (\bar{U}^0 \cdots \bar{U}^p)$ of X_u with $U^0 \cap \cdots \cap U^p \cap A^i \neq \emptyset$. Furthermore, from (i) of Definition 6.2, $A_u^{i_0} \cap \cdots \cap A_u^{i_q}$ consists of those simplexes τ^p of X_u with $U^0 \cap \cdots \cap U^p$ intersecting $A^{i_0} \cap \cdots \cap A^{i_q}$. In particular, the nerve $(X_u)_{\alpha_u}$ of α_u coincides with the nerve X_α . If $T_q = (\bar{A}^{i_0} \cdots \bar{A}^{i_q})$ is a simplex of X_α then $T_q^u = (\bar{A}_u^{i_0} \cdots \bar{A}_u^{i_q})$ is a simplex of $(X_u)_{\alpha_u}$ and $T_q = T_q^u$. Moreover, $\cap T_q^u = A_u^{i_0} \cap \cdots \cap A_u^{i_q}$ consists of all those simplexes τ^p of X_u with $U^0 \cap \cdots \cap U^p$ intersecting $\cap T_q$.

If $u, v \in U(\alpha)$ and v refines u , let π_{vu} denote a projection of X_v into X_u . It may be checked that π_{vu} maps $\cap T_q^v$ into $\cap T_q^u$; this map is denoted by $\pi_{vu}| \cap T_q^v$. It follows that (π_{vu}, id) , id the identity function, maps the simplicial pair (X_v, α_v) into (X_u, α_u) . Define $\tilde{F}^{vu} = (\pi_{vu}, id)$.

(6.4) For a simplex T_q of X_α , the limit group of the inverse system $[H_q(\cap T_q^u), (\pi_{vu}| \cap T_q^v)_*]$, indexed by $U(\alpha)$, is $H_p(\cap T_q)$.

Proof. We have that $\cap T_q^u$ consists of all simplexes of X_u whose intersection intersects $\cap T_q$. Also, $\pi_{vu}| \cap T_q^v$ is a projection of $\cap T_q^v$ into $\cap T_q^u$. Since $U(\alpha)$ is

cofinal in the set of all open coverings of X , the limit group may be identified with $H_p(\cap T_q)$.

(6.5) If $v, u \in U(\alpha)$ and v refines u , then $(\tilde{F}^{vu})_h = \tilde{F}_h^{vu}$ and $(\tilde{F}^{vu})_k = \tilde{F}_k^{vu}$ are independent of the projection π_{vu} . If also w refines v then $\tilde{F}_h^{vu} \tilde{F}_h^{wv} = \tilde{F}_h^{wu}$ and $\tilde{F}_k^{vu} \tilde{F}_k^{wv} = \tilde{F}_k^{wu}$.

Proof. If π_{vu} and π'_{vu} are projections of X_\bullet into X_u , then the maps (π_{vu}, id) and (π'_{vu}, id) satisfy the hypothesis of (5.8). Hence F_h^{vu} and F_k^{vu} are independent of the projection.

(6.6) The limit group of the inverse system $[H_{p,q}(\alpha_u), \tilde{F}_h^{vu}]$, indexed by $U(\alpha)$, is isomorphic with $H_{p,q}(\alpha)$. Henceforth we identify the two.

Proof. Consider an element $A_p \cdot T_q$ of $H_{p,q}(\alpha)$, where $A_p \in H_p(\cap T_q)$. According to (6.4), $A_p = (A_p(u) : u \in U(\alpha))$, where $A_p(u)$ is the coordinate of A_p in $H_p(\cap T_q^u)$, and where, if v refines u , then $(\pi_{vu} | \cap T_q^u) * A_p(v) = A_p(u)$. For each v , $A_p(v) \cdot T_q^v$ is an element of $H_{p,q}(\alpha_v)$ and if v refines u then \tilde{F}_h^{vu} maps $A_p(v) \cdot T_q^v$ into $A_p(u) \cdot T_q^u$. The correspondence

$$A_p \cdot T_q \rightarrow (A_p(u) \cdot T_q^u : u \in U(\alpha))$$

together with linearity, yields an isomorphism of $H_{p,q}(\alpha)$ onto the limit group.

(6.7) DEFINITION. For a pair (X, α) , $K_{p,q}(\alpha)$ is defined to be the limit group of the inverse system $[K_{p,q}(\alpha_u), \tilde{F}_k^{vu}]$, indexed by $U(\alpha)$. The maps

$$K_{p,q}(\alpha) \xrightarrow{l} H_{p,q}(\alpha) \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q}(\alpha)$$

are defined as limits of

$$K_{p,q}(\alpha_u) \xrightarrow{l} H_{p,q}(\alpha_u) \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q}(\alpha_u).$$

Theorem I then follows from (5.5), together with the fact that the groups of α_u are either compact or finite-dimensional vector spaces.

THEOREM II. We have $K_{p,-1}(\alpha) = H_p(X)$, $K_{-1,q}(\alpha) = H_q(\alpha)$.

Proof. Consider the inverse system $[K_{p,-1}(\alpha_u), \tilde{F}_k^{vu}]$. According to (5.3), $K_{p,-1}(\alpha_u) = H_p(u)$. According to (5.7), $F_k : K_{p,-1}(\alpha_v) \rightarrow K_{p,-1}(\alpha_u)$ coincides with $\pi_{vu} : H_p(v) \rightarrow H_p(u)$. Hence the inverse system coincides with $[H_p(u), \pi_{vu}]$ and $K_{p,-1} = H_p(X)$. Consider now the inverse system $[K_{-1,q}(\alpha_u), \tilde{F}_k^{vu}]$. Now $K_{-1,q}(\alpha_u) = H_q((X_u)\alpha_u) = H_q(u)$ and \tilde{F}_k^{vu} is the identity by (5.7). Hence $K_{-1,q}$ may be identified with $H_q(\alpha)$.

THEOREM III. The homomorphism $\eta : H_p(X) \rightarrow H_p(\alpha)$ is such that $\eta = (-1)^{p(p+1)/2} \pi_\alpha$.

Proof. Let W be an open covering (W^0, \dots, W^r) of X where $W^i \supset A^i$ and where $W^{i_0} \cap \dots \cap W^{i_r} \neq \emptyset$ if and only if $A^{i_0} \cap \dots \cap A^{i_r} \neq \emptyset$. Then $X_w = X_\alpha$ and $\pi_w = \pi_\alpha$. Consider any special covering u of X such that if U is

an element of u and $U \cap A^i \neq \emptyset$ then $U \subset W^i$. Since u is a special covering, if $U^0 \cap \cdots \cap U^p \neq \emptyset$, where the U^i 's are elements of u , there is an element U^i such that if $U^i \cap A^i \neq \emptyset$ then $U^0 \cap \cdots \cap U^p \cap A_i \neq \emptyset$. It follows that (X_u, α_u) satisfies the hypothesis of (5.6). Let f be a function assigning to each vertex U of X_u an element A_u^i of α_u with $U \cap A^i \neq \emptyset$ (that is, U is a vertex of A_u^i). According to (5.6), $f: X_u \rightarrow (X_u)\alpha_u = X_\alpha$ is simplicial and $f_*: H_p(X_u) \rightarrow H_p((X_u)\alpha_u)$ is such that $f_* = (-1)^{p(p+1)/2} \eta_u$ where η_u is the composition

$$H_p(X_u) = K_{p,-1}(\alpha_u) \xrightarrow{\eta^{(p)}} \cdots \xrightarrow{\eta^{(0)}} K_{-1,p}(\alpha_u) = H_p((X_u)\alpha_u).$$

In the diagram

$$X_u \xrightarrow{f} (X_u)\alpha_u = X_\alpha = X_w$$

note that u and w are chosen so that $f: X_u \rightarrow X_w$ is a projection π_{uw} of X_u into X_w . So $\pi_{uw*} = f_*$ and

$$\pi_{uw*} = (-1)^{p(p+1)/2} \eta_u, \quad \pi_w = (-1)^{p(p+1)/2} \eta.$$

Since $X_\alpha = X_w$ and $\pi_\alpha = \pi_w$, the theorem follows.

(6.8) DEFINITION. Suppose $F = (f, g)$ maps the compact pair (X, α) into the compact pair (Y, β) where $\beta = (B^1, \cdots, B^r)$. Denote by $P(F)$ the set of all pairs (u, v) , where $u \in U(\alpha)$ and $v \in U(\beta)$ and where if U is an element of u then there is an element V of v with $f(U) \subset V$. Let $f_{uv}: X_u \rightarrow X_v$ assign to each vertex U of X_u a vertex V of X_v with $f(U) \subset V$. Then f_{uv} is simplicial.

There are the coverings $\alpha_u = (A_u^1, \cdots, A_u^r)$ of X and $\beta_v = (B_v^1, \cdots, B_v^r)$ of Y . Moreover $(X_u)\alpha_u = X_\alpha$ and $(Y_v)\beta_v = Y_\beta$. It may be seen that $F^{uv} = (f_{uv}, g)$ maps the pair (X_u, α_u) into the pair (Y_v, β_v) . Moreover, by (5.8), $F_h^{uv}: H_{p,q}(\alpha_u) \rightarrow H_{p,q}(\beta_v)$ and F_k^{uv} are independent of the particular f_{uv} chosen. Also, if $u' \in U(\alpha)$ and u' refines u we have $\tilde{F}^{u'u} = (\pi_{u'u}, id)$ mapping $(X_{u'}, \alpha_{u'})$ into (X_u, α_u) . If v refines $v' \in U(\beta)$ there is $F^{vv'} = (\pi_{vv'}, id)$ mapping (Y_v, β_v) into $(Y_{v'}, \beta_{v'})$. Then $\tilde{F}^{vv'} F^{uv} \tilde{F}^{u'u} = F^{u'v'}$. Hence

$$(*) \quad \tilde{F}_h^{vv'} F_h^{uv} \tilde{F}_h^{u'u} = F_h^{u'v'}, \quad \tilde{F}_k^{vv'} F_k^{uv} \tilde{F}_k^{u'u} = F_k^{u'v'}.$$

Consider the inverse systems $[K_{p,q}(\alpha_u), \tilde{F}_k^{u'u}]$, indexed by $U(\alpha)$, and $[K_{p,q}(\beta_v), \tilde{F}_k^{vv'}]$, indexed by $U(\beta)$. For $(u, v) \in P(F)$, there is the map $F_k^{uv}: K_{p,q}(\alpha_u) \rightarrow K_{p,q}(\beta_v)$ such that $(*)$ holds. There is induced, in the limit, a map $F_k: K_{p,q}(\alpha) \rightarrow K_{p,q}(\beta)$.

(6.9) The limit of the map $F_h^{uv}: H_{p,q}(\alpha_u) \rightarrow H_{p,q}(\beta_v)$, $(u, v) \in P(F)$, is the map F_h defined earlier.

Proof. It is sufficient to check the equality of the two maps on elements $A_p \cdot T_q$, T_q an oriented q -simplex of α and $A_p \in H_p(\cap T_q)$. Now F_h maps $A_p \cdot T_q$ into $[(f| \cap T_q) * A_p] \cdot gT_q$. Also, according to (6.6), $A_p \cdot T_q = (A_p(u) \cdot T_q^u: u \in U(\alpha))$. Now F_h^{uv} maps $A_p(u) \cdot T_q^u$ into $[(f_{uv}| \cap T_q^u) * A_p(u)] \cdot gT_q^u$. This is the coordinate in $H_{p,q}(\beta_v)$ of $[(f| \cap T_q) * A_p] \cdot gT_q$.

We leave the proofs of Theorems IV, V, and VI to the reader.

7. Proof of the basic theorem.

THEOREM. Suppose that $A_{-1} \supset A_0 \supset \cdots \supset A_n$ is a sequence of closed subsets of a compact space X , and that $\alpha_{-1} \ll^n \cdots \ll^n \alpha_n$, where α_i is a closed covering of A^i . Then

(i) the kernel of $\pi_{\alpha_n}: H_j(A_n) \rightarrow H_j(\alpha_n)$ is contained in the kernel of the injection $I: H_j(A_n) \rightarrow H_j(A_{-1})$ for $j \leq n$;

(ii) the image of $\pi_{\alpha_n, \alpha_0}: H_j(\alpha_n) \rightarrow H_j(\alpha_0)$ is contained in the image of $\pi_{\alpha_0}: H_j(A_0) \rightarrow H_j(\alpha_0)$ for $j \leq n+1$.

Proof. For each i there is a projection $\pi = \pi_{\alpha_i, \alpha_{i-1}}$ of α_i into α_{i-1} such that $H_j(A^{i_0} \cap \cdots \cap A^{i_q}; \pi A^{i_0} \cap \cdots \cap \pi A^{i_q}) = 0$ for all $j \leq n$. Let $I'_i: A_i \rightarrow A_{i-1}$ denote the inclusion map. Then $F^i = (I'_i, \pi)$ maps (A_i, α_i) into (A_{i-1}, α_{i-1}) and $F^i_h: K_{p,q}(\alpha_i) \rightarrow K_{p,q}(\alpha_{i-1})$ is trivial for $0 \leq p \leq n$.

Consider the diagram

$$\begin{array}{ccccccc} \cdots \rightarrow H_{p,q}(\alpha_i) & \xrightarrow{m} & K_{p,q-1}(\alpha_i) & \xrightarrow{\eta^{(p)}} & K_{p-1,q}(\alpha_i) & \xrightarrow{l} & H_{p-1,q}(\alpha_i) \rightarrow \cdots \\ & \downarrow F^i_h & \downarrow F^i_k & & \downarrow F^i_k & & \downarrow F^i_h \\ \cdots \rightarrow H_{p,q}(\alpha_{i-1}) & \xrightarrow{m} & K_{p,q-1}(\alpha_{i-1}) & \xrightarrow{\eta^{(p)}} & K_{p-1,q}(\alpha_{i-1}) & \xrightarrow{l} & H_{p-1,q}(\alpha_{i-1}) \rightarrow \cdots \end{array}$$

Now (1) if $x \in K_{p,q-1}(\alpha_i)$, $p \leq n$, and $\eta^{(p)}(x) = 0$, then $F^i_k(x) = 0$. For if $\eta^{(p)}(x) = 0$ then $x = m(y)$ and $F^i_k(x) = F^i_k m(y) = m F^i_h(y) = 0$. Moreover, (2) if $x \in K_{p-1,q}(\alpha_i)$, $p \leq n+1$, then $F^i_k(x) \in K_{p-1,q}(\alpha_{i-1})$ is of the form $\eta^{(p)}(y)$ for some $y \in K_{p,q-1}(\alpha_{i-1})$.

To prove (i), suppose $\pi_{\alpha_n}(x) = 0$, $x \in H_j(A_n) = K_{j,-1}(\alpha_n)$. Then, by Theorem III, $\eta(x) = \eta^{(0)} \cdots \eta^{(j)}(x) = 0$ in $K_{-1,j}(\alpha_n) = H_j(\alpha_n)$. By (1), $F^n_k \eta^{(1)} \cdots \eta^{(j)}(x) = \eta^{(1)} \cdots \eta^{(j)} F^n_k(x) = 0$ in $K_{0,j-1}(\alpha_{n-1})$. By (1) again, $F^{n-1}_k \eta^{(2)} \cdots \eta^{(j)} F^n_k(x) = \eta^{(2)} \cdots \eta^{(j)} F^{n-1}_k F^n_k(x) = 0$ in $K_{1,j-2}(\alpha_{n-2})$. Continuing, $F^{n-j}_k \cdots F^{n-1}_k F^n_k(x) = 0$ in $K_{j,-1}(\alpha_{n-j-1})$. But $F^{n-j}_k \cdots F^n_k(x)$ is, by Theorem V, the injection I of $H_j(A_n)$ into $H_j(A_{n-j-1})$. Hence (i) follows.

To prove (2), suppose $x \in H_j(\alpha_n) = K_{-1,j}(\alpha_n)$. The map

$$K_{0,j-1} \xrightarrow{\eta^{(0)}} K_{-1,j}$$

is onto, by Theorem I, so $x = \eta^{(0)}(x_0)$, $x_0 \in K_{0,j-1}(\alpha_n)$. By (2), $F^n_k(x_0) \in K_{0,j-1}(\alpha_{n-1})$ is of the form $\eta^{(1)}(x_1)$ for some $x_1 \in K_{1,j-2}(\alpha_{n-1})$. Continuing by use of (2), $F^{n-j+2}_k(x_{j-1}) \in K_{j-1,0}(\alpha_{n-j+1})$ is of the form $\eta^{(j)}(x_j)$ for some $x \in K_{j,-1}(\alpha_{n-j+1})$. Then

$$\begin{aligned} \eta^{(0)} \cdots \eta^{(j)}(x_j) &= \eta^{(0)} \cdots \eta^{(j-1)} F^{n-j+2}_k(x_{j-1}) \\ &= F^{n-j+2}_k \eta^{(0)} \cdots \eta^{(j-2)} F^{n-j+3}_k(x_{j-2}) \\ &= F^{n-j+2}_k \cdots F^n_k(x). \end{aligned}$$

By Theorems IV and V, $F_k^{n-j+2} \cdots F_k^n$ is the projection π of $H_j(\alpha_n)$ into $H_j(\alpha_{n-j+1})$. Hence $\text{image } \pi_{\alpha_n, \alpha_{n-j+1}} \subset \text{image } \pi_{\alpha_{n-j+1}}$. For $j \leq n+1$, it follows readily that $\text{image } \pi_{\alpha_n, \alpha_0} \subset \text{image } \pi_{\alpha_0}$.

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