POSITIVITY IN H-SYSTEMS AND SUFFICIENT STATISTICS([1])

BY

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Introduction. In §1, Theorem 2.1 of [6]([3]) is improved slightly and then is used to show that nonzero, central positivity preserving projections in the H-system of a group arise in a simple way from compact normal subgroups. In §2 this theorem is applied to a problem in statistics.

1. Let G be a separable, locally compact, unimodular group, $H(G)$ its H-system (see [1] and [5] for the definition of an H-system), P the non-negative elements of $H(G)$, and $W(H(G))$ the weakly closed ring of operators generated by left convolution operators in $H(G)$. An element of $W(H(G))$ is *central* if it commutes with all right convolution operators. If x is an element of G or f is a function on G we write $l(x)$ or $l(f)$ $(r(x)$ or $r(f))$ for the corresponding left (right) translation or convolution operators.

If K is a compact normal subgroup of G and we define πf by $\pi f(x) = \int_K f(kx)dk$ where dk is normalized Haar measure on K then π is a central projection and $\pi P \subset P$. Moreover every πf corresponds to a function $i(\pi(f))$ on G/K and i gives a positivity preserving isomorphism between $\pi(H(G))$ and the H-system of G/K. The following theorem shows that every positive central projection arises from such a K in this way.

THEOREM 1.1. *Let π be a nonzero central projection in $W(H(G))$ with $\pi P \subset P$ and let $K = \{x : x \in G, \pi(x) = \pi\}$. Then K is a compact normal subgroup and $\pi f(x) = \int_K f(kx)dk$ almost everywhere for every bounded f.*

Proof. K is clearly a closed and normal subgroup. We can find a neighborhood U of the identity for which the closure S of UU^{-1} is compact and a function p with $\int_U (\pi p)^2 dx > 0$. If a sequence k_i from K is defined as follows: $k_1 = $ identity and $k_{i+1}U$ is disjoint from $\bigcup_{j=1}^i k_j U$ then we have

$$\int (\pi p)^2 dx \geq \sum_{j=1}^i \int_{k_j U} (\pi p)^2 dx = i \int_U (\pi p)^2 dx$$

so the sequence must end for some N and $K \subset \bigcup_{j=1}^N k_j S$ which is compact. Hence, if f is bounded and measurable $\rho f(x) = \int_K f(kx)dk$ is defined and ρ

can be extended to a central positivity preserving projection of $H(G)$ onto a subset isomorphic to $H(G/K)$ and satisfying $\pi\rho = \rho\pi = \pi$. $xK \to l(x)$ maps G/K onto a group of unitaries operating on $\pi H(G)$ and taking πP into itself. If we could verify the conditions of Theorem 2.1 of [6] we could infer that πH was isomorphic to $H(G/K)$ under the map $i: i(f(xK)) = f(x)$, and this would complete the proof since ρH is also isomorphic to $H(G/K)$ under the same map. However there seems no easy way to verify condition (3) in the definition of an HP system [6, p. 485] so we are forced to prove an alternate version of the theorem.

Let $C = \{ \sum_{i=1}^{n} p_i q_i; \; p_i, \; q_i \in \pi P$ and the products involved are defined$\}$ and $[\sum_{i=1}^{n} p_i q_i]$ be the function on G/K defined by:

$$\left[\sum_{i=1}^{n} p_i q_i \right](xK) = \sum_{i=1}^{n} (p_i, \, l(x)q_i).$$

LEMMA 1.1. (1) *πH can be represented as the L_2 of some measure space in such a way that πP corresponds to the almost everywhere non-negative functions in this L_2.*

(2) *$\pi P = (\pi P)^*$.*

(3) *If $f_i, \; h \in C$ and there is an $f \in C$ with $f_i \leq f$ $(f - f_i \in \pi P)$ for all i and $\sup [f_i] \geq [h]$ then $\sup f_i \geq h$.*

(4) *There are approximate identities for $\pi H(G)$ in πP and $\pi P \cap A$ is dense in πP, where A is the bounded algebra of $H(G)$ (see [1]).*

(5) *$C \subset \pi P$.*

Proof. If πp and πq are in πP then $(\pi p, \pi q) \geq 0$ since $\pi P \subset P$ and if $(\pi p, \pi x) \geq 0$ for all $\pi p \in \pi P$ then $(p, \pi x) \geq 0$ for all $p \in P$ so $\pi x \in P$ so $\pi x \in \pi P$. Hence by a theorem of Nagy [4] (1) will be verified if we show that whenever $\pi p_1, \pi p_2, \pi q_1$, and πq_2 are in πP and satisfy $\pi p_1 + \pi p_2 = \pi q_1 + \pi q_2$ we can find w_{ij} $(i, j = 1, 2)$ in πP to satisfy $\pi p_i = w_{i1} + w_{i2}$, $\pi q_i = w_{1i} + w_{2i}$. But since $\pi p_i, \pi q_i$ are in P we can find u_{ij} in P to satisfy these equations and then set $w_{ij} = \pi u_{ij}$.

For p in P, $(\pi(p^*)f, \, g) = (\pi(p^*x), \, y) = (p^*x, \, \pi y) = (x, \, p(\pi y)) = (x, \, \pi(p)y)$ whenever the products involved are defined which shows that $\pi(p^*) = \pi(p)^*$ so that $\pi P = \pi(P^*) = (\pi P)^*$ verifying (2). (3) and (5) are trivial and the approximate identities in (4) can be obtained by applying π to a set of approximate identities for $H(G)$ in P.

The proof of Theorem 1.1 is completed by:

THEOREM 1.2. *If H is an H-system with a subset P and G is a group of unitaries commuting with right multiplications in H, and H, P, G satisfy the conditions (1) through (5) imposed on $(\pi H, \pi P, G/K)$ by the previous lemma then H, P is isomorphic to the H-system of G under an extension of the map $[\;]$ of C onto functions on G.*

Proof. We first prove the following lemma.

LEMMA 1.2. *If D is a strongly closed subset of G and U is an element not in D then there is an f in C with $[f](U) = 1$ and $[f](V) \leq 1 - \alpha < 1$ for all V in D.*

Proof. We can take U to be the identity. For some p_i's in $P \cap A$ the strong neighborhood $\{ V; \| V p_i - p_i \| < \beta, i = 1, \cdots, n \}$ is disjoint from D and if we take

$$ f = \sum_{i=1}^{n} p_i p_i^* \bigg/ \sum_{i=1}^{n} \| p_i \|^2 $$

we have $[f](I) = 1$ and

$$ [f](V) = \sum_{i=1}^{n} (p_i, V p_i) \bigg/ \sum_{i=1}^{n} \| p_i \|^2 \leq \sum_{i=1}^{n} \left(\| p_i \|^2 - \frac{1}{2} \beta^2 \right) \bigg/ \sum_{i=1}^{n} \| p_i \|^2. $$

From now on the proof follows the proof of Theorem 2.1 in [6], using the definition of C given above instead of that in [6]. We shall only indicate the changes that are necessary. Lemma 2.1 is no longer needed and Lemma 2.2 is trivial. Lemma 2.3 requires the following result.

LEMMA 1.3. *If B and B' are closed, bounded, and disjoint then*

$$ \inf (d(B), d(B')) = 0. $$

Proof. For a V in B' choose an f in C with $[f](V) = 1 + \alpha$ and $[f](B) \leq 1 - \alpha$, and let $M(V) = \{ W; f(W) > 1 \}$. For each U in B choose a g in C with $[g](U) = 1$ and $[g](M(V)) \leq 1 - \delta$. Then if e_n is a set of approximate identities in P we have, on $M(V)$, $[(f - \inf (f, g)) e_n] \geq [(f - g) e_n]$ which converges uniformly to $[f] - [g] \geq \delta$ and at U, $[(g - \inf (f, g)) e_n] \geq [(g - f) e_n]$ which converges to $[g] - [f] \geq \alpha$ so that $2/\delta (f - \inf (f, g)) e_n$ covers $M(V)$ and $2/a (g - \inf (f \cdot g)) e_n$ covers a neighborhood $N(U)$ of U for large enough n. Hence $(d(M(V)), d(N(U))) \leq (4/\alpha\delta)((f - \inf (f, g)) e_n, (g - \inf (f, g)) e_n)$ which converges to 0 so that if $N(U_i)$ is a countable subcovering of $B(d(M(V)), d(B)) \leq (d(M(V)),$ $\sup d(N(U_i))) = 0$ and then finally by a similar argument $(d(B'), d(B)) = 0$.

Lemma 2.4 and 2.5 go as before but for Lemma 2.6 we need to know that if S is open and bounded then $d(S) \neq 0$. This is an obvious corollary of the following lemma.

LEMMA 1.4. *If S is open and bounded, p is in P and for every U in G $(p, Ud(S)) = 0$ then $p = 0$.*

Proof. It will be sufficient to prove $(p, f) = 0$ for all f in C. We can find a countable covering S_n of G with $S_n \subset U_n S$ for some U_n in G and $a_n = \sup f(x)$ for x in $S_n \leq 2 \inf f(x)$ for x in S_n. Then $a_n d(S_n) \leq 2f$ so $a = \sup_n a_n d(S_n) \leq 2f$. Choose a set g_{nj} from C so that $(g_{nj})_{j=1}^{\infty}$ covers S_n and $\| \sup_j g_{nj} - d(S_n) \| \leq (\beta/2^n)$ so that

$$\sup_{n,j} a_n g_{nj} \leq 2f + \sum_{n=1}^{\infty} a_n \left(\sup_j g_{nj} - d(S_n) \right)$$

where the last sum exists and has norm $\leq \beta$. Hence $f \leq \sup a_n g_{nj}$ so

$$(p, f) \leq (p, \sup a_n g_{nj}) = \lim_{N \to \infty} \left(p, \sup_{n \leq N, j=1 \text{ to } \infty} a_n g_{nj} \right)$$

$$\leq \liminf_{N \to \infty} \sum_{j=1}^{N} a_n (p, U_n d(S)) + a_n \left(p, \sup_j g_{nj} - d(S_n) \right)$$

$$\leq \liminf_{N \to \infty} \sum (\sup f(x)) \|p\| \frac{\beta}{2} n \leq \beta (\sup f(x)) \|p\|$$

for every β.

Finally in the proof of Theorem 2.1 only the second paragraph where it is proved that ST is the identity needs revision. For this we show as before that if $E = ST$ then $E = E^2$ so that $(E^*)^2 = E^*$ and for any x and bounded B $(x - E^*x, Ud(B)) = (x - E^*x, E(S(c(UB)))) = (E^*x - (E^*)^2 x, S(c(UB))) = 0$ so by the above lemma E^* is the identity so E is also.

We can restate Theorem 1.1 using ideals instead of projections as follows.

THEOREM 1.3. *If I is a two-sided ideal in $H(G)$ which can be represented as the L_2 of a measure space in such a way that $I \cap P$ corresponds to the almost everywhere non-negative functions then I consists of all the functions which are invariant under $l(k)$ for all k in the compact normal subgroup $K = \{k; l(k)$ is the identity on $I\}$.*

Proof. The projection π of $H(G)$ onto I is central since I is two sided and if p is in P and q is in $P \cap I$ then $(\pi p, q) = (p, q) \geq 0$ which shows that πp is in $P \cap I$, i.e. that $\pi P \subset P$. Hence Theorem 1.1 applies and completes the proof.

2. Let S be a field[4] of subsets of a set X and for any subfield S_1 of S let $B(S_1)$ be the set of bounded S_1 measurable functions on X.

DEFINITION. If S_1 is a subfield of S, f is in $B(S)$, and m is a probability measure on S then the *conditional expectation of f with respect to S_1 and m* which we write $E(f | S_1, m)$ is the element of $B(S_1)$ satisfying for every g in $B(S_1) \int E(f | S_1, m) g dm = \int f g dm$. The existence of $E(f | S_1, m)$ and its uniqueness to within sets of m measure zero follow from the Radon Nikodym theorem. If M is a set of probability measures and there is an element h of $B(S_1)$ satisfying $h = E(f | S_1, m)$ almost everywhere (m) for every m in M we write $h = E(f | S_1, M)$. If $E(f | S_1, M)$ exists for every f in $B(S)$ we say S_1 is a *sufficient subfield* for M.

For most statistical purposes it is sufficient to consider only functions measurable S_1 if S_1 is sufficient for the set of measures in question (see [2]),

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[4] All fields and subfields used are assumed to be Borel fields.

and hence it is often a considerable simplification of a statistical problem to find nontrivial sufficient subfields and in particular to find the minimal sufficient subfield when one exists.

If M is dominated, i.e. if there is a measure n with respect to which each m in M is absolutely continuous then it can be shown (see [3, Corollary 4]) that S_1 is sufficient if and only if all the derivatives $dm_1/d(m_1+m_2)$ for m_1 and m_2 in M are measurable S_1. From this it follows that there is a best sufficient subfield for a dominated M, namely the smallest field with respect to which all the above derivatives are measurable, and that any other subfield is measurable if and only if it contains this one up to sets of m measure zero for every m in M.

The following facts are easily proved.

LEMMA 2.1. *The map $f \to E(f \mid S_1, m)$ is linear and has a unique extension to a projection in $L_2(m)$. $E(f \mid S_1, m)$ can always be so defined that $\inf f(x) \leq E(f \mid S_1, m) \leq \sup f(x)$ everywhere and if S_1 is sufficient for a set M of measures then $E(f \mid S_1, M)$ can be so defined that $\inf f(x) \leq E(f \mid S_1, M) \leq \sup f(x)$.*

In many statistical problems X is a topological group, S is the field of Borel sets, and M is a dominated set of measures closed under translation. The problem of finding the mean of a Gaussian distribution of known or unknown or partly restricted variance on the real line is of this type. In this section we find an explicit expression for the best sufficient subfield in this case.

Let G be a locally compact, separable, unimodular group, S_0 the field of Borel sets of G and $V(G)$ the set of bounded Borel measures make into a Banach algebra with convolution for multiplication and norm given by $m = \sup \int f dm$ where the sup is taken over all f in $B(S_0)$ with $\sup |f(x)| \leq 1$. We will write V^P for the set of non-negative measures in $V(G)$ and m_0 for the Haar measure of G.

DEFINITION. If x is in G, we write $l(x)f$ and $r(x)f$ for the functions defined by $(l(x)f)(y) = f(xy)$, $(r(x)f)(y) = f(yx)$, $l(x)m$ and $r(x)m$ for the measures defined by $\int f d(l(x)m) = \int (l(x)f) dm$ and $\int f d(r(x)m) = \int (r(x)f) dm$. If S is contained in S_0 we write $l(x)S = \{x^{-1}A ; A \text{ is in } S\}$ and $r(x)S = \{Ax^{-1}; A \text{ is in } S\}$. The following facts are easily proved.

LEMMA 2.2. *If f is in $B(S)$ then $l(x)f$ is in $B(l(x)S)$ and $r(x)f$ is in $B(r(x)S)$.*

$$E(l(x)f \mid l(x)S, m) = l(x)E(f \mid S, l(x)m)$$

and

$$E(r(x)f \mid r(x)S, m) = r(x)E(f \mid S, r(x)m).$$

If a subfield S is sufficient for a set M in V^P then $l(x^{-1})S$ is sufficient for $l(x)M$ and $r(x^{-1})S$ is sufficient for $r(x)M$.

A set M of measures or a subfield S will be called *invariant* if $l(x)M$ $=r(x)M=M$ or $l(x)S=r(x)S=S$ for every x in G.

LEMMA 2.3. *If M is invariant, I is the smallest two sided ideal containing M and I^P is the set of non-negative measures in I then a subfield is sufficient for M if and only if it is sufficient for I^P.*

Proof. Since M is contained in I^P, sufficiency for I^P clearly implies sufficiency for M. Suppose S is sufficient for M, then for any m in M, n_1 and n_2 in $V(G)$, f in $B(S_0)$, and g in $B(S)$ we have, where $\pi f = E(f|S, M)$

$$\int (\pi f)g\, d(n_1 m n_2) = \iiint \pi f(xyz)g(xyz)\, dn_1(x)\, dm(y)\, dn_2(z)$$

$$= \int dn_1(x) \int dn_3(z) \int (\pi f)g\, d(l(x)r(z)m)$$

$$= \int dn_1(x) \int dn_3(z) \int fg\, d(l(x)r(z)m)$$

$$= \int fg\, d(n_1 m n_2).$$

It is easy to extend this to finite sums completing the proof.

From now on we deal with a fixed nontrivial invariant dominated subset M of V^P and define I and I^P as in the above lemma.

LEMMA 2.4. *There are elements m_n in I^P such that for every non-negative, measurable function f, $\int f\, dm_0 = \lim \int f\, dm_n$.*

Proof. Choose an m in M and an increasing sequence A_n of Borel sets of finite Haar measure whose union is G. Let g_n be the characteristic function of A_n and d_n the absolutely continuous measure $g_n(x)\, dm_0(x)$. Then $m_n = m d_n$ is in I^P and $\int f\, dm_n = \int dm(x)\int_{A_n} f(yx)\, dm_0(y) = \int F_n\, dm \to \int (\lim F_n)\, dm = \int f\, dm_0$ since F_n is increasing.

Since M is dominated it has a best sufficient subfield S and S is invariant by Lemma 2.1. If f is any non-negative function in $L_2(m_0)$ and $\pi f = E(f|S, M)$ $= E(f|S, I^P)$ then $\int \pi f^2\, dm_0 = \lim \int \pi f^2\, dm_n \leq \lim \int f^2\, dm_n = \int f^2\, dm_0$ so πf is also in $L_2(m_0)$ and π has a unique extension to a projection in $L_2(m_0)$. Now $l(x)\pi f$ $= l(x)E(f|S, M) = E(l(x)f|l(x)S, l(x^{-1})M) = \pi l(x)f$ and similarly $r(x)\pi f$ $= \pi r(x)f$ so π is central and since it also preserves non-negativeness we can apply Theorem 1.1 to it.

THEOREM 2.1. *Let M be an invariant dominated subset of V^P and $K(M)$ $= \{x; x \in G \text{ and } l(x)m = m \text{ for all } m \in M\}$. Then $K(M)$ is a compact normal subgroup of G and $S(M) = \{A; A \in S \text{ and } l(k)A = A \text{ for all } k \in K(M)\}$ is the best sufficient subfield for M. Moreover if f is in $B(S_0)$ and we define πf by*

$\pi f(x) = \int_{K(M)} f(xk) dm_1(k)$ *where m_1 is the Haar measure of $K(M)$ then πf*
$= E(f \mid S(M), m)$ *for every m in M.*

Proof. We only need to identify $K(M)$ with the K of Theorem 1.1, that is
with $\{x; x \in G$ and $l(x)\pi = \pi\}$. If $l(k)\pi = \pi$ and m is in M then, using the fact
that $\pi 1 = 1$ where 1 is the function which is identically 1 on G, we have

$$\int fd(l(k)m) = \int f1d(l(k)m) = \int (\pi f)1d(l(k)m) = \int (l(k)\pi f)1 dm$$

$$= \int (\pi f)1 dm = \int f dm$$

which proves that K is contained in $K(M)$. $K(M)$ is clearly closed and it is
also compact for if we choose m in M and compact C with $m(C) > 0$ then a
disjoint sequence $(k_1C, \cdots, k_NC)$ must be maximal for some $N \le m(C)^{-1}$
proving that K is contained in the compact set $\cup k_i C C^{-1}$. Hence the equation
defines a πf and it is easily verified that $S(M)$ is a sufficient subfield for I^P with
$\pi f = E(f \mid S(M), I^P)$ so that $S(M)$ contains S and hence $K(M)$ is contained in
K.

The assumption that M is dominated or at least admits a minimal suffi-
cient statistic cannot be dropped as the following example shows. Let G be
the reals mod one and let M be all the translates of the measure m defined by

$$\int f dm = \frac{1}{2} \int_0^1 f(x)\phi(x)dx + \frac{1}{4} f(0) + \frac{1}{4} f(1/2)$$

for some ϕ. Then for each x the subfield S_x consisting of all sets either con-
taining both x and $x+1/2$ or neither is sufficient for M with

$$E(f \mid S_x, M)(y) = \begin{cases} f(y) & \text{if } y \ne x \text{ or } x + 1/2, \\ 1/2(f(x) + f(x + 1/2)) & \text{if } y = x \text{ or } x + 1/2. \end{cases}$$

If S is contained in all S_x and we write d_x for the characteristic function of
the point x then

$$E(f \mid S, M) \ge E(f(x)d_x \mid S, M) = \frac{1}{2} f(x)(d_x + d_{x+1}$$

so

$$E(f \mid S, M) \ge \frac{1}{2} f(x) + f(x + 1/2)$$

and substituting this in $\int E(f \mid S, M)dm = \int f dm$ gives

$$\int f(x)\phi(x)dx \ge \int (f(x) + f(x + 1/2)\phi(x)dx$$

which is a contradiction since f and ϕ are arbitrary.

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