It is the purpose of this paper to establish necessary and sufficient conditions that a function \( f(x) \), defined for all real values of \( x \), admit representation as a Poisson transform

\[
(1) \quad f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(t)}{1 + (x - t)^2}, \quad -\infty < x < \infty,
\]

where \( A \) is a nondecreasing function. If we ask in addition that \( A \) be bounded, the problem has already been solved by Standish [2] in terms of an operator described in the paper [1]. The removal of the condition of boundedness seems to require techniques quite different from his. In §5 we establish a result concerning harmonic function which may be of independent interest.

1. The function \( \hat{f}(z) \). In [1] it is established that a function \( f(x) \) defined by the formula (1) necessarily admits analytic continuation into the strip \( |y| < 1 \) of the complex \( z \)-plane. If we denote the continuation of \( f(x) \) by \( f(z) \) then (1) continues to hold if \( x \) is replaced by \( z \).

It is also shown in [1] that if \( f(x) \) is defined by (1) then the function \( \hat{f}(z) \) defined by

\[
(1.1) \quad \hat{f}(z) = -\frac{1}{\pi} \int_{0+}^{\infty} u^{-2} [f(z + u) - 2f(z) + f(z - u)] \, du
\]

exists for all real \( z \) and also admits continuation into the strip \( |y| < 1 \). It is important for the sequel to know that the formula (1.1) persists for complex values of \( z \) in the strip \( |y| < 1 \).

For this purpose it is clearly enough that the integral

\[
\int_{-\infty}^{\infty} \frac{f(z - u)}{1 + u^2} \, du
\]

converges uniformly in compact subsets of the strip \( |y| < 1 \). Since (1) holds with \( x \) complex it suffices to verify this uniform convergence for the integral

\[
I = \int_{-\infty}^{\infty} \frac{du}{1 + u^2} \int_{-\infty}^{\infty} \frac{dA(t)}{1 + (z - u - t)^2}.
\]

We are assuming of course that \( A \) is nondecreasing.

Received by the editors June 15, 1956.

(1) The research of this author was supported by the United States Air Force under Contract No. AF18(600)-685 monitored by the Office of Scientific Research.
After an inversion of the order of integration and a change of variable $I$ takes the form

$$I = \int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2][1 + (z - u)^2]}.$$ 

If $z = x + iy$ lies in some compact subset $S$ of the strip $|y| < 1$, then

$$|y|^2 \leq 1 - \epsilon^2, \quad |x| \leq B,$$

where $\epsilon$ and $B$ depend only on $S$. For such $z$

$$|1 + (z - u)^2| \geq \epsilon^2 + (x - u)^2,$$

so that $I$ is dominated by

$$\int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2][\epsilon^2 + (x - u)^2]}.$$

Apart from a multiplicative factor this is the same as

$$\int_{-\infty}^{\infty} \frac{dA(t)}{(t - x)^2 + (1 + \epsilon)^2}$$

which converges uniformly in $|x| \leq B$, since

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(t)}{t^2 + 1}$$

exists by hypothesis.

We have proved that the integral in (1.1) converges uniformly in compact subsets of the strip $|y| < 1$.

2. The representation theorem. Let $f(z)$ be analytic in the strip $|y| < 1$, and let $\hat{f}(z)$ be defined by (1.1). We define an operator $T_{if}$ for $-1 < t < 1$ and $-\infty < x < \infty$ by

$$(T_{if})(x) = \frac{1}{2} [f(x + it) + f(x - it)] + \frac{1}{2i} \int_{z-it}^{z+it} \hat{f}(u) du$$

whenever it has meaning.

The main result can now be stated.

Theorem 2.1. Let $f(x)$ be defined for all real values of $x$. In order that it admit the representation (1) with $A$ a nondecreasing function it is necessary and sufficient that

(i) $f(x)$ admits analytic continuation into the strip $|y| < 1$ of the $z$-plane;

(ii) the integral in (1.1) converges uniformly in compact subsets of $|y| < 1$ to a function $\hat{f}(z)$;

(iii) $(T_{if})(x) \geq 0, 0 \leq t < 1, -\infty < x < \infty$;
(iv) \( f(x+iy) = o(x^2), \ |x| \to \infty, \) uniformly in the substrips \(|y| \leq \delta\) for each \( \delta < 1.\)

3. **Proof of the necessity.** Suppose that \( f(x) \) has the form (1). The necessity of condition (i) is established in [1] and that of (ii) in the preceding section of this paper.

We turn our attention to (iii). In view of the analyticity of \( f \) and \( \widehat{f} \) in the strip \(|y| < 1, \) \((T_{tf})(x)\) can be written as

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} f^{(2k)}(x) + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \widehat{f}^{(2k)}(x).
\]

By formula (4.1) of [1] this is in turn equal to

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v dA(u)}{v^2 + (x-u)^2}, \quad v = 1 - t
\]

for \( 0 \leq t < 1. \) (iii) follows since \( A \) is nondecreasing.

As for condition (iv), we shall establish it under the condition \( x \to -\infty; \) the proof for \( x \to +\infty \) is similar. We begin by recalling formula (1.3) of [1] which states (if we set \( z_0 = 0 \)) that \( f(z) \) can be written in the form

\[
f(z) = f(0) - 2z \int_{-\infty}^{\infty} F(t) \frac{1 - (t - z)t}{[1 + (z - t)^2]^2} dt,
\]

where \( F(t) \) is the bounded function

\[
F(t) = \frac{1}{\pi} \int_{-\infty}^{t} \frac{dA(u)}{1 + u^2}.
\]

Since \(|y| \leq \delta < 1\) it suffices by (3.1) to prove that

\[
J(z) = \int_{-\infty}^{\infty} F(t) \frac{1 - (t - z)t}{[1 + (z - t)^2]^2} dt = o(x), \quad x \to -\infty,
\]

uniformly in \(|y| \leq \delta. \) \( J(z) \) can be written as \( K(z) - zL(z), \) where

\[
K(z) = \int_{-\infty}^{\infty} F(t) \frac{1 - (t - z)^2}{[1 + (z - t)^2]^2} dt
\]

and

\[
L(z) = \int_{-\infty}^{\infty} F(t) \frac{t - z}{[1 + (z - t)^2]^2} dt.
\]

We shall prove that each of these functions is \( o(1) \) as \( x \to -\infty, \) uniformly in \(|y| \leq \delta, \) and thus will complete the proof of (iv).

In each of the integrals defining \( K \) and \( L \) let \( z = x + iy \) and then make the
change of variable \( u = t - x \). We get

\[
K(z) = \int_{-\infty}^{\infty} F(u + x) \frac{1 - (u - iy)^2}{[1 + (u - iy)^2]^2} \, du
\]

and

\[
L(z) = \int_{-\infty}^{\infty} F(u + x) \frac{u - iy}{[1 + (u - iy)^2]^2} \, du.
\]

Then, since \( |y| \leq \delta < 1 \),

\[
|K(z)| \leq \int_{-\infty}^{\infty} |F(u + x)| \frac{1 + (|u| + \delta)^2}{(u^2 + 1 - \delta^2)^2} \, du
\]

and

\[
|L(z)| \leq \int_{-\infty}^{\infty} |F(u + x)| \frac{|u| + \delta}{(u^2 + 1 - \delta^2)^2} \, du.
\]

Because \( F \) is bounded and \( F(-\infty) = 0 \) each of these is \( o(1) \) as \( x \to -\infty \).

4. The sufficiency. Suppose now that conditions (i)-(iv) of the theorem hold.

It is a consequence of them that the integral

\[
(4.1) \quad \int_{-\infty}^{\infty} \frac{|f(u)|}{1 + u^2} \, du
\]

is finite. For by (ii) if we let \( z = 0 \) the integral

\[
\int_{1}^{\infty} \frac{f(u) - 2f(0) + f(-u)}{u^2} \, du
\]

converges and by (iii) with \( t = 0 \) we have \( f(u) \geq 0 \), \(-\infty < u < \infty \). This establishes that (4.1) is finite.

Consequently the function \( u(x, t) \) defined by

\[
(4.2) \quad u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(-t)f(u)}{(x - u)^2 + t^2} \, du
\]

exists and defines a function harmonic for \( t < 0 \), \(-\infty < x < \infty \). Since \( f \) is continuous \( u \) takes on the boundary value \( f(x) \) as \( t \to 0 - \), and since \( f \geq 0 \) the function \( u \) is non-negative for \( t < 0 \).

By (i) and (ii) both \( f \) and \( \hat{f} \) are analytic in the strip \( |y| < 1 \). Consequently the function \( (T_f)(x) \) defined by (2.1) is harmonic in the region \(-1 < t < 1, -\infty < x < \infty \). We shall show in §5 that \( u(x, t) \) is the harmonic continuation of \( (T_f)(x) \) into the half-plane \( t < 0 \). Consequently the two functions together form a function \( v(x, t) \) which is harmonic in \( t < 1 \) and which by condition
(iii) is non-negative for \( t < 1 \). Hence by Herglotz' theorem for a half-plane \( v(x, t) \) takes the form

\[
(4.3) \quad v(x, t) = K(1 - t) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - t)dA(u)}{(1 - t)^2 + (x - u)^2}
\]

for \( t < 1 \), where \( A \) is nondecreasing and \( K \) is a non-negative constant. If we let \( t = 0 \) and use the fact that \( v(x, 0) = u(x, 0) = f(x) \) we obtain from (4.3) that

\[
(4.4) \quad f(x) = K + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(u)}{1 + (x - u)^2}.
\]

Also by (4.3)

\[
K = \lim_{t \to -\infty} \frac{v(\omega, t)}{1 - t} = \lim_{t \to -\infty} \frac{u(\omega, t)}{1 - t}
\]

so by (4.2)

\[
K = \lim_{t \to -\infty} \int_{-\infty}^{\infty} \frac{f(u)}{u^2 + t^2} du.
\]

Since (4.1) is finite we must have \( K = 0 \). (4.4) reduces to (1) and the proof of sufficiency is complete.

It remains only to establish the preceding italicized remark concerning harmonic continuation.

5. **Proof of the harmonic continuation.** This requires that for \(-1 < t < 0\) we have \( u(x, t) = (T_t f)(x) \), or replacing \( t \) by \(-t\) that for \( 0 < t < 1 \) the following formula holds:

\[
(5.0) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(u)}{(x - u)^2 + t^2} du = \frac{1}{2} \left[ f(x + it) + f(x - it) \right] - \frac{1}{2i} \int_{-it}^{iit} \tilde{f}(u) du.
\]

The verification of this constitutes the remainder of the paper. For future applications we state the result explicitly.

**Theorem 5.1.** Under hypotheses (i), (ii) and (iv) of Theorem 2.1 and the finiteness of (4.1) the formula (5.0) holds in the region \( 0 < t < 1, \ -\infty < x < \infty \).

Let \( x \) be fixed and define \( f(x + u) = g(u), \ -\infty < u < \infty \). By (1.1) \( \tilde{f}(x + u) = \tilde{g}(u) \). Therefore it is enough to establish (5.0) when \( x = 0 \). We write this out explicitly:

\[
(5.1) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(u)}{u^2 + t^2} du = \frac{1}{2} \left[ f(it) + f(-it) \right] - \frac{1}{2i} \int_{-it}^{it} \tilde{f}(u) du,
\]

for \( 0 < t < 1 \).

To prove this write the left-hand side of (5.1) as the limit as \( R \to \infty \) of
\[ I(R) = \frac{1}{2\pi i} \int_{-R}^{R} \left( \frac{1}{u-it} - \frac{1}{u+it} \right) f(u) du \]

which can be written as

\[ I(R) = \frac{1}{2\pi i} \int_{-R-it}^{R-it} f(w+it) \frac{1}{w} dw + \frac{1}{2\pi i} \int_{R-it}^{R+it} f(w-it) \frac{1}{w} dw. \]

(5.2)

We rewrite the first term on the right-hand side as follows. Consider the rectangular contour with vertices at \(-R, -R-it, R-it, R\) and indented by a semi-circle of radius \(\rho\) below the origin. Since \(f(w+it)/w\) is analytic in and on this contour we have

\[ \frac{1}{2\pi i} \int_{-R-it}^{R-it} f(w+it) \frac{1}{w} dw = \frac{1}{2\pi i} \int_{-R-it}^{R-it} f(w+it) \frac{1}{w} dw + \frac{1}{2\pi i} \int_{R}^{-R} f(it) \frac{1}{w} dw, \]

where \(C\) denotes the part of the contour running from \(R\) to \(-R\) and indented below the origin. Now let \(\rho\rightarrow 0\). Then \(\int_{C}\) becomes the principal value integral \(\text{P.V.} \int_{-R}^{R} + 2\pi i\) times half the residue of \(f(w+it)/w\) at \(w=0\). Hence

\[ \frac{1}{2\pi i} \int_{-R-it}^{R-it} f(w+it) \frac{1}{w} dw = \frac{1}{2\pi i} \int_{-R-it}^{R-it} f(w+it) \frac{1}{w} dw + \frac{1}{2\pi i} \int_{-R}^{R} f(it) \frac{1}{w} dw. \]

There is a similar formula for the second term on the right-hand side of (5.2). Adding the two we get

\[ I(R) = \frac{1}{2} \left[ f(it) + f(-it) \right] + \frac{1}{2\pi i} \text{P.V.} \int_{-R}^{R} \frac{f(w+it) - f(w-it)}{w} dw \]

\[ + \frac{1}{2\pi i} \int_{-R-it}^{R-it} \frac{f(w+it)}{w} dw + \frac{1}{2\pi i} \int_{-R-it}^{R+it} \frac{f(w-it)}{w} dw \]

\[ + \frac{1}{2\pi i} \int_{R}^{R} \frac{f(w+it)}{w} dw + \frac{1}{2\pi i} \int_{R}^{-R} \frac{f(w-it)}{w} dw. \]

According to hypothesis (iv) we can in the denominators of the last two integrals replace \(w\) by \(R\) and in the preceding two by \(-R\), with an error of \(o(1)\) as \(R\rightarrow\infty\). Hence

\[ I(R) = \frac{1}{2} \left[ f(it) + f(-it) \right] + \int_{-R}^{R} \frac{f(w+it) - f(w-it)}{w} dw + I_1 + I_2 + I_3 + o(1), \quad R\rightarrow\infty, \]

where

\[ I_1 = \frac{1}{2\pi i} \text{P.V.} \int_{-R}^{R} \frac{f(w+it) - f(w-it)}{w} dw, \]

\[ I_2 = -\frac{1}{2\pi i R} \left\{ \int_{-R-it}^{R-it} f(w+it) dw + \int_{-R}^{R} f(w-it) dw \right\} \]
and
\[ I_3 = \frac{1}{2\pi i R} \left\{ \int_{R}^{R-i\ell} f(w + it)\,dw + \int_{R}^{R+i\ell} f(w - it)\,dw \right\}, \]
respectively. Since \( I(\infty) \) is the left-hand side of (5.1) what remains to be proved is that \( L = I_1 + I_2 + I_3 + o(1), R \to \infty, \) where
\[ L = -\frac{1}{2i} \int_{-i\ell}^{i\ell} \hat{f}(u)\,du. \]

By formula (1.1) and an integration by parts we have
\[ \hat{r}(z) = \frac{1}{\pi R} \left[ f(z + R) + f(z - R) \right] - \frac{1}{\pi} \int_{0+}^{R} \frac{1}{u} \left[ f'(z + u) - f'(z - u) \right] du \]
\[ + o(1), \quad R \to \infty, \]
uniformly in compact subsets of \(|y| < 1\). Hence, integrating from \(-i\ell\) to \(i\ell\) we have
\[ L = -\frac{1}{2\pi i} \frac{1}{R} \int_{0+}^{R} \frac{1}{u} \left[ f(it + u) - f(it - u) + f(-it + u) - f(-it - u) \right] du \]
\[ + o(1), \quad R \to \infty. \]
The third term on the right-hand side is the same as \( I_4 \). We shall identify the first with \( I_3 \), and the second with \( I_2 \). This will complete the proof.

The first term can be written
\[ -\frac{1}{2\pi i} \frac{1}{R} \left\{ \int_{0}^{R} f(z + R)\,dz + \int_{-it}^{it} f(z + R)\,dz \right\} \]
\[ = -\frac{1}{2\pi i} \frac{1}{R} \left\{ \int_{R}^{R} f(w + it)\,dw + \int_{R}^{R+i\ell} f(w - it)\,dw \right\} \]
which is clearly the same as \( I_3 \).

A similar argument shows the second term is the same as \( I_2 \).

References


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