# REPRESENTATION AS A POISSON TRANSFORM

BY

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It is the purpose of this paper to establish necessary and sufficient conditions that a function $f(x)$, defined for all real values of $x$, admit representation as a Poisson transform

$$(1) \qquad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dA(t)}{1+(x-t)^2}, \qquad -\infty < x < \infty,$$

where $A$ is a nondecreasing function. If we ask in addition that $A$ be bounded the problem has already been solved by Standish [2] in terms of an operator described in the paper [1]. The removal of the condition of boundedness seems to require techniques quite different from his. In §5 we establish a result concerning harmonic function which may be of independent interest.

**1. The function $\hat{f}(z)$.** In [1] it is established that a function $f(x)$ defined by the formula (1) necessarily admits analytic continuation into the strip $|y| < 1$ of the complex $z$-plane. If we denote the continuation of $f(x)$ by $f(z)$ then (1) continues to hold if $x$ is replaced by $z$.

It is also shown in [1] that if $f(x)$ is defined by (1) then the function $\hat{f}(z)$ defined by

$$(1.1) \qquad \hat{f}(z) = -\frac{1}{\pi}\int_{0+}^{\infty} u^{-2}[f(z+u) - 2f(z) + f(z-u)]du$$

exists for all *real* $z$ and also admits continuation into the strip $|y| < 1$. It is important for the sequel to know that the formula (1.1) persists for complex values of $z$ in the strip $|y| < 1$.

For this purpose it is clearly enough that the integral

$$\int_{-\infty}^{\infty} \frac{f(z-u)}{1+u^2}\, du$$

converges uniformly in compact subsets of the strip $|y| < 1$. Since (1) holds with $x$ complex it suffices to verify this uniform convergence for the integral

$$I = \int_{-\infty}^{\infty} \frac{du}{1+u^2}\int_{-\infty}^{\infty} \frac{dA(t)}{|1+(z-u-t)^2|}.$$

We are assuming of course that $A$ is nondecreasing.

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After an inversion of the order of integration and a change of variable $I$ takes the form

$$I = \int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2] \left| 1 + (z - u)^2 \right|} .$$

If $z = x + iy$ lies in some compact subset $S$ of the strip $|y| < 1$, then

$$|y|^2 \leqq 1 - \epsilon^2, \qquad |x| \leqq B,$$

where $\epsilon$ and $B$ depend only on $S$. For such $z$

$$\left| 1 + (z - u)^2 \right| \geqq \epsilon^2 + (x - u)^2,$$

so that $I$ is dominated by

$$\int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2][\epsilon^2 + (x - u)^2]} .$$

Apart from a multiplicative factor this is the same as

$$\int_{-\infty}^{\infty} \frac{dA(t)}{(t - x)^2 + (1 + \epsilon)^2}$$

which converges uniformly in $|x| \leqq B$, since

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(t)}{t^2 + 1}$$

exists by hypothesis.

We have proved that *the integral in* (1.1) *converges uniformly in compact subsets of the strip* $|y| < 1$.

**2. The representation theorem.** Let $f(z)$ be analytic in the strip $|y| < 1$, and let $\hat{f}(z)$ be defined by (1.1). We define an operator $T_t f$ for $-1 < t < 1$ and $-\infty < x < \infty$ by

$$(2.1) \qquad (T_t f)(x) = \frac{1}{2} [f(x + it) + f(x - it)] + \frac{1}{2i} \int_{x-it}^{x+it} \hat{f}(u) du$$

whenever it has meaning.

The main result can now be stated.

THEOREM 2.1. *Let $f(x)$ be defined for all real values of $x$. In order that it admit the representation* (1) *with $A$ a nondecreasing function it is necessary and sufficient that*

(i) *$f(x)$ admits analytic continuation into the strip $|y| < 1$ of the $z$-plane*;

(ii) *the integral in* (1.1) *converges uniformly in compact subsets of $|y| < 1$ to a function $\hat{f}(z)$*;

(iii) $(T_t f)(x) \geqq 0$, $0 \leqq t < 1$, $-\infty < x < \infty$;

(iv) *$f(x+iy)=o(x^2)$, $|x|\to\infty$, uniformly in the substrips $|y|\leqq\delta$ for each $\delta<1$.*

3. **Proof of the necessity.** Suppose that $f(x)$ has the form (1). The necessity of condition (i) is established in [1] and that of (ii) in the preceding section of this paper.

We turn our attention to (iii). In view of the analyticity of $f$ and $\hat{f}$ in the strip $|y|<1$, $(T_tf)(x)$ can be written as

$$\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}f^{(2k)}(x)+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}\hat{f}^{(2k)}(x).$$

By formula (4.1) of [1] this is in turn equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{vdA(u)}{v^2+(x-u)^2}, \qquad v=1-t$$

for $0\leqq t<1$. (iii) follows since $A$ is nondecreasing.

As for condition (iv), we shall establish it under the condition $x\to-\infty$; the proof for $x\to+\infty$ is similar. We begin by recalling formula (1.3) of [1] which states (if we set $z_0=0$) that $f(z)$ can be written in the form

$$f(z)=f(0)-2z\int_{-\infty}^{\infty}F(t)\frac{1-(t-z)t}{[1+(z-t)^2]^2}dt, \tag{3.1}$$

where $F(t)$ is the bounded function

$$F(t)=\frac{1}{\pi}\int_{-\infty}^{t}\frac{dA(u)}{1+u^2}.$$

Since $|y|\leqq\delta<1$ it suffices by (3.1) to prove that

$$\begin{aligned}J(z)&\equiv\int_{-\infty}^{\infty}F(t)\frac{1-(t-z)t}{[1+(z-t)^2]^2}dt\\&=o(x), \qquad x\to-\infty,\end{aligned}$$

uniformly in $|y|\leqq\delta$. $J(z)$ can be written as $K(z)-zL(z)$, where

$$K(z)=\int_{-\infty}^{\infty}F(t)\frac{1-(t-z)^2}{[1+(z-t)^2]^2}dt$$

and

$$L(z)=\int_{-\infty}^{\infty}F(t)\frac{t-z}{[1+(z-t)^2]^2}dt.$$

We shall prove that each of these functions is $o(1)$ as $x\to-\infty$, uniformly in $|y|\leqq\delta$, and thus will complete the proof of (iv).

In each of the integrals defining $K$ and $L$ let $z=x+iy$ and then make the change of variable $u=t-x$. We get

$$K(z) = \int_{-\infty}^{\infty} F(u+x) \frac{1-(u-iy)^2}{[1+(u-iy)^2]^2} du$$

and

$$L(z) = \int_{-\infty}^{\infty} F(u+x) \frac{u-iy}{[1+(u-iy)^2]^2} du.$$

Then, since $|y| \leqq \delta < 1$,

$$|K(z)| \leqq \int_{-\infty}^{\infty} |F(u+x)| \frac{1+(|u|+\delta)^2}{(u^2+1-\delta^2)^2} du$$

and

$$|L(z)| \leqq \int_{-\infty}^{\infty} |F(u+x)| \frac{|u|+\delta}{(u^2+1-\delta^2)^2} du.$$

Because $F$ is bounded and $F(-\infty)=0$ each of these is $o(1)$ as $x\to-\infty$.

4. **The sufficiency.** Suppose now that conditions (i)–(iv) of the theorem hold.

It is a consequence of them that the integral

$$\int_{-\infty}^{\infty} \frac{|f(u)|}{1+u^2} du \tag{4.1}$$

is finite. For by (ii) if we let $z=0$ the integral

$$\int_{1}^{\infty} \frac{f(u)-2f(0)+f(-u)}{u^2} du$$

converges and by (iii) with $t=0$ we have $f(u) \geqq 0$, $-\infty < u < \infty$. This establishes that (4.1) is finite.

Consequently the function $u(x, t)$ defined by

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(-t)f(u)}{(x-u)^2+t^2} du \tag{4.2}$$

exists and defines a function harmonic for $t<0$, $-\infty < x < \infty$. Since $f$ is continuous $u$ takes on the boundary value $f(x)$ as $t\to 0-$, and since $f\geqq 0$ the function $u$ is non-negative for $t<0$.

By (i) and (ii) both $f$ and $\hat{f}$ are analytic in the strip $|y|<1$. Consequently the function $(T_t f)(x)$ defined by (2.1) is harmonic in the region $-1<t<1$, $-\infty < x < \infty$. We shall show in §5 that *$u(x, t)$ is the harmonic continuation of $(T_t f)(x)$ into the half-plane $t<0$.* Consequently the two functions together form a function $v(x, t)$ which is harmonic in $t<1$ and which by condition

(iii) is non-negative for $t<1$. Hence by Herglotz' theorem for a half-plane $v(x, t)$ takes the form

$$v(x, t) = K(1-t) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(1-t)dA(u)}{(1-t)^2 + (x-u)^2} \tag{4.3}$$

for $t<1$, where $A$ is nondecreasing and $K$ is a non-negative constant. If we let $t=0$ and use the fact that $v(x, 0) = u(x, 0-) = f(x)$ we obtain from (4.3) that

$$f(x) = K + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dA(u)}{1 + (x-u)^2}\,. \tag{4.4}$$

Also by (4.3)

$$K = \lim_{t\to-\infty} \frac{v(o, t)}{1-t} = \lim_{t\to-\infty} \frac{u(o, t)}{1-t}$$

so by (4.2)

$$K = \lim_{t\to\infty} \int_{-\infty}^{\infty} \frac{f(u)}{u^2 + t^2}\, du.$$

Since (4.1) is finite we must have $K=0$. (4.4) reduces to (1) and the proof of sufficiency is complete.

It remains only to establish the preceding italicized remark concerning harmonic continuation.

5. **Proof of the harmonic continuation.** This requires that for $-1<t<0$ we have $u(x, t) = (T_t f)(x)$, or replacing $t$ by $-t$ that for $0<t<1$ the following formula holds:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{tf(u)}{(x-u)^2 + t^2}\, du = \frac{1}{2}\,[f(x+it) + f(x-it)] - \frac{1}{2i}\int_{x-it}^{x+it} \hat{f}(u)du. \tag{5.0}$$

The verification of this constitutes the remainder of the paper. For future applications we state the result explicitly.

THEOREM 5.1. *Under hypotheses* (i), (ii) *and* (iv) *of Theorem* 2.1 *and the finiteness of* (4.1) *the formula* (5.0) *holds in the region* $0<t<1$, $-\infty<x<\infty$.

Let $x$ be fixed and define $f(x+u) = g(u)$, $-\infty<u<\infty$. By (1.1) $\hat{f}(x+u) = \hat{g}(u)$. Therefore it is enough to establish (5.0) when $x=0$. We write this out explicitly:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{tf(u)}{u^2 + t^2}\, du = \frac{1}{2}\,[f(it) + f(-it)] - \frac{1}{2i}\int_{-it}^{it} \hat{f}(u)du, \tag{5.1}$$

for $0<t<1$.

To prove this write the left-hand side of (5.1) as the limit as $R\to\infty$ of

$$I(R) = \frac{1}{2\pi i}\int_{-R}^{R}\left\{\frac{1}{u - it} - \frac{1}{u + it}\right\} f(u)du$$

which can be written as

$$(5.2) \qquad I(R) = \frac{1}{2\pi i}\int_{-R-it}^{R-it} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i}\int_{R+it}^{-R+it} \frac{f(w - it)}{w}\, dw.$$

We rewrite the first term on the right-hand side as follows. Consider the rectangular contour with vertices at $-R$, $-R-it$, $R-it$, $R$ and indented by a semi-circle of radius $\rho$ below the origin. Since $f(w+it)/w$ is analytic in and on this contour we have

$$\frac{1}{2\pi i}\int_{-R-it}^{R-it} = \frac{1}{2\pi i}\int_{-R-it}^{-R} + \frac{1}{2\pi i}\int_{R}^{R-it} + \frac{1}{2\pi i}\int_{C},$$

where $C$ denotes the part of the contour running from $R$ to $-R$ and indented below the origin. Now let $\rho \to 0$. Then $\int_C$ becomes the principal value integral P.V. $\int_{-R}^{R}$ plus $2\pi i$ times half the residue of $f(w+it)/w$ at $w=0$. Hence

$$\frac{1}{2\pi i}\int_{-R-it}^{R-it} \frac{f(w + it)}{w}\, dw = \frac{1}{2\pi i}\int_{-R-it}^{-R} + \frac{1}{2\pi i}\int_{R}^{R-it} + \frac{1}{2\pi i}\text{ P.V. }\int_{-R}^{R} + \frac{f(it)}{2}\cdot$$

There is a similar formula for the second term on the right-hand side of (5.2). Adding the two we get

$$\begin{aligned} I(R) = {} & \frac{1}{2}\left[f(it) + f(-it)\right] + \frac{1}{2\pi i}\text{ P.V. }\int_{-R}^{R} \frac{f(w + it) - f(w - it)}{w}\, dw \\ & + \frac{1}{2\pi i}\int_{-R-it}^{-R} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i}\int_{-R}^{-R+it} \frac{f(w - it)}{w}\, dw \\ & + \frac{1}{2\pi i}\int_{R}^{R-it} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i}\int_{R+it}^{R} \frac{f(w - it)}{w}\, dw. \end{aligned}$$

According to hypothesis (iv) we can in the denominators of the last two integrals replace $w$ by $R$ and in the preceding two by $-R$, with an error of $o(1)$ as $R\to\infty$. Hence

$$I(R) = \frac{1}{2}\left[f(it) + f(-it)\right] + I_1 + I_2 + I_3 + o(1), \qquad R \to \infty,$$

where

$$I_1 = \frac{1}{2\pi i}\text{ P.V. }\int_{-R}^{R} \frac{f(w + it) - f(w - it)}{w}\, dw,$$

$$I_2 = -\frac{1}{2\pi i R}\left\{\int_{-R-it}^{-R} f(w + it)dw + \int_{-R}^{-R+it} f(w - it)dw\right\}$$

and

$$I_3 = \frac{1}{2\pi i R}\left\{\int_R^{R-it} f(w+it)dw + \int_{R+it}^{R} f(w-it)dw\right\},$$

respectively. Since $I(\infty)$ is the left-hand side of (5.1) what remains to be proved is that $L=I_1+I_2+I_3+o(1)$, $R\to\infty$, where

$$L = -\frac{1}{2i}\int_{-it}^{it}\hat{f}(u)du.$$

By formula (1.1) and an integration by parts we have

$$\hat{f}(z) = \frac{1}{\pi R}[f(z+R)+f(z-R)] - \frac{1}{\pi}\int_{0+}^{R}\frac{1}{u}[f'(z+u)-f'(z-u)]du$$
$$+ o(1), \qquad R\to\infty,$$

uniformly in compact subsets of $|y|<1$. Hence, integrating from $-it$ to $it$ we have

$$L = -\frac{1}{2\pi i}\frac{1}{R}\int_{-it}^{it} f(z+R)dz - \frac{1}{2\pi i}\frac{1}{R}\int_{-it}^{it} f(z-R)dz$$
$$+\frac{1}{2\pi i}\int_{0+}^{R}\frac{1}{u}[f(it+u)-f(it-u)-f(-it+u)+f(-it-u)]du$$
$$+ o(1), \qquad R\to\infty.$$

The third term on the right-hand side is the same as $I_1$. We shall identify the first with $I_3$, and the second with $I_2$. This will complete the proof.

The first term can be written

$$-\frac{1}{2\pi i}\frac{1}{R}\left\{\int_0^{it} f(z+R)dz + \int_{-it}^{0} f(z+R)dz\right\}$$
$$= -\frac{1}{2\pi i}\frac{1}{R}\left\{\int_{R-it}^{R} f(w+it)dw + \int_{R}^{R+it} f(w-it)dw\right\}$$

which is clearly the same as $I_3$.

A similar argument shows the second term is the same as $I_2$.

## References

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