ON THE REPRESENTATION OF α-COMPLETE BOOLEAN ALGEBRAS[1]

BY

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Let α be an infinite cardinal. A Boolean algebra A is *α-complete* if every subset of A with power (cardinality) at most α possesses a *least upper bound* in A. An ideal I in a Boolean algebra is *α-complete* in case the least upper bound (if it exists) of every subset of I with power at most α belongs to I. A Boolean algebra that is $\aleph_0$-complete is also called σ-complete. A *field of sets* B is a Boolean algebra where the operations $+$, $\cdot$, and $^-$ are respectively the operations of set-union, set-intersection, and complementation with respect to the unit element of B. A field of sets B is *α-complete* if the *union* of any subset of B with power at most α belongs to B.

By a theorem of Stone [7], every Boolean algebra is isomorphic to a field of sets. On the other hand, not every α-complete Boolean algebra is isomorphic to some α-complete field of sets; a necessary and sufficient condition for such a representation is that every principal ideal of the algebra be contained in an α-complete maximal ideal (cf. [5]). In 1947, Loomis [3] proved that every σ-complete Boolean algebra A is isomorphic to a σ-complete field of sets B modulo a σ-complete ideal of B. The question was raised as to whether this result holds for all infinite cardinals α. In 1948, Sikorski [5] showed that the Boolean algebra L of Lebesgue measurable subsets of the unit interval modulo the sets of measure zero is $2^{\aleph_0}$-complete but not isomorphic to any $2^{\aleph_0}$-complete field of sets modulo a $2^{\aleph_0}$-complete ideal.

It is the object of this note to give a necessary and sufficient condition for α-complete Boolean algebras A to be *α-representable*, i.e., to be isomorphic to an α-complete field of sets B modulo an α-complete ideal of B[2]. It turned out that to each α-complete Boolean algebra A there is associated an ideal $R_\alpha(A)$ which plays the role of a radical with respect to α-representation, i.e., a homomorphic image of A is α-representable if, and only if, the kernel of the homomorphism includes $R_\alpha(A)$ (cf. Theorem 3). Our characterization, presented in Theorem 2, may be regarded as a generalization of the theorem of Loomis since every σ-complete Boolean algebra satisfies our condition with

Presented to the Society, April 30, 1955 (cf. [2]); received by the editors June 5, 1956.

(1) The preparation of this paper was supported in part by the United States Navy under Contract No. NONR 401(20)-NR 043-167 monitored by the Office of Naval Research; reproduction in whole or in part is permitted for any purpose of the United States government.

(2) This result was first announced in the abstract [2]. It represents the first known characterization of α-representable Boolean algebras and gives a complete solution to Problem 80 in [1, p. 168].

$\alpha=\aleph_0$ (cf. Theorem 4). Recently, other characterizations of α-representable Boolean algebras have been found (Scott and Tarski [4]) where the proofs given are metamathematical in nature. We shall give a purely algebraic and direct proof (Theorem 6) of the equivalence of the characterization given here and the one given in [4]. This equivalence can be established by analyzing certain properties which individual elements of a Boolean algebra must possess; in so doing, we have proved a theorem (Theorem 5) which, aside from its use in the problem of α-representation, is of some interest in itself.

Let A be a Boolean algebra. We shall denote by $\sum_{i\in I}a_i$ ($\prod_{i\in I}a_i$) the least upper bound (greatest lower bound) in A (if it exists) of the set $\{a_i; i\in I\}$. If I is of power at most α, then an element of the form $\sum_{i\in I} a_i$ ($\prod_{i\in I} a_i$) is called an *α-sum* (*α-product*). A *system of elements* $a_{i,j}$ indexed by the sets I and J (i.e., $a_{i,j}$ is an element of the Boolean algebra for $i\in I$ and $j\in J$) is call an *α-system* in case the sets I and J have powers at most α. As usual, the complement of an element a of A shall be denoted by $\bar{a}$. For typographical reasons, we shall denote the complement of a group of letters by enclosing the group of letters in square brackets followed by a bar, e.g., $[\sum_{i\in I} a_i]^-$. Whenever there is no possibility of confusion, we simply let $\bar{a}_i=[a_i]^-$ and $\bar{a}_{i,j}=[a_{i,j}]^-$. 0 and 1 shall denote respectively the zero and unit elements of A. For arbitrary sets A, $P(A)$ denotes the set of all functions f with domain A and such that $f(x)\in x$ for each $x\in A$. We let J^I denote the set of all functions f with domain I and range included in J. If f is a function and X is a set, $f^*(X)$ is the image of X under f. We assume that ordinals have been defined in such a way that every ordinal coincides with the set of smaller ordinals. A cardinal can be understood as an ordinal which has larger power than every smaller ordinal.

DEFINITION. *If A is a Boolean algebra and α an infinite cardinal, then $R_\alpha(A)$ shall denote the set of all elements $x\in A$ for which there exists an α-system of elements $a_{i,j}\in A$ indexed by the sets I and J such that*

(i) $$\prod_{j\in J} a_{i,j} = 0 \textit{ for each } i\in I,$$

and

(ii) *for each function f, $f\in J^I$, the set of elements $\{a_{i,f(i)}; i\in I\}$ either contains x or else contains some complementary pair of elements b and $\bar{b}$.*

We see readily from the definition that $0\in R_\alpha(A)$.

THEOREM 1. *If A is an α-complete Boolean algebra, then $R_\alpha(A)$ is an α-complete ideal of A and $A/R_\alpha(A)$ is α-representable.*

Proof. We shall first prove that there exists a homomorphism f of A onto an α-complete field of sets B modulo an α-complete ideal N of B, then prove that this homomorphism f preserves α-sums, and, finally, that the kernel of this homomorphism is $R_\alpha(A)$.

For each $x \in A$, we let $x^* = \{x, \bar{x}\}$ and $A^* = \{x^*; x \in A\}$. We define a function g on the elements of A to subsets of $P(A^*)$ such that

$$g(x) = \{h; h \in P(A^*) \text{ and } h(x^*) = x\} \text{ [3]}.$$

It is clear that for every $x \in A$, $g(x) \cap g(\bar{x}) = \phi$ and $g(x) \cup g(\bar{x}) = P(A^*)$. We let B be the α-complete field of sets generated by the elements of $g^*(A)$ in the (complete) field of all subsets of $P(A^*)$. Furthermore, let $M = \{\bigcap_{i \in I} g(x_i);$ I has at most power α, $x_i \in A$ for each $i \in I$, and $\prod_{i \in I} x_i = 0\}$ and N be the α-complete ideal generated by M in B. We now wish to show that B/N is a homomorphic image of A by the following mapping f:

$$f(x) = g(x)/N.$$

It is clear that $f^*(A)$ generates B/N. Now $f(\bar{x}) = g(\bar{x})/N = [P(A^*) \sim g(x)]/N = [g(x)]^-/N$ and hence f preserves complementation. Let $a = \sum_{i \in I} a_i$ be an α-sum of elements of A. In order to show $f(a) = \sum_{i \in I} f(a_i)$ it is sufficient to prove that the symmetric difference of $g(a)$ and $\bigcup_{i \in I} g(a_i)$ is an element of N. Since I has power at most α and $a \cdot \prod_{i \in I} \bar{a}_i = 0$, we obtain $g(a) \cap \bigcap_{i \in I} g(\bar{a}_i) \in N$. But

$$g(a) \cap \bigcap_{i \in I} g(\bar{a}_i) = g(a) \cap \bigcap_{i \in I} [g(a_i)]^- = g(a) \cap \left[\bigcup_{i \in I} g(a_i) \right]^-,$$

hence

$$g(a) \cap \left[\bigcup_{i \in I} g(a_i) \right]^- \in N. \tag{1}$$

On the other hand, $\bar{a} \cdot a_i = 0$ for each $i \in I$, thus $g(\bar{a}) \cap g(a_i) = [g(a)]^- \cap g(a_i) \in N$ for each $i \in I$. Since N is α-complete, $\bigcup_{i \in I} ([g(a)]^- \cap g(a_i)) \in N$, and

$$[g(a)]^- \cap \bigcup_{i \in I} g(a_i) \in N. \tag{2}$$

It follows from (1) and (2) that the symmetric difference of $g(a)$ and $\bigcup_{i \in I} g(a_i)$ belongs to N. Thus f preserves α-sums and $f^*(A)$ is an α-complete subalgebra of B/N. Hence $f^*(A) = B/N$ and f maps A homomorphically onto B/N preserving all α-sums of elements of A.

It remains to prove that the kernel of f is the set $R_\alpha(A)$. If $f(x) = 0$, then $g(x) \in N$. We see that the condition $g(x) \in N$ is equivalent to the following: there exists an α-system of elements $a_{i,j}$ indexed by the sets I and J such that

$$\prod_{j \in J} a_{i,j} = 0 \quad \text{for each } i \in I, \tag{i}$$

$$\bigcap_{j \in J} g(a_{i,j}) \in M \quad \text{for each } i \in I, \tag{ii}$$

[3] The idea of using the elements of $P(A^*)$ as points in the representation was discussed in [3].

and

(iii) $$g(x) \subseteq \bigcup_{i\in I} \bigcap_{j\in J} g(a_{i,j}).$$

By the set-theoretical distributive law,

(3) $$\bigcup_{i\in I} \bigcap_{j\in J} g(a_{i,j}) = \bigcap_{h\in J^I} \bigcup_{i\in I} g(a_{i,h(i)}).$$

Hence (iii) together with (3) imply

(4) $$g(x) \subseteq \bigcup_{i\in I} g(a_{i,h(i)}) \text{ for each } h \in J^I.$$

If the set $\{a_{i,h(i)}; i\in I\}$ does not contain a complementary pair of elements, then any two different elements $a_{i,h(i)}$ and $a_{j,h(j)}$, with $i\neq j$, belong to different elements of A^*. If, in addition, the set $\{a_{i,h(i)}; i\in I\}$ does not contain x, then clearly there exists a function $k\in P(A^*)$ such that $k(x^*)=x$ and $k(a_{i,h(i)}{}^*) = \bar{a}_{i,h(i)}$ for each $i\in I$, i.e.,

(5) $$k \in g(x) \text{ and } k \notin g(a_{i,h(i)}) \text{ for each } i \in I.$$

(5) is a contradiction to (4). Hence $x\in R_\alpha(A)$. On the other hand, let $x\in R_\alpha(A)$ and let $a_{i,j}$ be the associated α-system of elements indexed by the sets I and J. Clearly conditions (i), (ii), and (4) are satisfied by the elements $a_{i,j}$. By (3) and (i) we see that $\bigcap_{h\in J^I}\bigcup_{i\in I} g(a_{i,h(i)})\in N$, which, together with (4) imply (iii). Thus $g(x)\in N$ and $f(x)=0$. The theorem has been proved[4].

THEOREM 2. *Let A be an α-complete Boolean algebra. Then A is α-representable if, and only if, $R_\alpha(A)=\{0\}$.*

Proof. Obviously, if $R_\alpha(A)=\{0\}$, then by Theorem 1 A is α-representable. Let f be a homomorphism of an α-complete field of sets B onto A and such that f preserves all α-sums of B. Let $x\in R_\alpha(A)$ and let $a_{i,j}$ be the associated α-system of elements. We choose an inverse f^{-1} to the function f satisfying the condition: that $f^{-1}(\bar{y}) = [f^{-1}(y)]^-$ for each $y\in A$. It is evident that such an inverse can always be chosen. Since $\prod_{j\in J} a_{i,j}=0$ for each $i\in I$ and since f preserves all α-sums (and hence all α-products), we obtain

(1) $$\bigcap_{j\in J} f^{-1}(a_{i,j}) \in B \text{ and } f\left(\bigcap_{j\in J} f^{-1}(a_{i,j})\right) = 0 \text{ for every } i \in I.$$

From (1), it follows that $\bigcup_{i\in I} \bigcap_{j\in J} f^{-1}(a_{i,j})\in B$, $f(\bigcup_{i\in I} \bigcap_{j\in J} f^{-1}(a_{i,j}))=0$, and, by an application of the set-theoretical distributive law,

(2) $$f\left(\bigcap_{h\in J^I} \bigcup_{i\in I} f^{-1}(a_{i,h(i)})\right) = 0.$$

[4] The fact that the ideal $R_\alpha(A)$ is α-complete can be proved without resorting to the homomorphism f and without even the assumption of the α-completeness of A.

By our choice of the inverse function f^{-1}, we see that

$$\text{(3)}\qquad \begin{gathered}\text{for every } h \in J^I, \text{ either } f^{-1}(x) \subseteq \bigcup_{i\in I} f^{-1}(a_{i,h(i)}), \\ \text{or else } \bigcup_{i\in I} f^{-1}(a_{i,h(i)}) = 1.\end{gathered}$$

Clearly (3) leads to the condition

$$\text{(4)}\qquad f^{-1}(x) \subseteq \bigcup_{i\in I} f^{-1}(a_{i,h(i)}) \text{ for each } h \in J^I.$$

Applying now the function f to both sides of the inclusion of (4) and by the use of (2), we obtain the desired conclusion $x=0$. The theorem has been proved.

The condition $R_\alpha(A)=\{0\}$, as has been proved in Theorem 2, is both necessary and sufficient for A to be α-representable. Earlier, Smith [6] gave a sufficient condition for A to be α-representable and which he has shown to be not necessary. Furthermore, he pointed out (in [6]) that all those α-complete Boolean algebras in which the so-called *α-distributive law* holds satisfy his sufficient condition and, consequently, are α-representable. We see quite easily from our definition of $R_\alpha(A)$ that if $R_\alpha(A)\neq\{0\}$ then clearly A will not satisfy the α-distributive law. One can also give a simple and direct argument that the condition $R_\alpha(A)=\{0\}$ is implied by his sufficient condition; however, we point out here that our condition was obtained without the knowledge of the results to be found in [6] and that the two approaches are entirely different.

The next theorem studies more closely the role that the ideals $R_\alpha(A)$ play in the problem of α-representation.

THEOREM 3. *Let A be an α-complete Boolean algebra and let N be an α-complete ideal of A. Then A/N is α-representable if, and only if, $R_\alpha(A)\subseteq N$.*

Proof. Assume that A/N is α-representable, i.e., $R_\alpha(A/N)=\{0/N\}$. Let $x\in R_\alpha(A)$ and $a_{i,j}$ be the associated α-system of elements. It is evident that the elements $a_{i,j}/N$ of A/N satisfy

$$\text{(i)}\qquad \prod_{j\in J} [a_{i,j}/N] = 0/N \text{ for each } i \in I,$$

and

(ii) for every $h\in J^I$, the set of elements $\{a_{i,h(i)}/N;\ i\in I\}$ contains either x/N or a complementary pair.

Thus, it follows from (i) and (ii) that $x/N\in R_\alpha(A/N)$ and $x\in N$.

On the other hand, assume that $R_\alpha(A)\subseteq N$. We shall prove that $R_\alpha(A/N)=\{0/N\}$. Let $x/N\in R_\alpha(A/N)$ and let $(a/N)_{i,j}$ be the associated α-system of elements. Let us now pick representatives $a_{i,j}$ out of the cosets $(a/N)_{i,j}$ such that if $(a/N)_{i,j}=x/N$, then $a_{i,j}=x$, and such that if $(a/N)_{i,j}=[(a/N)_{i',j'}]^-$, then $a_{i,j}=\bar{a}_{i',j'}$. From this choice of representatives, it follows that

$$(1) \qquad \prod_{j\in J} a_{i,j} \in N \text{ for every } i \in I$$

and

(2) for each $h\in J^I$, the set of elements $\{a_{i,h(i)}; i\in I\}$ either contains x or else contains a complementary pair.

Let now $y=x\cdot\prod_{i\in I}\sum_{j\in J}\bar{a}_{i,j}$ and let us pick a $j'\notin J$ and set $J'=J\cup\{j'\}$. We define an α-system of elements $b_{i,j}$ indexed by the sets I and J' as follows:

$$\text{(i)} \qquad b_{i,j} = a_{i,j} \text{ if } j\neq j' \text{ and } a_{i,j}\neq x,$$

$$\text{(ii)} \qquad b_{i,j} = y \quad \text{if } j\neq j' \text{ and } a_{i,j} = x,$$

and

$$\text{(iii)} \qquad b_{i,j'} = y.$$

It follows from (1), (2), and the definition of $b_{i,j}$ that

$$(3) \qquad \prod_{j\in J'} b_{i,j} = 0 \text{ for each } i \in I,$$

and

(4) for every $h\in J'^I$, the set of elements $\{b_{i,h(i)}; i\in I\}$ either contains the element y or else contains a complementary pair.

Conditions (3) and (4) show that the element $y\in R_\alpha(A)$ and hence, by our hypothesis, $y\in N$. However, $x=x\cdot\bar{y}+y=x\cdot\sum_{i\in I}\prod_{j\in J}a_{i,j}+y$ and whence, by (2), $x\in N$ and $x/N=0/N$. The proof is now complete. (It actually follows from the proof of Theorem 3 that $R_\alpha(A/N)=R_\alpha(A)/N$ for any α-complete ideal N of A.)

Due to Theorem 3 we may now justly regard the ideal $R_\alpha(A)$ as the *α-radical* of an α-complete Boolean algebra with respect to α-representation. $R_\alpha(A)$ is unique in the sense that any α-complete ideal N' of A satisfying Theorem 3 with $R_\alpha(A)$ replaced by N' must be identical with $R_\alpha(A)$, i.e., $R_\alpha(A)=N'$. Furthermore, we see that if α and β are infinite cardinals and $\beta\leq\alpha$, then $R_\beta(A)\subseteq R_\alpha(A)$. It follows then for each α-complete Boolean algebra A either A is α-representable or else there exists a least $\beta\leq\alpha$ for which $R_\beta(A)$ does not vanish. The problem is open whether for all cardinals α and β with $\aleph_0<\beta\leq\alpha$ there exists an α-complete Boolean algebra A for which β is the *least* cardinal such that $R_\beta(A)$ does not vanish. We shall see from Theorem 4 that if $\beta=\aleph_0$, then $R_\beta(A)=\{0\}$.

From the results in [5] and Theorem 2, the algebra L of Lebesgue measurable sets modulo the sets of measure zero is such that $R_\gamma(L)\neq\{0\}$, where for the discussion in this paragraph we let $\gamma=2^{\aleph_0}$. Since the algebra L is known to be of the power of the continuum, complete, and homogeneous(5), we see immediately that $R_\gamma(L)$ is a principal ideal of L and, what is more interesting, $R_\gamma(L)$ simply coincides with L. Another interesting example is the Boolean algebra B of the Borel sets modulo the sets of first category in a separable complete metric space S. This Boolean algebra is also known as the algebra of *regular open sets* of S(6). It is known that B is complete and is of the power of the continuum. Hence $R_\gamma(B)$ is again a principal ideal. It is not difficult to see that for any regular open set x there exists a sequence of sets $\{x_{i_0,i_1,\dots,i_n}\}$ where each i_j is either 0 or 1,

$$x = x_0 + x_1,$$

and

$$x_{i_0,\dots,i_n} = x_{i_0,\dots,i_n,0} + x_{i_0,\dots,i_n,1} \text{ for each } n,$$

and such that for every choice of the index i

$$\prod_n x_{i_0,i_1,\dots,i_n} = 0\text{(7)}.$$

From the above and Theorem 3.1 in [5] we see that again $R_\gamma(B)=B$. Thus in the above two instances, not only are the algebras themselves not $2^{\aleph_0}$-representable, but any nontrivial $2^{\aleph_0}$-complete homomorphic image is also not $2^{\aleph_0}$-representable.

It should also be mentioned that Theorem 3 may be obtained in a metamathematical fashion by using Theorem 2 and the fact that the class of all α-complete Boolean algebras which are α-representable forms an *equational class of algebras*. As a matter of fact, Scott and Tarski have shown that the characterization given in Theorem 2 can be transformed in a natural way to yield a set of characterizing equations for the class of all α-representable Boolean algebras(8).

The connection between the result of Loomis [3] concerning σ-complete Boolean algebras and Theorem 2 will be made clear by the following theorem.

THEOREM 4. *For any Boolean algebra A, $R_{\aleph_0}(A)=\{0\}$.*

Proof. Let $x\in R_{\aleph_0}(A)$ and let $a_{i,j}$ be the associated $\aleph_0$-system of elements indexed by the sets I and J where we may assume $I=J=$the set of all natural numbers. Suppose that $x\neq 0$, thus $\bar{x}\neq 1$. Hence $1\neq\bar{x}+\prod_{j\in J} a_{0,j}$ and $1\neq\prod_{j\in J}(\bar{x}+a_{0,j})$. We can now pick a j_0 such that $\bar{x}+a_{0,j_0}\neq 1$. If we proceed

(5) For some details on the algebra L, cf. [1, pp. 168–169 and p. 184].

(6) For some details on the algebra B, cf. [1, pp. 176–179].

(7) Cf. footnote 5.

(8) This result may be found in [4, Theorem 1].

in this fashion, we will pick an infinite sequence of elements a_{0,j_0}, a_{1,j_1}, $a_{2,j_2}, \cdots$ such that

$$\bar{x} + a_{0,j_0} + a_{1,j_1} + \cdots + a_{i,j_i} \neq 1 \text{ for each } i \in I.$$

This clearly means that the function h defined by the condition $h(i) = j_i$ for each $i \in I$ will yield a set of elements $\{a_{i,h(i)}; i \in I\}$ which will not contain x and will not contain a complementary pair. Hence $x = 0$ and the theorem is proved.

For our subsequent discussion we introduce the following notion. An ideal N of an α-complete Boolean algebra A *preserves the α-system of elements* $a_{i,j}$ (of A) indexed by the sets I and J if, and only if,

$(*)$ for each $i \in I$, $\sum_{j \in J} a_{i,j} \notin N$ if, and only if, $\bar{a}_{i,j} \in N$ for some $j \in J$.

We see that if, in particular, N is a maximal ideal, then condition $(*)$ can be replaced by thc condition

$(**)$ for each $i \in I$, $\sum_{j \in J} a_{i,j} \in N$ if, and only if, $a_{i,j} \in N$ for every $j \in J$.

In general, we see that condition $(*)$ implies the corresponding notion defined in [4] and which in turn implies condition $(**)$; however, for maximal ideals N all three notions are equivalent. The following lemma will require no proof.

LEMMA. *An ideal N preserves the α-system of elements $a_{i,j}$ indexed by I and J if, and only if, N preserves the α-system of elements $b_{i,j}$ indexed by the sets I and $J \cup \{j'\}$ $(j' \notin J)$ where $b_{i,j} = a_{i,j}$ if $j \neq j'$ and $b_{i,j'} = [\sum_{j \in J} a_{i,j}]^-$ for each $i \in I$.*

It follows from the lemma that if an ideal N preserves all α-systems of elements $a_{i,j}$ where $1 = \sum_{j \in J} a_{i,j}$ for each $i \in I$, then N preserves all α-systems of elements.

THEOREM 5. *For any element x of an α-complete Boolean algebra A the following four conditions are equivalent.*

(i) $x \notin R_\alpha(A)$.

(ii) *For any α-system of elements of A, there exists a proper ideal N containing $\bar{x}$ and preserving the α-system of elements and which is β-complete for every cardinal β such that α^β has at most the power α.*

(iii) *For any α-system of elements of A, there exists a proper ideal N containing $\bar{x}$ and preserving the α-system of elements.*

(iv) *For any α-system of elements of A, there exists a maximal ideal M containing $\bar{x}$ and preserving the α-system of elements.*

Proof. The equivalence of (iii) and (iv) follows from the fact that if N preserves an α-system of elements, then any of its maximal extensions M will also preserve the same α-system of elements. The implication (ii) to (iii) is obvious. We shall now show the implication of (iii) to (i) by contradiction. Suppose $x \in R_\alpha(A)$ and let $a_{i,j}$ be the associated α-system of elements. Consider now a proper ideal N which preserves the α-system of elements $\bar{a}_{i,j}$ and which contains $\bar{x}$. Since $1 = \sum_{j\in J} \bar{a}_{i,j}$ for each $i \in I$, it follows that

$$\text{(1)} \qquad \text{for each } i \in I,\ a_{i,j} \in N \text{ for some } j \in J.$$

By (1), we define a function h, $h \in J^I$, such that

$$\text{(2)} \qquad a_{i,h(i)} \in N \text{ for each } i \in I.$$

Using (2) and the fact that N is a proper ideal containing $\bar{x}$, we see that the set of elements $\{a_{i,h(i)};\ i \in I\}$ cannot contain the element x nor a complementary pair. Hence we have a contradiction and $x \notin R_\alpha(A)$.

Next we prove (ii) from (i). Let $a_{i,j}$ be an α-system of elements indexed by I and J and such that

$$\text{(3)} \qquad 1 = \sum_{j\in J} a_{i,j} \text{ for each } i \in I.$$

We may assume without loss of generality that the sets I and J have precisely the power α. It follows from the lemma that it is sufficient if we can prove the existence of an ideal N preserving α-systems of the above special form. Notice that (3) leads to

$$\text{(4)} \qquad 0 = \prod_{j\in J} \bar{a}_{i,j} \text{ for each } i \in I.$$

Let β be any cardinal such that α^β has at most the power α. We let

$$I_\beta = \{\bar{a}_{i,j};\ i \in I, j \in J\}^\beta,$$

and

$$I' = I \cup \bigcup (I_\beta;\ \alpha^\beta \text{ has the power at most } \alpha).$$

Since the set $\{a_{i,j};\ i \in I, j \in J\}$ has the power α, it is clear that each set I_β has the power at most α and the set I' also has power at most α. We now define (in any manner we wish) an α-system of elements $b_{i,j}$ indexed by the sets $I' \cup \{i'\}$ $(i' \notin I')$ and J and satisfying the following conditions:

$$\text{(5)} \qquad b_{i,j} = \bar{a}_{i,j} \text{ for } i \in I \text{ and } j \in J.$$

$$\text{(6)} \qquad b_{i',j} = 0 \text{ for all } j \in J.$$

$$\text{(7)} \quad \{\bar{y};\ y \in f^*(\beta)\} \cup \{x\} \cup \left\{ \sum_{\rho\in\beta} f(\rho) + \bar{x} \right\} = \{b_{f,j};\ j \in J\} \text{ for each } f \in I_\beta.$$

It follows readily from (4)–(7) that

$$\text{(8)} \qquad \prod_{j\in J} b_{i,j} = 0 \text{ for each } i \in I' \cup \{i'\}.$$

Since $x \notin R_\alpha(A)$ and $b_{i,j}$ is an α-system of elements satisfying (8), there exists a function $h \in J^{I' \cup \{i'\}}$ such that

(9) the set of elements $K=\{b_{i,h(i)};\, i\in I'\cup\{i'\}\}$ does not contain x and does not contain a complementary pair.

From (6) and (9), we see that

$$0 \in K \text{ and } 1 \notin K. \tag{10}$$

Let $L=\{b_{i,h(i)};\, i\in I\}=\{\bar{a}_{i,h(i)};\, i\in I\}\subseteq K$. For any β such that α^β has power at most α and for any subset L' of L with power β we can find a function f, $f\in I_\beta$, such that $L'=f^*(\beta)$. From (7) and (9) we see that

$$\sum_{\rho\in\beta} f(\rho) + \bar{x} = b_{f,h(f)} \in K. \tag{11}$$

(10) and (11) clearly imply that the least upper bound of any subset L' of $L\cup\{\bar{x}\}$ of power β is different from 1. We now simply let

$$N = \left\{ y;\ \text{for some } L' \subseteq L \cup \{\bar{x}\},\ L' \text{ has power } \beta, \text{ and } \alpha^\beta \text{ has power } \alpha, \text{and } y \leqq \sum_{x\in L'} x \right\}. \tag{12}$$

Obviously, N is a proper ideal containing $\bar{x}$ and preserving the α-system of elements $a_{i,j}$. Suppose α^β has power α. For each $\xi\in\beta$, let $y_\xi\in N$ and such that for some β_ξ, α^{β_ξ} has power α and

$$y_\xi \leqq \sum_{\rho\in\beta_\xi} y_{\xi,\rho} \tag{13}$$

where

$$y_{\xi,\rho} \in L \cup \{\bar{x}\} \text{ for each } \rho \in \beta_\xi. \tag{14}$$

Let C be the *cardinal sum* of all the sets β_ξ as $\xi\in\beta$. By the set-theoretical law on exponents, we see that α^C is set-theoretically equivalent to $P(\{\alpha^{\beta_\xi};\, \xi\in\beta\})$. Since α^{β_ξ} has the power α for each $\xi\in\beta$, we see that $P(\{\alpha^{\beta_\xi};\, \xi\in\beta\})$ is simply set-theoretically equivalent to α^β which again has the power α. Hence α^C has power α. From this and (14) it follows that there exists a subset L' of $L\cup\{\bar{x}\}$ with power β' where $\alpha^{\beta'}$ has power at most α such that

$$\sum_{\xi\in\beta}\sum_{\rho\in\beta_\xi} y_{\xi,\rho} = \sum_{x\in L'} x. \tag{15}$$

Hence by (13) and (15), $\sum_{\xi\in\beta} y_\xi \leq \sum_{x\in L'} x$ and, by (12), $\sum_{\xi\in\beta} y_\xi\in N$. Thus we see that N is β-complete for every β such that α^β has the power α. The theorem is proved.

Using Theorem 5 we can now present several characterizations of α-representable Boolean algebras in the following(9):

(9) The equivalence of 6(i) with 6(iv) is precisely a condition given in [4, Theorem 2]. This is easily seen from our remarks concerning the equivalence of conditions (*) and (**) in case N is maximal.

THEOREM 6. *For any α-complete Boolean algebra A the following four conditions are equivalent*:

(i) *A is α-representable.*

(ii) *For any $x \neq 1$ and any α-system of elements, there exists a proper ideal N containing x and preserving the α-system of elements and which is β-complete for every β such that α^β has power at most α.*

(iii) *For any $x \neq 1$ and any α-system of elements, there exists a proper ideal N containing x and preserving the α-system of elements.*

(iv) *For any $x \neq 1$ and any α-system of elements, there exists a maximal ideal M containing x and preserving the α-system of elements.*

Proof. By Theorem 2 and Theorem 5.

In conclusion we point out the significance of condition 6(ii) in the following special application: If a Boolean algebra A is *continuously-representable* (i.e. $2^{\aleph_0}$-representable), then for any $x \neq 1$ and any continuum-system of elements there exists a *denumerably-complete* proper ideal N containing x and preserving the continuum-system of elements.

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