THE BOUNDARY INTEGRAL OF LOG $|\phi|$ FOR GENERALIZED ANALYTIC FUNCTIONS

BY

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1. **Introduction.** In classical analytic function theory, the boundary integral

$$-\int \log |\phi(\rho e^{i\theta})| P(\zeta, \theta) d\theta \equiv \phi(\rho; \zeta),$$

where $P(\zeta, \theta)$ is Poisson's kernel, has been extensively studied (Compare, for example, the Jensen-Poisson formula). We study it here, much less extensively of course, in a greatly generalized situation (see [II]).

By way of introduction, we state results only for the "archimedean" case. (The archimedean case deals with that analogue of holomorphic functions

$$\phi(z) = a_0 + a_1\zeta + \cdots + a_n\zeta^n + \cdots \qquad (|\zeta| \leqq 1)$$

where the exponents are non-negative real numbers instead of integers).

First of all, for $\rho > |\zeta|$, the integral $\phi(\rho; \zeta)$ is finite, unless $\phi(z) = 0$ identically. It therefore provides (for each pair of the parameters $\rho > |\zeta|$) a real valued homomorphism of the multiplicative set of nonzero elements of the integral domain A_0 (of all functions holomorphic in the sense of [II] on the unit disc Δ, generalized as in [II]).

The second result is that this homomorphism depends continuously on ϕ, in the sense that if $\phi_n(z)$ converges uniformly to $\phi(z)$ on (suitable) compact sets in Δ, (which are never required to touch the boundary), then $\phi_n(\rho; \zeta) \to \phi(\rho; \zeta)$.

We make use of some properties of equicontinuous families of holomorphic functions. These properties were first utilized in connection with almost periodic functions by H. Bohr [VI], and later used by B. Jessen [III] in studying a function which we show below to be $\phi(\rho; 0)$.

Jessen and Tornehave [IV] establish convexity properties of $\phi(\rho; 0)$, and connect the points of increase of the derivative with the zeros of $\phi(z)$. We shall explore these matters in general, elsewhere.

2. **Measures for semi-groups of functions.** We review briefly some matters discussed in [I, §§5–6]. There S is a topological space and H is a class of real valued non-negative continuous functions, closed under multiplication, and containing all positive constants. Moreover, there is a compact set B in S such that every element of H attains its maximum value (for all of S) on B.

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Upon these hypotheses, there exist [I, 5.2] for each point s of S, measures m_s "representing s" (which is to say)

2.1 $$\int \log g(\alpha) m_s(d\alpha) = \log g(s) \quad \text{when} \quad g, g^{-1} \in H$$

and supported by B. Moreover, for each singular element h of H (s having already been selected) there is at least one representing measure for s, supported by B, for which

2.2 $$\log h(s) \leqq \int \log h(\beta) m_s(d\beta)$$

and indeed the right hand side of 2.2 can be made as large as, but no larger than [I, 6.25, 6.4]

2.21 $$\inf \frac{1}{n} g(s)$$

where (g, n) runs over all nonsingular g in H and n over all positive integers such that

2.22 $$\frac{1}{n} \log g \geqq \log h \text{ on } B.$$

2.3. Lemma. *Let S, H, B be as above, and let s, h be selected as above. Then a (regular Baire) measure m_s representing s, supported by B, can be found such that for any other (regular Baire) measure n_s with compact support not necessarily in B, but representing s, one has*

2.31 $$\int_S \log h(\alpha) n_s(d\alpha) \leqq \int_B \log h(\beta) m_s(d\beta).$$

Proof. For m_s we take anyone giving the value 2.21, and supported by B. Let B_1 be B plus the (compact) support of n_s. Then S, H, B_1 satisfy all the conditions imposed on S, H, B. In particular [I, 6.4] or the remarks above

$$\int_{B_1} \log h(\alpha) n_s(d\alpha) \leqq 2.21(B_1)$$

where 2.21 (B_1) means the value 2.21 with "B" in 2.22 replaced by "B_1". But the predicate 2.22 of g is *independent* of B, since it says precisely that $g^{-1}h^n \leqq 1$ on S. Therefore 2.21 $(B_1) = 2.21$ (B), which value m_s was constructed to give. This suffices to give 2.31.

By way of elucidation we shall derive a well-known result (see, for example, [VIII]).

2.4. THEOREM. *Let f be continuous on the closed unit disc, and analytic inside. Let $0 \leqq r \leqq 1$. Then*

$$\int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \leqq \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta.$$

We shall give the proof as it illuminates the lemma.

Let A_0 be the algebra of all f of the specified sort. Let H be the class of all $|f|$. Let S be the disc and B its boundary. If g is nonsingular in H, $g = |g_1|$, $g_1 \in A_0$, then $g_1 = e^{u+iv}$, $u + iv \in A_0$; and $\log g = u$. For such a u,

2.5 $$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta \equiv I_r(u) = u(0).$$

Thus the integral I_r "represents" the origin in the sense used above. For $r = 1$, the measure involved is supported by B. Because the measure is determined by its Fourier Stieltjes coefficients, it is the only measure on B representing 0. We therefore know that m_s (see 2.31) is the measure corresponding to the integral I_1, and that I_r represents 0. From 2.31 we obtain

2.6 $$I_r(\log |f|) \leqq I_1(\log |f|).$$

We insert the definition of I_r, I_1, multiply by 2π, and arrive at 2.41 as desired.

Taking $r = 0$, we obtain

2.7 $$\log |f(0)| \leqq \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta.$$

This special inequality holds also in the general case [I, 6.41] where it can be deduced from Lemma 2.3 by noting that the point s is itself a measure representing s.

These methods can be applied to any commutative Banach algebra [I] but nothing interesting is yet known to result. We apply them therefore to a special situation generalizing 2.4.

3. **A generalization of analytic functions.** Let [I, 4.21–4.23] G be a discrete abelian group with a distinguished multiplicative subset G_+ generating G, and containing the 1 of G. Such a system will be called an A-system (G, G_+). The elements of G will be denoted by $x, y, \cdots$. Let Δ be the space of homomorphisms (onto the complex numbers C) of the algebra A_0 of continuous functions ϕ on Γ (the character group of G) whose Fourier transforms

$$\int_\Gamma \phi(\alpha)\overline{\alpha(x)} d\alpha$$

vanish for x not in G_+. Then Δ coincides with the class of complex homomorphisms of G_+, and the Šilov boundary of Δ is naturally identifiable with Γ [II, 4.6]. Following the ideas and notation of [II, 5.52], we let, for each

ζ in Δ, m_ζ denote the harmonic measure on Γ. Among all regular Baire measures *supported by* Γ there is only one which represents ζ.

We shall now consider a proposition (3.2, below) which involves the multiplication of points, say ζ and η of Δ. The product $\zeta\eta$ is that point of Δ for which $\zeta\eta(x)=\zeta(x)\eta(x)$ for each x in G_+. Having selected ζ and η from Δ, we can define an elementary integral

3.1 $$J(\Phi) = \int_\Gamma \Phi(\zeta\alpha)m_\eta(d\alpha)$$

where Φ runs through the space of complex-valued continuous functions on Δ. Now this integral represents the point $s=\zeta\eta$, in the sense that $J(\phi)=\phi(s)$ for each ϕ in A_0, as can be checked by selecting $\Phi(\delta)=\phi(\delta)=\delta(x)$ for each of the x in G_+. This, it will be noted, is precisely what is required of the integral on the left side of 2.31. For the measure on the right, one can and must take the harmonic measure m_s, because of the uniqueness mentioned a few lines above. An application of 2.3 now yields the desired result.

3.2. LEMMA. *For ϕ in A_0, and ζ, η in Δ,*

3.21 $$\int_\Gamma \log |\phi(\zeta\alpha)| m_\eta(d\alpha) \leqq \int_\Gamma \log |\phi(\alpha)| m_{\zeta\eta}(d\alpha).$$

It pays to complicate this a bit. Replace $\phi(\alpha)$ by $\phi(\theta\alpha)$. Then we obtain

3.22 $$\int_\Gamma \log \frac{1}{|\phi(\theta\alpha)|} m_{\zeta\eta}(d\alpha) \leqq \int_\Gamma \log \frac{1}{|\phi(\theta\zeta\alpha)|} m_\eta(d\alpha)$$

It is convenient to introduce the following

3.3. *Notation*. For ϕ in A_0 define

$$\phi(\theta;\eta) = \int_\Gamma \log \frac{1}{|\phi(\theta\alpha)|} m_\eta(d\alpha).$$

A priori, this might be $+\infty$ in some cases, e.g., $\phi=0$. In this notation, 3.22 says just that

3.31 $$\phi(\theta;\zeta\eta) \leqq \phi(\theta\zeta;\eta);$$

while for $\|\phi\|\leqq 1$,

3.32 $$\phi(\theta;\eta) \geqq 0.$$

From [II, 5.73] we obtain

3.33 $$-\log |\phi(\theta\eta)| \geqq \phi(\theta;\eta).$$

Our main objective is to see when 3.32 is positively infinite. (We remark in passing that, by 3.33, this makes ϕ vanish at the point $\theta\eta$).

Let ρ be a non-negative element of Δ (which means that $\rho(x)\geqq 0$ for all x in G_+). Then we can define ρ^u for all $u\geqq 0$ [II, 7.1].

3.4. THEOREM. *For ϕ in A_0 with $\|\phi\|\leqq 1$, and θ, ρ, u as above,*

3.41 $$\phi(\theta;\rho) \leqq e^u\phi(\theta\rho^u;\rho).$$

The proof consists of three parts: first a proof of

3.42 $$I_\rho \leqq uI_{\rho^u} \qquad (u \geqq 1),$$

where I_ρ is the Poisson (-like) integral on Γ associated with the point ρ [II, 5.3, 5.51]; then an application of 3.31; and finally a limit process.

Let h be a non-negative Baire function on Γ. By [II; *loc. cit.*]

3.43 $$I_\rho(h) = \int_{-\infty}^{\infty}\int_{\Gamma_\rho} h(\gamma r^{iv})c(v)d\gamma dv$$

where $c(v)=\pi^{-1}(1+v^2)^{-1}$. Further Γ_ρ is a certain compact subgroup (and $d\gamma$ means integrate with respect to the Haar measure). Of this, all that is relevant here is that if $\rho=\tau^a$ $(a>0)$ then $\Gamma_\tau=\Gamma_\rho$. The r is any real valued homomorphism of G which agrees with ρ wherever ρ is not 0, and itself never vanishes. If we let t serve as the r for τ, and take $\rho=\tau^a$, then 3.43 takes the form

$$I_{\tau^a}(h) = \int_{-\infty}^{\infty}\int_{\Gamma_\tau} h(\gamma t^{aiv})c(v)d\gamma dv.$$

Let $av=u$. Then

$$I_{\tau^a}(h) = \int_{-\infty}^{\infty} K(u)\left(a\left[1+\frac{u^2}{a^2}\right)\right]^{-1} du$$

where $K(u)$ is independent of a, and non-negative. Let

$$\chi(a) = \int_{-\infty}^{\infty} K(u)(a^2+u^2)^{-1}du.$$

Then

$$aI_{\tau^a}(h) = \chi(a).$$

Now

$$-\chi'(a) = \int K(u)2a(a^2+u^2)^{-2}du \leqq 2a^{-1}\chi(a).$$

From this $a^2\chi'+2a\chi\geqq 0$ or $a^2\chi$ is increasing. Replacing a by 1 and then by $u\geqq 1$ gives 3.42.

Now insert

$$-\log |\phi(\theta\rho^{na}\alpha)|$$

as an integrand in 3.42, where $u=1+a$; a, $n\geqq 0$. Then

$$\begin{aligned}\phi(\theta\rho^{na};\rho) &\leqq (1+a)\phi(\theta\rho^{na};\rho^{1+a})\\ &\leqq (1+a)\phi(\theta\rho^{(n+1)a};\rho) \qquad \text{(by 3.31).}\end{aligned}$$

Consequently

$$(1+a)^n\phi(\theta\rho^{na};\rho)$$

increases as the integer n increases, and is not less than $\phi(\theta;\rho)$. Let $a=u/n$ and make n go to infinity. This yields

$$\phi(\theta;\rho) \leqq e^u\phi(\theta\rho^u;\rho).$$

3.5. THEOREM. *Let ϕ belong to A_0, and let ζ be an element of Δ whose polar decomposition is $\zeta=\rho\beta$. Suppose*

3.51 $$\int_\Gamma \log |\phi(\alpha)| m_\zeta(d\alpha) = -\infty.$$

Then for all complex w (with non-negative real part)

3.52 $$\phi(\beta\rho^w) = 0.$$

Equivalently, ϕ vanishes on the subset $\beta\rho^0\Gamma^\rho$ of Δ (see [II, 5, just above 5.6] for Γ^ρ.)

If ρ maps G_+ in a 1-1 way into the (multiplicative) non-negative reals then $\phi=0$. If G_+ is the set of elements greater than or equal to the identity in an archimedean ordering of G, and (3.51 holds where) ζ is not a boundary point, then $\phi=0$.

Proof. We can clearly factor a constant out of ϕ achieving $\|\phi\|\leqq 1$ and retaining 3.51. Then 3.51 says that $\infty=\phi(1;\rho\beta)$. Now

$$\phi(1;\rho\beta) \leqq (\beta;\rho) \leqq e^u\phi(\beta\rho^u;\rho).$$

Then 3.33 tells us that $\phi(\beta\rho^{1+u})=0$. It is known [II, 7.4] that $\phi(\beta\delta^w)$ is holomorphic in w, so 3.52 must hold. The next line of 3.5 is established by invoking [II, 7.4, $u=0$].

If ρ (or ζ) never vanishes then $\rho^0=1$ so that ϕ vanishes on $\beta\Gamma^\rho$, which is a coset of the subgroup Γ^ρ in Γ. Now Γ^ρ is the set of α which are 1 wherever $\rho=1$ (on G_+). If ρ is 1-1, this is only for $x=1$ in G_+; and every α in Γ is 1 there, so that $\Gamma^\rho=\Gamma$. Finally, if ρ is 1-1 then it cannot ever vanish. Hence $\phi=0$.

In the *archimedean case*, if ζ is not on Γ then $\rho(x)\neq 1$ for some x in G_+. Let us take first the case where $\rho(x)\neq 1$ for some $x\neq 1$. Then $\rho(x)\neq 0$ for all x in G_+, because x^n is co-final in G_+ and (because $\rho\leqq 1$) $\rho(x)$ never increases as we ascend in G_+. For the same reason $\rho(x)\neq 1$ except for $x=1$. Thus ρ is 1-1 (and so $\phi=0$). If $\rho(x)=0$ for some $x\neq 1$ then $\rho(x)=0$ for all $x\neq 1$ in G_+, and ρ (and also ζ) is the "center" of the disc (which we might call 0):

3.53 $$\phi(0) = \int_\Gamma \phi(\alpha)d\alpha, \qquad \text{(Haar integral).}$$

We have now to deduce that $\phi=0$ if

3.54 $$\phi(1;0) = \infty.$$

This says (of course) that

3.55 $$\int \log |\phi(\alpha)| d\alpha = -\infty.$$

Noting that 0 in 3.54 can be written as $\rho\cdot 0$, we have (by 3.31) for all ρ

3.56 $$\phi(\rho; 0) = \infty.$$

We shall present two demonstrations of the fact that 3.56 implies $\phi=0$. Our first proof (§4) the more complicated, but has the merit of establishing a connection with some work of B. Jessen.

The second treatment is given in §5. It involves some of the ideas used by Jessen in the paper quoted in §4.

4. **Connection of $\phi(\rho; 0)$ in the archimedean case with a function of B. Jessen.** B. Jessen [III, 500–502] has proved the following lemma.

4.1 (Jessen). *Let $F(u+iv)$ be almost periodic (in u), holomorphic and uniformly bounded for $u_0<u<u_3$, and nonzero. Suppose u_1, u_2, and c are given with $u_0<u_1<u_2<u_3$ and $c>0$. Then there exists a number m such that if $b>1$ then for all real a and all u such that $u_1\leqq u\leqq u_2$, we have*

4.2 $$0 \leqq \int_a^{a+b} (\log |F(u+iv)|_m - \log |F(u+iv)|)dv < bc$$

where $|n|_m$ means max $(|n|, m)$.

With the aid of this we can show that 3.54 implies $\phi=0$.

As we have observed before, $F(u+iv)=\phi(\rho^{u+iv})$, (if it doesn't vanish identically) satisfies all the conditions of 4.1 with u_0 chosen arbitrarily but positive, and ρ chosen arbitrarily. We select $\rho\neq 0$, 1. If $\phi\neq 0$ then $F\neq 0$, since $\{\rho^{iv}\}$ is dense in Γ as $\rho^0=1$ and $\Gamma^\rho=\Gamma$.

Let "lm" stand for either lim sup or lim inf, $b\to\infty$. From 4.1 we obtain (with any value of c),

4.3 $$0 \leqq \lim \frac{1}{b}\int_a^{a+b} \log |\phi(\rho^{u+iv})|_m dv - \operatorname{lm} \frac{1}{b}\int_a^{a+b} \log |\phi(\rho^{u+iv})| dv \leqq c.$$

Consider that $\log|\phi(\rho^u\alpha)|_m$ is a continuous function on Γ. It may therefore be uniformly approximated by linear combinations of elements of G, and for each element x of G

$$\lim \frac{1}{b}\int_a^{a+b} x(\rho^{iv})dv = 0 \text{ or } 1 \tag{4.4}$$

according to whether $x=1$ or $\neq 1$ in G. Hence 4.4 gives the Haar integral of x, and consequently the lim term of 4.3 is

$$\int_\Gamma \log|\phi(\rho^u\alpha)|_m d\alpha = I(m). \tag{4.5}$$

On the other hand, 4.3 shows that

$$M = \operatorname{lm}\frac{1}{b}\int_a^{a+b}\log|\phi(\rho^{u+iv})|\,dv$$

is finite ($-\infty$ was the only danger), since

$$0 \leqq I(m) - M \leqq c. \tag{4.6}$$

As m approaches 0 from above, the integral $I(m)$ tends downward to

$$\int_\Gamma \log|\phi(\rho^u\alpha)|\,d\alpha = -\phi(\rho^u;0) \tag{4.7}$$

by Levi's theorem, and by 4.6 this is finite too. This destroys the possibility of 3.56 when $\phi\neq 0$, and finishes the proof of 4.1.

However, it is worthwhile to exploit 4.1 somewhat more. We evidently have

$$0 \leqq -\phi(\rho^u;0) - M < c$$

from which we can conclude

4.8. COROLLARY.

$$-\phi(\rho^u;0) = \lim_{b\to\infty}\frac{1}{b}\int_a^{a+b}\log|\phi(\rho^{u+iv})|\,dv.$$

Here we intend to assert also that the "mean" on the right exists.

It should not be supposed that

$$\int_\Gamma \psi(\alpha)d\alpha = \lim_{b\to\infty}\int_a^{a+b}\psi(\rho^{iv})dv$$

holds for *every* summable ψ on Γ, or that the right side necessarily exists, since it depends merely on the behavior of ψ on a set of Haar measure 0.

Jessen and Tornehave [IV] have studied the behavior of (what amounts to) the function in 4.8 and its dependence on u. They do not consider the (perhaps more elementary) functions $\phi(\rho^u; \zeta)$, $\zeta \neq 0$.

5. **Second proof that $\phi(\rho; 0) < \infty$ if $\phi \neq 0$ in the archimedean case.** Select an x from G_+, x not equal the identity. This determines a character of Γ; and the kernel ${}_x\Gamma$ of that character has Haar measure 0 in Γ. Consequently the set

$$E = \{\alpha; \alpha(x) \neq -1\}$$

has measure 1 in Γ.

Select $\rho \in \Delta$, $\rho \geqq 0$, such that $\rho(x) \neq 1$. Let

$$b = -\pi/\log \rho(x).$$

It is easy to see that E is homeomorphic with

$$(-b, b) x {}_x\Gamma$$

under the mapping

5.1 $$(v, \beta) \to \alpha = \rho^{iv}\beta.$$

If we use the proper Haar measure in ${}_x\Gamma$, and normalize so that the whole measure of ${}_x\Gamma$ is $1/2b$ then 5.1 is also measure-preserving [V], and (therefore)

5.2 $$\int_E \log |\phi(\rho\alpha)| \, d\alpha = \int_{{}_x\Gamma} \int_{-b}^{b} \log |\phi(\rho^{1+iv}\beta)| \, dv d\beta.$$

Since E has measure 1 in Γ, the integral over E in 5.2 is equal to $-\phi(\rho; 0)$.

Let $F_\beta(w) = \phi(\rho^w\beta)$ ($w = u + iv$, $u \geqq 0$). Let S be the segment $u = 1$, $-b \leqq v \leqq b$. Let W_1 be the set of points not more than $1/3$ away from S, and let W_2 be the set of points not more than $2/3$ away from S. Using the fact that $\{F_\beta\}$ is a totally bounded family of functions bounded away from the zero function, one can prove (*cf.* [VI])

5.3. *There is an N such that each F_β has no more than N zeros on W_1.*

5.4. *There is a positive number p such that for each β, if $w_1, \cdots, w_n$ are the zeros of F_β on W_1, then*

$$\left| \frac{F_\beta(z)}{(z - w_1) \cdots (z - w_n)} \right| \geqq p \text{ for } z \in S.$$

In §6 we shall use the following

5.5. *For every $r > 0$ there is an $m > 0$ such that for each β, if z in W_1 is not less than r removed from all the zeros of F_β on W_1, then*

$$|F_\beta(z)| \geqq m.$$

Now for $F = F_\beta$,

$$\log |F(w)| \geqq \log p + \sum \log |w - w_i| \qquad \text{(on } S\text{)}$$

whence

$$5.6 \qquad \int_{-b}^{b} \log |F(1+iv)| \, dv \geqq 2b \log p + \sum \int_{-b}^{b} \log |w - w_i| \, dv.$$

It is not hard to verify that for w_i in W_1,

$$5.7 \qquad \int_{-b}^{b} \log |1 + iv - w_i| \, dv \geqq 2b(\log b - 1).$$

We insert this into 5.6, and observe that there are not more than N zeros, so that

$$\int_{-b}^{b} \log |F(1+iv)| \, dv \geqq 2b(\log p + N(\log b - 1)).$$

Inserting this in 5.2 gives $\phi(\rho; 0) \neq \infty$ since

$$\int_{\Gamma} \log |\phi(\rho\alpha)| \, d\alpha = -\phi(\rho; 0) \geqq \log p + N(\log b - 1).$$

6. **The continuity of $\phi(\theta; \zeta)$ as a function of ϕ.** In this section we assume that G is *archimedean-ordered*, just as we did in §5.

We have just proved that, for ϕ in A_0, $\phi \neq 0$,

$$6.1 \qquad K(\phi) \equiv \int_{\Gamma} \log |\phi(\rho\alpha) d\alpha \neq -\infty.$$

(The symbol $K(\phi)$ is introduced for convenience at this time).

6.2. Lemma. *Let ϕ in A_0 be given. Let $\epsilon > 0$, and $\rho \in \Delta$ be given, $0 < \rho < 1$. Then a compact set in the interior of Δ can be found, and an $m > 0$, such that if $\psi \in A_0$ and $|\psi(\zeta) - \phi(\zeta)| < m$ on the compact set, then*

$$|K(\phi) - K(\psi)| < \epsilon.$$

Proof. Let ψ be any element of A_0, besides the given ϕ. Define $F_\beta(w) = \phi(\rho^w \beta)$, and define H_β similarly in terms of ψ. By 5.2 we have

$$6.21 \qquad |K(\phi) - K(\psi)| \leqq \max_{\beta \in_x \Gamma} \frac{1}{2b} \int_{-b}^{b} |\log F_\beta(1+iv) - \log H_\beta(1+iv)| \, dv.$$

Hence we will achieve our end if we show that the integral in 6.21 can be made uniformly less than ϵ.

We will not bother to write the indices β on F, H henceforth, but we intend the indices to be the same. We use the closed sets S, W_1, W_2 as in §5. Let $x_1, \cdots, x_n$ be the zeros of F on W_1, and let

$$X(w) = (w - x_1) \cdots (w - x_n).$$

By 5.5 and Rouché's theorem, there is, for each positive r (which we shall specify later), an $m>0$ such that if $|F-H|<m$ on W_1 then there is one zero of H in the r-neighborhood of each zero of F. Denote by y_i that zero of H which is near x_i. (This need not exhaust the zeros of H.) Let

$$Y(w) = (w - y_1) \cdots (w - y_n).$$

Let $X/Y-1=f$, and $H-F=g$. Then

6.22 $$\frac{H}{Y} - \frac{F}{X} = X^{-1}[fF + g(f+1)].$$

We consider the modulus of 6.22 on the boundary B_2 of W_2. The modulus of X^{-1} depends essentially only on n, and that is fixed. The modulus of f becomes as small as we wish if r is small enough; and g becomes small if $|F-H|$ is small enough on W_2. For any positive q, we select an r so small, and an m so small as to both serve as the m (for that r) in 5.5 and also so small so that if

6.23 $$|F(w) - H(w)| < m \textit{ for } w \in W_2$$

then the modulus of 6.22 is less than q on B_2, and hence on W_2:

6.24 $$\left|\frac{H}{Y} - \frac{F}{X}\right| < q \qquad (\textit{on } W_2).$$

Our first requirement on q is that it should be smaller than the p of 5.4; and we then have

6.25 $$\left|\frac{H}{F}\frac{X}{Y} - 1\right| < \frac{q}{p} < 1 \qquad (\textit{on } S).$$

Let us denote the rational function X/Y by Z. Then

6.26 $$\bigl|\log|F| - \log|H|\bigr| \leqq q/(p-q) + \bigl|\log|Z|\bigr| \qquad (\textit{on } S).$$

Now the integrals defined by the various y:

$$\int_E \log|1 + iv - y|\, dv,$$

regarded as functions of the set $E\subset[-b, b]$ are easily seen to be uniformly absolutely continuous [VII, 29.4]; whence by [VII, 29.6]

6.27 $$\int_{-b}^{b}\left|\log\left|\frac{1+iv-x}{1+iv-y}\right|\right| dv$$

is uniformly small for all y sufficiently close to x.

By 6.21, if we take the integral mean

$$\frac{1}{2b}\int_{-b}^{b} \cdots dv \qquad (w = 1 + iv)$$

of the right side of 6.26, we get an estimate of $|K(\phi)-K(\psi)|$, namely $q/(p-q)$ plus n terms like 6.27. The latter go to zero with r. Now we finally select r, q to make this estimate as small as desired. In order to ensure 6.23, we need only have $|\phi(\zeta)-\psi(\zeta)|<m$ on the "sub-disc" $\rho\Delta$.

We now turn to the measures m_σ associated with points σ not at the center of the disc. We wish to study the continuity of $\phi(\rho;\sigma)$. We can better adapt the earlier formulas if we study $\phi(\tau;\rho)$. Analogously to 6.1 we define

6.3 $$L(\phi) = \int_\Gamma \log |\phi(\tau\alpha)| m_\rho(d\alpha) = -\phi(\tau;\rho).$$

Now we suppose $|\rho|\neq 1$, $|\tau|\neq 1$; and by a rotation we can obtain $0<\tau, \rho<1$. Moreover, by the archimedean ordering, we can write $\tau=\rho^a$, $a>0$. Consequently (by 3.43)

$$L(\phi) = \sum_n \int_{-b}^{b} F_n(a + iv)c(2nb + v)dv$$

where

$$F_n(w) = \phi(\rho^{2nbi+w}).$$

For $\psi\in A_0$ we define H_n analogously. It is now a matter of studying F, H on and near the segment from $a-bi$ to $a+bi$. The previous method of estimating $|\log|F|-\log|H||$ leads in the same way to the corresponding result. We combine it with 6.2.

6.4. THEOREM. *Let ϕ in A_0 be given (where G_+ is supposed archimedean ordered). Let τ in Δ be given, $|\tau|\neq 0, 1$, and suppose $\phi\neq 0$. Let ρ be given in Δ, $|\rho|\neq 1$. Let $\epsilon>0$ be given. Then there exists a compact set in $\Delta-\Gamma$, and an $m>0$, such that if (for any other ψ in A_0)*

$$|\phi(\zeta) - \psi(\zeta)| < m \text{ for all } \zeta \text{ in that compact set}$$

then

$$\left|\int_\Gamma \log|\phi(\tau\alpha)| m_\rho(d\alpha) - \int_\Gamma \log|\psi(\tau\alpha)| m_\rho(d\alpha)\right| < \epsilon.$$

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