SOME LIMIT THEOREMS FOR NONHOMOGENEOUS MARKOFF PROCESSES

BY

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Summary. We intend to study some problems related to the asymptotic behaviour of a physical system the evolution of which is markovian. The typical example of such an evolution is furnished by an homogeneous discrete chain with a finite number of possible states considered first by A. A. Markoff. In §1 we recall briefly the main results of this theory and in §2 we treat its obvious generalization to the continuous parameter case. In §3 we pass to the proper object of this paper and we establish a limit theorem for time-homogeneous Markoff processes. This limit theorem is then extended to the nonhomogeneous case under some supplementary conditions (§4). Finally we give an application of this theory to random functions connected with a Markoff process (§5).

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1. **Finite discrete case.** Suppose we are given a system in evolution which can assume a finite number r of distinct states; suppose in addition that the random changes of states occur at fixed instants of time equally spaced on the time axis. Let us denote by:

$$P_{ij}^{n} \qquad\qquad i, j = 1, 2, \cdots, r, n = 0, 1, 2, \cdots$$

the probability that the system passes from the state i to the state j in n consecutive steps. The hypothesis of homogeneity of the chain is revealed by the fact that this probability depends only on the number n of steps and not on their position.

We are interested in the asymptotic behaviour of this quantity as $n \to \infty$. The main result in this direction is that, for fixed i and j, P_{ij}^{n} tends always towards a limit Π_{ij} in the sense of Cesaró, i.e. we always have:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{k} = \Pi_{ij}.$$

Cesaró convergence is an averaging effect which hides an important subjacent reality viz. the existence of cyclic groups inside one and the same final group. It is evident that if such groups exist, P_{ij}^{n} cannot tend towards a limit in the ordinary sense. The existence of such a limit is therefore intimately connected with the absence of cyclic groups and we speak then of the *acyclic*

case. Analytically this is expressed by the fact that the matrix $P = \{P_{ij}\}$ of the transition probabilities in one step does not admit any proper value of modulus 1 other than unity. In this case we have thus:

$$\lim_{n \to \infty} P_{ij}^n = \Pi_{ij}$$

and the limit is attained exponentially.

Suppose now that the chain has begun at a certain instant taken as the origin and let us denote by $\bar{\omega}_i^0 \geqq 0$, $i = 1, \cdots, r$, $\sum_{i=1}^r \bar{\omega}_i^0 = 1$ a system of arbitrary initial probabilities. The distribution of a priori probabilities after n consecutive steps is then given by

$$(1.1) \qquad\qquad \bar{\omega}_j^n = \sum_{i=1}^r \bar{\omega}_i^0 P_{ij}^n.$$

Letting $n \to \infty$ we have

$$(1.2) \qquad\qquad \lim_{n \to \infty} \bar{\omega}_j^n = \sum_{i=1}^r \bar{\omega}_i^0 \Pi_{ij} = P_j(\bar{\omega}_k^0).$$

In other words the a priori probabilities themselves tend towards a limit as $n \to \infty$ and this limit depends linearly on the initial probability distribution. If we take this limit distribution as the initial distribution of our chain, it is easily seen that it is "stable," i.e. that it is preserved after every step. In other words we have:

$$P_j(\bar{\omega}_k^0) = \sum_{i=1}^r P_i(\bar{\omega}_k^0) P_{ij}^n$$

for each integer $n \geqq 0$.

With this initial probability distribution the chain is not only time-homogeneous, but also stationary. The limits Π_{ij} being attained exponentially, the evolution of the system is in fact very soon indiscernible from the stationary evolution which we have just characterized.

In the general acyclic case there are infinitely many stationary evolutions of the type just studied, each of them being obtained as the limit of an evolution of the chain having begun with a given initial distribution $\bar{\omega}_k^0$. But in physical problems we do not know the initial distribution $\bar{\omega}_k^0$ and this fact makes the general acyclic case rather uninteresting. Thus we look for acyclic chains the evolution of which tends towards a stable limit independent of the initial distribution $\bar{\omega}_k^0$. It is easily seen that this condition is fulfilled if the limits Π_{ij}, which are attained in the ordinary sense, are independent of the first index i, i.e. are independent of the initial state, such that we can then write $\Pi_{ij} = P_j$.

If this condition is fulfilled we say that we are in the *"regular case."* Hence, in the regular case:

$$\lim_{n \to \infty} P_{ij}^{n} = P_j$$

the limit P_j being always attained exponentially.

Numerous papers have been devoted to the study of the regular case. A. A. Markoff [1] was the first in 1906 to find a necessary and sufficient condition in order that it be realized. His condition is the following:

Condition of Markoff. A necessary and sufficient condition for the regular case to be realized is that there exist positive integers $n_0 > 0$, $j_0 \in (1, 2, \cdots, r)$ and an $\epsilon > 0$ such that $P_{ij_0}^{n_0} \geqq \epsilon$ for every index $i \in (1, 2, \cdots, r)$.

However this condition is very cumbersome to verify in application, because it needs the iteration of the matrix $P = \{P_{ij}\}$ of transition probabilities in one step. Later on several sufficient conditions have been given which are quite simple to verify in a given case but are often too restrictive. (See an account in the exhaustive book of M. Fréchet [2].) But among them we must mention the sufficient condition of J. Kaucky [3] which was found to be also necessary by M. Konečny [4].

Condition of Kaucky-Konečny. A necessary and sufficient condition for the regular case to be realized is that the matrix $P = \{P_{ij}\}$ of the transition probabilities in one step does not admit any proper value of modulus 1 other than unity (acyclic case) and further that unity is a simple root of the characteristic equation of P.

In the regular case the a priori probabilities $\bar{\omega}_j^n$ given by (1.1) tend, as $n \to \infty$, towards a limit which is independent of the initial distribution $\bar{\omega}_k^0$. In fact in this case (1.2) becomes

$$\lim_{n \to \infty} \bar{\omega}_j^n = \sum_{i=1}^{r} \bar{\omega}_i^0 P_j = P_j \sum_{i=1}^{r} \bar{\omega}_i^0 = P_j.$$

Thus the evolution of the system tends towards a stationary evolution which, in this case, is unique and independent of the initial distribution. Any initial disturbance of the chain is smoothed out as $n \to \infty$ and that exponentially. As a first generalization of the chains we have just studied we consider the case of continuous parameter chains, with a finite number of states.

2. **Finite continuous parameter case.** In this case the process can be defined by the function

$$P_{ij}(t) \qquad\qquad i, j = 1, 2, \cdots, r, t \geqq 0$$

which denotes the probability that the system passes from the state i to the state j during the time-interval t.

It can be shown that, in this case, the probability $P_{ij}(t)$ tends always towards a limit in the ordinary sense as $t \to \infty$, this limit being again obtained exponentially. In other words we have always

$$\lim_{t \to \infty} P_{ij}(t) = \Pi_{ij}.$$

One can say that we are always in the acyclic case.

This result may seem surprising. But in fact, in the continuous parameter case we impose the Chapman-Kolmogoroff relation $P(t+\tau) = P(t)P(\tau) = P(\tau)P(t)$, t, $\tau \geqq 0$ for all pairs of non-negative numbers t, τ whereas in the discrete case we impose this condition only for non-negative integral values. Thus we impose much more stringent conditions in the continuous parameter case than in the discrete case and it is understandable that there may exist phenomena in the discrete case which disappear in the continuous case. The mathematical reason is the following:

In the finite discrete case the existence of cyclic groups is intimately connected with the existence of proper values of the transition matrix $P = \{P_{ij}^1\}$ in one step of modulus one and distinct from unity. If λ is such a proper value, $|\lambda| = 1$, $\lambda \neq 1$, it is known that it is necessarily a root of unity. But the number λ^n, which is the proper value of the transition matrix $P^n = \{P_{ij}^n\}$ is also a root of unity, and this for each integer $n \geqq 1$.

In the finite continuous parameter case similar considerations can be made. If λ is a proper value of the transition matrix $P(1) = \{P_{ij}(1)\}$ of modulus one and distinct from unity, it must be a root of unity. But then λ^t which is a proper value of $P(t) = \{P_{ij}(t)\}$ is also of modulus one and has to be a root of unity for each $t \geqq 0$. This can be the case only if $\lambda = 1$. Thus in the finite continuous parameter case every proper value of modulus one of the matrix $P(1)$ is necessarily equal to unity and the cyclic groups disappear.

It can be shown as in the discrete case that the a priori probability distribution itself tends towards a limit as $t \to \infty$ and that if this limit is taken as initial probability distribution the evolution becomes stationary.

This limit, of course, depends on the initial distribution unless the Π_{ij} are independent of the first index i, that is of the initial state.

We shall not insist on the generalization where the number of possible states is denumerable. This theory is mainly due to A. A. Kolomogoroff [5] and J. L. Doob [6]. An interesting account of this subject has recently been given by P. Lévy [7].

We pass now to the proper object of this paper where we suppose that the set of possible states is the real line.

3. **Time homogeneous Markoff processes.** Let $X(t)$ be a time-homogeneous Markoff random function and let

$$F(t; x, E) = P\{X(t + \tau) \in E \mid X(\tau) = x\} \qquad\qquad t, \tau \geqq 0$$

be the probability that the system passes from the point $x \in R^1$ to the Borel set $E \subset R^1$ (R^1 being the real line) in the time-interval t. The process being supposed time-homogeneous, this function depends only upon the length t of this time-interval and not upon its position on the real axis. It should be noted that the time-homogeneity is a condition imposed on the transition

probability law and has to be distinguished from the stationarity which postulates the invariance of the joint probability law under translations of the time-axis.

The a priori probability law of the process at time t will be denoted by

$$\phi(t, E) = P\{X(t) \in E\},$$

E being, as always, a Borel subset of R^1.

Using some results of J. L. Doob [8], we shall prove the following theorem:

THEOREM I. *Suppose the process has begun at a fixed instant taken as the origin. If it satisfies the Doeblin condition* (D), *which we shall define later, then, the a priori probability law* $\phi(t, E)$ *tends, as* $t \to \infty$, *towards a limiting probability distribution* $\Lambda(E)$ *which is independent of any initial probability distribution, the limit* $\Lambda(E)$ *being attained exponentially and uniformly in E. In other words it is possible to give a bound of the following type*:

$$\left| \phi(t, E) - \Lambda(E) \right| < h e^{-kt}$$

where h and k are strictly positive constants independent of E. The limiting distribution $\Lambda(E)$ *is in addition stable, i.e. it is preserved throughout the course of time. In other words it satisfies the integral equation*:

$$\Lambda(E) = \int_{-\infty}^{+\infty} \Lambda(dx) F(t; x, E) \qquad \text{for each } t \geqq 0.$$

If we take $\Lambda(E)$ as the initial distribution, the process is not only time-homogeneous, but also stationary. The limit $\Lambda(E)$ being attained exponentially the evolution of the system is in fact very soon indiscernible from the stationary evolution which we have just characterized.

Note: J. Neveu, in his thesis [9], has proved an analogous theorem by an entirely different method going back to K. Yosida and S. Kakutani [12]. But all these methods suppose that the set of possible states of $X(t)$ is a *compact* subset of R^1. In our approach, resembling Markoff's original method, we need not make this hypothesis.

Doeblin-condition D. Let $\mu(E)$ be a bounded positive measure, it is well known that $F(t; x, E)$ (which is a probability measure in E) can be decomposed into a component absolutely continuous with respect to $\mu(E)$ and a singular component (Lebesgue decomposition). In other words we can always write:

$$F(t; x, E) = \int_E p(t; x, y) \mu(dy) + \Delta(t; x, E),$$

$p(t; x, y)$ being a Borel-measurable function of y and $\Delta(t; x, E)$ the singular component of $F(t; x, E)$ i.e. a measure in E which is zero everywhere except

on a set of Borel-measure zero (dependent on t, x)(¹).

The Doeblin condition D consists then in supposing that there exist a measure $\mu(E)$ of the above kind, a strictly positive number $\delta > 0$ and a Borel set C, such that, for a sufficiently large t_0 we have:

$$\mu(C) > 0,$$

$$p(t_0; x, y) \geqq \delta > 0 \quad \text{uniformly in } x \in R^1, y \in C.$$

Proof of the Theorem I. The proof will be given in several steps:

(i) We begin by establishing a fundamental property of the transition probability function $F(t; x, E)$. We start from the Chapman-Kolmogoroff equation:

$$F(t + \tau; x, E) = \int_{-\infty}^{+\infty} F(t; x, dy)F(\tau; y, E) \qquad \text{for each } t, \tau \geqq 0.$$

Since $\int_{-\infty}^{+\infty} F(t; x, dy) = 1$ we have immediately

$$F(t + \tau; x, E) \leqq \sup_{y \in R^1} F(\tau; y, E) \int_{-\infty}^{+\infty} F(t; x, dy) = \sup_{y \in R^1} F(\tau; y, E).$$

This bound being uniform in x, we have

$$\sup_{x \in R^1} F(t + \tau; x, E) \leqq \sup_{y \in R^1} F(\tau; y, E).$$

In other words the function

$$M(t, E) = \sup_{x \in R^1} F(t; x, E)$$

is an nonincreasing function of t. It can be shown in the same manner that the function

$$m(t, E) = \inf_{x \in R^1} F(t; x, E)$$

is a nondecreasing function of t and we have of course:

$$0 \leqq m(t, E) \leqq M(t, E) \leqq 1.$$

These monotonic properties combined with the boundedness of the functions assure the existence of the following limits

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(¹) The Borel-measure of a Borel-measurable linear set E is defined in the following constructive way: one begins by defining the measure of intervals of R^1, without distinguishing between open and closed intervals. The measure of an interval is the length of this interval, measured with a certain arbitrary unity. A Borel-measurable linear set E is then defined as the union of a finite or a denumerably infinite number of disjoint intervals and the Borel-measure of E is by definition the sum of the measures (lengths) of these intervals.

For this notation, see e.g., E. Borel, *Leçons sur la théorie des functions*, Paris, Gauthier-Villars, 1950.

$$\lim_{t \to \infty} M(t, E) = L(E), \qquad \lim_{t \to \infty} m(t, E) = l(E)$$

and we have

$$0 \leqq l(E) \leqq L(E) \leqq 1.$$

(ii) Now we return to the a priori distribution $\phi(t, E)$ at time $t \geqq 0$, which is connected with an arbitrary initial distribution $\phi(0, E)$ by the relation

$$\phi(t, E) = \int_{-\infty}^{+\infty} \phi(0, dx)F(t; x, E).$$

By the definition of the functions $M(t, E)$ and $m(t, E)$ we have

$$0 \leqq m(t, E) \leqq \phi(t, E) \leqq M(t, E) \leqq 1.$$

Taking the limit as $t \to \infty$

$$0 \leqq l(E) \leqq \liminf_{t \to \infty} \phi(t, E) \leqq \limsup_{t \to \infty} \phi(t, E) \leqq L(E) \leqq 1.$$

So far we have made no hypotheses on the transition probability function $F(t; x, E)$. The main role of the Doeblin condition D will then be to assure the equality of the 2 limits $L(E)$ and $l(E)$.

Denoting their common value by $\Lambda(E)$ we will then have:

$$\lim_{t \to \infty} \phi(t, E) = \Lambda(E).$$

In addition the limit $\Lambda(E)$ is attained exponentially and uniformly in E, $\Lambda(E)$ being the limit, uniformly in E, of a family of probability measures is itself a probability measure, i.e. $\Lambda(R^1) = 1$. Further it is independent of the initial distribution $\phi(0, E)$.

(iii) We shall now establish that the Doeblin condition D implies that $L(E) = l(E) = \Lambda(E)$. We can write, t_0 being a positive constant to be determined later

$$F(t + t_0; x, E) - F(t + t_0; y, E) = \int_{-\infty}^{+\infty} F(t; \lambda, E)[F(t_0; x, d\lambda) - F(t_0; y, d\lambda)].$$

The form of the second member leads us to introduce the function

$$\psi_{x,y}(E) = F(t_0; x, E) - F(t_0; y, E).$$

For fixed x and y this function is a σ-additive set function in E and $\psi_{x,y}(R^1) = 0$. There exists therefore a Borel set S^+ such that for each Borel subset $E^+ \subset S^+$ we have $\psi_{x,y}(E^+) \geqq 0$ and for each Borel subset E^- of its complement $S^- = R^1 - S^+$ we have $\psi_{x,y}(E^-) \leqq 0$. (Hahn decomposition of a signed measure.) Whence

$$\psi_{x,y}(S^+) + \psi_{x,y}(S^-) = \psi_{x,y}(R^1) = 0, \qquad \psi_{x,y}(S^+) = -\psi_{x,y}(S^-).$$

Thus we can set

$$\left| \psi_{x,y}(S^+) \right| = \left| \psi_{x,y}(S^-) \right| = \theta(x, y).$$

The function $\theta(x, y)$ lies between 0 and 1, since:

$$0 \leqq \theta(x, y) = F(t_0; x, S^+) - F(t_0; y, S^+) \leqq F(t_0; x, S^+) \leqq F(t_0; x, R^1) = 1.$$

We shall see that under the Doeblin condition D, $\sup_{(x,y)\in R^1\times R^1} \theta(x, y) < 1 - \epsilon$, ϵ being a certain strictly positive number. In fact:

$$\theta(x, y) = \psi_{x,y}(S^+) = F(t_0; x, S^+) - F(t_0; y, S^+)$$

$$= 1 - F(t_0; x, S^-) - F(t_0; y, S^+)$$

whence, using the Lebesgue decomposition of $F(t; x, E)$

$$\leqq 1 - \int_{S^-} p(t_0; x, \lambda)\mu(d\lambda) - \int_{S^+} p(t_0; y, \lambda)\mu(d\lambda)$$

whence, C being the set occurring in the Doeblin condition D:

$$\leqq 1 - \int_{S^-\cap C} p(t_0; x, \lambda)\mu(d\lambda) - \int_{S^+\cap C} p(t_0; y, \lambda)\mu(d\lambda).$$

Supposing now that t_0 is the t_0 occurring in the Doeblin condition D

$$\leqq 1 - \delta\mu(S^- \cap C) - \delta\mu(S^+ \cap C) = 1 - \delta\mu(C).$$

This bound being independent of x, y we have thus:

$$\sup_{(x,y)\in R^1\times R^1} \theta(x, y) \leqq 1 - \delta\mu(C) < 1.$$

Now we write:

$$F(t + t_0; x, E) - F(t + t_0; y, E)$$

$$= \int_{S^+} F(t; \lambda, E) \left| \psi_{x,y}(d\lambda) \right| - \int_{S^-} F(t; \lambda, E) \left| \psi_{x,y}(d\lambda) \right|$$

$$\leqq M(t, E) \int_{S^+} \left| \psi_{x,y}(d\lambda) \right| - m(t, E) \int_{S^-} \left| \psi_{x,y}(d\lambda) \right|$$

$$= [M(t, E) - m(t, E)]\theta(x, y) \leqq [M(t, E) - m(t, E)][1 - \delta\mu(C)].$$

This bound being independent of x, y we have thus:

$$M(t + t_0, E) - m(t + t_0, E) \leqq [M(t, E) - m(t, E)][1 - \delta\mu(C)] \leqq 1 - \delta\mu(C)$$

where $0 \leqq 1 - \delta\mu(C) < 1$ or:

$$\text{Oscillation}_{x\in R^1} F(t + t_0; x, E) \leqq [1 - \delta\mu(C)]\text{Oscillation}_{x\in R^1} F(t; x, E).$$

By iteration we have then, n being a positive integer:

(3.1) $$M(t + nt_0, E) - m(t + nt_0, E) \leqq [1 - \delta\mu(C)]^n.$$

Supposing now t and t_0 fixed and letting $n \to \infty$ the first member tends exponentially towards 0, and this uniformly in E.

But in (i) we have already established the existence of the limits:

$$\lim_{t \to \infty} M(t, E) = L(E),$$

$$\lim_{t \to \infty} m(t, E) = l(E).$$

Every partial sequence extracted from the positive t-axis must give the same limits, i.e. we have also:

$$\lim_{n \to \infty} M(t + nt_0, E) = L(E),$$

$$\lim_{n \to \infty} m(t + nt_0, E) = l(E).$$

The inequality (3.1) shows then that $L(E) = l(E) = \Lambda(E)$ and further that the common limit $\Lambda(E)$ is attained exponentially and uniformly in E.

We have already said that the t_0 is fixed by the Doeblin condition D. The value being thus fixed, we can write, t being an arbitrary positive value:

$$t = nt_0 + r, \qquad\qquad\qquad 0 \leqq r < t_0$$

whence from (3.1) for $t \geqq t_0$ (in order that $n \geqq 1$)

$$(3.2) \qquad M(t, E) - m(t, E) \leqq \left[1 - \delta\mu(C)\right]^{(t-r)/t_0} = K\theta^t$$

where

$$K = \left[1 - \delta\mu(C)\right]^{-r/t_0}, \qquad \theta = \left[1 - \delta\mu(C)\right]^{1/t_0}.$$

But $0 \leqq r/t_0 < 1$, hence $K \leqq \left[1 - \delta\mu(C)\right]^{-1} = h > 0$. Further $0 \leqq \theta < 1$, hence $\theta = e^{-k}$, $k = -\log\theta > 0$. We can therefore write (3.2):

$$(3.3) \qquad M(t, E) - m(t, E) \leqq he^{-kt}, \qquad\qquad (t \geqq t_0),$$
$$h = \left[1 - \delta\mu(C)\right]^{-1} > 0,$$
$$k = -\log\left[1 - \delta\mu(C)\right]^{1/t_0} > 0$$

t_0 being the minimum value for which the Doeblin condition D holds. Finally:

$$(3.4) \qquad \left|\phi(t, E) - \Lambda(E)\right| \leqq M(t, E) - m(t, E) \leqq he^{-kt} \qquad (t \geqq t_0).$$

(iv) It remains to show that the limiting distribution $\Lambda(E)$ is stable i.e. that it satisfies the integral equation

$$\Lambda(E) = \int_{-\infty}^{+\infty} \Lambda(dx)F(t; x, E) \qquad\qquad \text{for each } t \geqq 0.$$

To prove that we need the following lemma which we give without proof:

LEMMA. *If, as $t \to \infty$, the family of probability measures $\phi(t, E)$ tends towards a limit $\Lambda(E)$, uniformaly in E we have, λ being a bounded Borel measurable function:*

$$\lim_{t \to \infty} \int_{-\infty}^{+\infty} \phi(t, dy)\lambda(y) = \int_{-\infty}^{+\infty} \left[\lim_{t \to \infty} \phi(t, dy) \right] \lambda(y) = \int_{-\infty}^{+\infty} \Lambda(dy)\lambda(y).$$

We can now prove the stability of $\Lambda(E)$. We have:

$$\int_{-\infty}^{+\infty} \Lambda(dy)F(t; y, E) = \int_{-\infty}^{+\infty} \left[\lim_{\tau \to \infty} \phi(\tau, dy) \right] F(t; y, E)$$

hence, by the lemma:

$$= \lim_{\tau \to \infty} \int_{-\infty}^{+\infty} \phi(\tau, dy)F(t; y, E) = \lim_{\tau \to \infty} \phi(t + \tau, E) = \Lambda(E).$$

The theorem is completely proved.

P.S. As has been pointed out by Professor M. Kac, the Doeblin condition D, which is an obvious analogue to the Markoff's original condition for the regular case, is rather difficult to verify for a given homogeneous process. Thus it would be very interesting to substitute it by another more accessible condition but we have not succeeded in doing so. However, as has been pointed out by Professor R. Fortet, Doeblin's condition is, in a certain way, necessary for the convergence of $F(t; x, E)$ as $t \to \infty$. Suppose in fact, that

$$\lim_{t \to \infty} F(t; x, E) = \Lambda(E) \qquad \text{uniformly in } x \text{ and } E,$$

where $\Lambda(E)$ is a probability measure. Then, to every $\epsilon > 0$ we can associate $t_0(\epsilon)$ such that, for $t > t_0$:

$$\left| F(t; x, E) - \Lambda(E) \right| < \epsilon$$

whence $F(t; x, E) > \Lambda(E) - \epsilon$ uniformly in x and E. But, $\Lambda(E)$ being a probability measure, there exists a Borel-measurable subset C of R^1, of nonzero Borel-measure, and such that:

$$\Lambda(C) > 2\epsilon \qquad\qquad (\text{if } \epsilon < 1/2).$$

Then

$$F(t; x, E) > \epsilon \qquad\qquad \text{for } t > t_0$$

and, finally, for $t > t_0$:

(1) $$F(t; x, E) > \delta\Lambda(C) \qquad\qquad \delta > 0$$

with $\delta < \epsilon/\Lambda(C)$. But (1) is a weaker form of Doeblin's condition (D) with

$\mu(E) = \Lambda(E)$ and we see that condition (1) is necessary for convergence in the specified sense.

4. Nonhomogeneous Markoff processes. So far we have studied time-homogeneous Markoff processes only. Time homogeneity being a rather restrictive condition in the long run, it would be desirable to have limit theorems of the above kind in the most general case. In fact the problem becomes very complicated and we have succeeded in establishing asymptotic properties only under severe conditions. This problem was suggested to me by the thesis of J. P. Vigier (unpublished) (10). We set, as always:

$$F(t, x; \tau, E) = P\{X(\tau) \in E \mid X(t) = x\} \qquad t < \tau,$$
$$\phi(t, E) = P\{X(t) \in E\},$$

E being, as always, a Borel-measurable subset of R^1. We have seen in §3 that in the time-homogeneous case the Doeblin condition (D) suffices to assure the existence of a stable limiting distribution $\Lambda(E)$, i.e. of a distribution satisfying the integral equation

$$\Lambda(E) = \int_{-\infty}^{+\infty} \Lambda(dx) F(t; x, E).$$

In the general case we shall suppose explicitly the existence of a strictly positive stable distribution, i.e. we suppose that there exists a distribution $\Lambda(E)$ satisfying the conditions:

(4.1)
$$\Lambda(E) > 0 \qquad\qquad \text{if the Borel-measure of } E \text{ is } > 0,$$
$$\Lambda(E) = \int_{-\infty}^{+\infty} \Lambda(dx) F(t, x; \tau, E) \qquad \text{for all } t, \tau > 0, t < \tau.$$

We shall then prove the following theorem:

THEOREM II. *Suppose the process has begun at a certain instant taken as the origin. If there exists a strictly positive stable distribution $\Lambda(E)$ and if in addition the transition probability distribution $F(t, x; \tau, E)$, $t < \tau$ satisfies the condition D' (analogous to the Doeblin condition D) which we shall define later, the a priori distribution $\phi(t, E)$ tends, as $t \to \infty$, towards the stable distribution $\Lambda(E)$. For the theorem to hold we must however take the initial probability distributions $\phi(0, E)$ from a certain family $\mathcal{G}$ defined by*

$$\mathcal{G} : \phi(0, E) \leqq H\Lambda(E),$$

H being a positive constant independent of E.

Under these conditions the limit is again attained exponentially and uniformly in E, i.e. we have again a bound of the following type:

$$\left| \phi(t, E) - \Lambda(E) \right| \leqq \mathbf{\cdot} h e^{-kt},$$

h and k being positive constants independent of E.

Condition D'. There exists a Borel set C, a number $\delta > 0$ such that, for t_0 sufficiently large we have simultaneously:

$$\Lambda(C) > 0,$$

$$F(t, x; t + t_0, E) \geqq \delta\Lambda(E)$$

for each $t \in R^1$ and each $x \in C$.

First it is clear that the hypotheses of Theorem II imply:

$$\sup_{E \in B} \frac{\phi(t, E)}{\Lambda(E)} \leqq H \qquad \text{for every } t > 0,$$

B being the class of all Borel measurable subsets of R^1. We now proceed to prove the theorem.

Proof of the theorem. It will follow the patterns of the proof of Theorem I and will be given in several steps.

(i) We can write:

$$\frac{\phi(\tau, E)}{\Lambda(E)} = \int_{-\infty}^{+\infty} \frac{\phi(t, dx)}{\Lambda(dx)} \frac{\Lambda(dx)}{\Lambda(E)} F(t, x; \tau, E) \qquad t < \tau$$

$$\leqq \sup_{G \in B} \frac{\phi(t, G)}{\Lambda(G)} \int_{-\infty}^{+\infty} \frac{\Lambda(dx)}{\Lambda(E)} F(t, x; \tau, E)$$

$$= \sup_{G \in B} \frac{\phi(t, G)}{\Lambda(G)} \text{ by (4.1)}$$

and the supremum is finite owing to the above lemma. The bound on the right being independent of E, we have also:

$$\sup_{E \in B} \frac{\phi(\tau, E)}{\Lambda(E)} \leqq \sup_{G \in B} \frac{\phi(t, G)}{\Lambda(G)}, \qquad t < \tau.$$

In other words the function

$$M(t) = \sup_{E \in B} \frac{\phi(t, E)}{\Lambda(E)}$$

is a nonincreasing function of t. It is shown in the same manner that the function

$$m(t) = \inf_{E \in B} \frac{\phi(t, E)}{\Lambda(E)}$$

is a nondecreasing function of t and we have of course

$$0 \leqq m(t) \leqq M(t) \leqq H < \infty.$$

These monotonic properties combined with the boundedness of the functions assure the existence of the following limits

$$\lim_{t \to \infty} M(t) = L, \qquad \lim_{t \to \infty} m(t) = l$$

and we have

$$0 \leq l \leq L \leq H < \infty.$$

(ii) Now we return to the a priori distribution $\phi(t, E)$ at time $t \geq 0$. We have of course:

$$0 \leq m(t) \leq \frac{\phi(t, E)}{\Lambda(E)} \leq M(t) \leq H < \infty.$$

Taking the limit as t

$$0 \leq l \leq \liminf_{t \to \infty} \frac{\phi(t, E)}{\Lambda(E)} \leq \limsup_{t \to \infty} \frac{\phi(t, E)}{\Lambda(E)} \leq L \leq H < \infty.$$

So far we have made no hypotheses on the transition probability function $F(t, x; \tau, E)$. The main role of the condition D′ will again be to assure the equality of the two limits L and 1. Denoting their common value by Λ, we have then:

$$\lim_{t \to \infty} \frac{\phi(t, E)}{\Lambda(E)} = \Lambda.$$

But then Λ is necessarily equal to 1 because, if we take $E = R^1$, $\phi(t, R^1) = \Lambda(R^1) = 1$. Hence, under condition D′ we will see that

$$\lim_{t \to \infty} \frac{\phi(t, E)}{\Lambda(E)} = 1.$$

In addition the limit 1 is attained exponentially and uniformly in E.

(iii) We shall now show that condition D′ implies $L = 1 = \Lambda = 1$. We can write, t_0 being a positive constant to be determined later and E, G being two Borel-measurable sets:

$$\frac{\phi(t + t_0, E)}{\Lambda(E)} - \frac{\phi(t + t_0, G)}{\Lambda(G)}$$

$$= \int_{-\infty}^{+\infty} \frac{\phi(t, dx)}{\Lambda(dx)} \left[\frac{\Lambda(dx)}{\Lambda(E)} F(t, x; t + t_0, E) - \frac{\Lambda(dx)}{\Lambda(G)} F(t, x; t + t_0, G) \right].$$

The form of the second member leads us to introduce the differential expression:

$$\psi_{t,E,G}(x,\,dx) = \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E) - \frac{\Lambda(dx)}{\Lambda(G)}\,F(t,\,x;\,t+t_0,\,G).$$

For fixed t, E, G we have then, owing to (4.1):

$$\int_{-\infty}^{+\infty} \psi_{t,E,G}(x,\,dx) = 0.$$

Let us denote by S^+ the set of x-values (depending on t, E, G), for which $\psi_{t,E,G}(x,\,dx) \geqq 0$ and by S^- its complement. We have then

$$0 = \int_{-\infty}^{+\infty} \psi_{t,E,G}(x,\,dx) = \int_{S^+} \psi_{t,E,G}(x,\,dx) + \int_{S^-} \psi_{t,E,G}(x,\,dx)$$

whence:

$$\int_{S^+} \psi_{t,E,G}(x,\,dx) = -\int_{S^-} \psi_{t,E,G}(x,\,dx).$$

Thus we may set

$$\int_{S^+} \left| \psi_{t,E,G}(x,\,dx) \right| = \int_{S^-} \left| \psi_{t,E,G}(x,\,dx) \right| = \theta(t,\,E,\,G).$$

The function $\theta(t,\,E,\,G)$ lies between 0 and 1, since

$$0 \leqq \theta(t,\,E,\,G) = \int_{S^+} \psi_{t,E,G}(x,\,dx)$$

$$= \int_{S^+} \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E) - \int_{S^+} \frac{\Lambda(dx)}{\Lambda(G)}\,F(t,\,x;\,t+t_0,\,G)$$

$$\leqq \int_{S^+} \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E) \leqq \int_{-\infty}^{+\infty} \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E) = 1.$$

We shall see that under condition D$'$

$$\sup_{t\in R^1;\,(E,\,G)\in B\times B} \theta(t,\,E,\,G) < 1 - \epsilon,$$

ϵ being a certain strictly positive number. In fact:

$$\theta(t,\,E,\,G) = \int_{S^+} \psi_{t,E,G}(x,\,dx) = \int_{S^+} \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E)$$

$$- \int_{S^+} \frac{\Lambda(dx)}{\Lambda(G)}\,F(t,\,x;\,t+t_0,\,G)$$

$$= 1 - \int_{S^-} \frac{\Lambda(dx)}{\Lambda(E)}\,F(t,\,x;\,t+t_0,\,E) - \int_{S^+} \frac{\Lambda(dx)}{\Lambda(G)}\,F(t,\,x;\,t+t_0,\,G)$$

whence, C being the set occurring in the condition (D'):

$$\leqq 1 - \int_{S^- \cap C} \frac{\Lambda(dx)}{\Lambda(E)} F(t, x; t + t_0, E) - \int_{S^+ \cap C} \frac{\Lambda(dx)}{\Lambda(G)} F(t, x; t + t_0, G)$$

whence if t_0 is chosen as required in the condition (D'):

$$\leqq 1 - \delta\Lambda(S^- \cap C) - \delta\Lambda(S^+ \cap C) = 1 - \delta\Lambda(C).$$

The bound on the right being independent of t, E, G, we have:

$$\sup_{t \in R^1; (E, G) \in B \times B} \theta(t, E, G) < 1 - \delta\Lambda(C) < 1.$$

We now write:

$$\frac{\phi(t + t_0, E)}{\Lambda(E)} - \frac{\phi(t + t_0, G)}{\Lambda(G)}$$

$$= \int_{S^+} \frac{\phi(t, dx)}{\Lambda(dx)} \left| \psi_{t, E, G}(x, dx) \right| - \int_{S^-} \frac{\phi(t, dx)}{\Lambda(dx)} \left| \psi_{t, E, G}(x, dx) \right|$$

$$\leqq [M(t) - m(t)]\theta(t, E, G) \leqq [M(t) - m(t)][1 - \delta\Lambda(C)].$$

The bound on the right being independent of E, G we have, taking the supremum on the left:

$$M(t + t_0) - m(t + t_0) \leqq [M(t) - m(t)][1 - \delta\Lambda(C)] \leqq H[1 - \delta\Lambda(C)]$$

or

$$\operatorname*{Oscillation}_{E \in B} \frac{\phi(t + t_0, E)}{\Lambda(E)} \leqq [1 - \delta\Lambda(C)] \times \operatorname*{Oscillation}_{E \in B} \frac{\phi(t, E)}{\Lambda(E)}.$$

By iteration, we have then, n being a positive integer:

(4.2)	$$M(t + nt_0) - m(t + nt_0) \leqq H[1 - \delta\Lambda(C)]^n.$$

Supposing now t and t_0 fixed and letting $n \to \infty$, the first member tends exponentially towards 0.

But in (i) we have already established the existence of the limits:

$$\lim_{t \to \infty} M(t) = L, \qquad \lim_{t \to \infty} m(t) = l.$$

Hence

$$\lim_{n \to \infty} M(t + nt_0) = L, \qquad \lim_{n \to \infty} m(t + nt_0) = l.$$

The inequality (4.2) shows then that $L = l$. But we have already seen that in this case the common value of L, l must be 1.

The value t_0 is fixed by the condition D'. t being an arbitrary positive

value, we can write:

$$t = nt_0 + r \qquad\qquad 0 \leqq r < t_0$$

whence from (4.2), for $t \geqq t_0$ (in order that $n \geqq 1$):

(4.3) $$M(t) - m(t) \leqq H[1 - \delta\Lambda(C)]^{(t-r)/t_0} = K\theta^t$$

where

$$K = H[1 - \delta\Lambda(C)]^{-r/t_0}, \qquad \theta = [1 - \delta\Lambda(C)]^{1/t_0}.$$

But $0 \leqq r/t_0 < 1$ hence $K \leqq H[1 - \delta\Lambda(C)]^{-1} = h > 0$. Further $0 \leqq \theta < 1$ hence $\theta = e^{-k}$, $k = -\log \theta > 0$ and finally (4.3) can be written:

$$M(t) - m(t) \leqq he^{-kt},$$

(4.4) $$h = H[1 - \delta\Lambda(C)]^{-1} > 0,$$

$$k = -\log [1 - \delta\Lambda(C)]^{1/t_0} > 0,$$

t_0 being the minimum value for which condition D′ holds. Finally we have:

$$\left| \frac{\phi(t, E)}{\Lambda(E)} - 1 \right| \leqq M(t) - m(t) \leqq he^{-kt}$$

whence

$$|\phi(t, E) - \Lambda(E)| = \Lambda(E) \left| \frac{\phi(t, E)}{\Lambda(E)} - 1 \right|$$

$$\leqq \Lambda(E)he^{-kt} \leqq \Lambda(R^1)he^{-kt} = he^{-kt}$$

i.e. as $t \to \infty$, $\phi(t, E)$ tends exponentially and uniformly in E, towards the stable distribution $\Lambda(E)$.

5. Application: Random functions connected with a Markoff process. Let $\lambda(\cdot)$ be a bounded nonrandom Borel measurable function defined on the whole axis. $X(t)$ being an arbitrary general Markoff process, we consider the function $\lambda[X(t)]$ which is then a random function defined at every instant t and for each realization of $X(t)$. Following Blanc-Lapierre and Fortet (11) we call it a random function connected with the original Markoff process $X(t)$.

We are interested in the study of such random functions. We always suppose that the original Markoff process $X(t)$ has begun at a certain instant taken as the origin.

Let us denote by M the Banach space of (real or complex) σ-additive set functions defined on the Borel-field of Borel measurable linear sets and being of bounded total variation; further by $\overline{M}$ the Banach space of (real or complex) bounded Borel-measurable functions defined on the whole real axis.

It is then known (see [12]) that the transition probability function $F(t, x; \tau, E)$ of the original Markoff process $X(t)$ allows us to define on M and $\overline{M}$ 2 families of endomorphisms $T_{t,\tau}$ and $\overline{T}_{\tau,t}$ in the following manner:

$$\phi \in M\phi \to T_{t,\tau}\phi = \psi, \quad \psi(E) = \int_{-\infty}^{+\infty} \phi(dx)F(t; x; \tau, E), \qquad t < \tau;$$

$$\bar{\phi} \in \overline{M}\bar{\phi} \to \overline{T}_{\tau,t}\bar{\phi} = \bar{\psi}, \quad \bar{\psi}(x) = \int_{-\infty}^{+\infty} \bar{\phi}(y)F(t, x; \tau, dy), \qquad t < \tau.$$

Further, if $\phi \in M$, $\bar{\phi} \in \overline{M}$ a scalar product can be defined by setting:

$$\langle \phi, \bar{\phi} \rangle = \int_{-\infty}^{+\infty} \phi(dx)\bar{\phi}(x)$$

and then the operators $T_{t,\tau}$ and $\overline{T}_{\tau,t}$ are adjoint with respect to this scalar product.

It is easily seen that if $\lambda \in \overline{M}$, $E\{\lambda[X(t)]\}$ exists and is finite for every $t \geq 0$.

We are interested in the asymptotic behavior of $E\{\lambda[X(t)]\}$ as $t \to \infty$.

THEOREM III. *Under the conditions of Theorem* II, *the limit of* $E\{\lambda[X(t)]\}$ *as* $t \to \infty$, *exists and is finite; more precisely we have*:

$$\lim_{t \to \infty} E\{\lambda[X(t)]\} = \langle \Lambda, \lambda \rangle = \mu < \infty,$$

$\Lambda(E)$ *being the limit of* $\phi(t, E)$ *as* $t \to \infty$.

Proof. We have

$$E\{\lambda[X(t)]\} = \int_{-\infty}^{+\infty} \phi(t, dx)\lambda(x)$$

whence

$$\lim_{t \to \infty} E\{\lambda[X(t)]\} = \lim_{t \to \infty} \int_{-\infty}^{+\infty} \phi(t, dx)\lambda(x),$$

$\phi(t, E)$ tending towards the limit $\Lambda(E)$ uniformly in E, we have, owing to the lemma of §3:

$$= \int_{-\infty}^{+\infty} \left[\lim_{t \to \infty} \phi(t, dx) \right] \lambda(x) = \int_{-\infty}^{+\infty} \Lambda(dx)\lambda(x) = \langle \Lambda, \lambda \rangle = \mu$$

and the proof is complete.

Now we turn to the study of the dispersion of $\lambda[X(t)]$ but we shall restrict ourselves to the time-homogeneous case. We begin by defining two integrals which will be of importance in the following. We have seen that under the conditions of Theorem I of §3 we have:

$$|\phi(t, E) - \Lambda(E)| \leq M(t, E) - m(t, E) \leq he^{-kt}; \qquad h, k > 0.$$

Hence the integral

(5.1) $$s(E) = \int_0^\infty [\phi(t, E) - \Lambda(E)]dt$$

is absolutely convergent. Further we have:

$$m(t, E) \leqq F(t; x, E) \leqq M(t, E),$$
$$m(t, E) \leqq \Lambda(E) \qquad \leqq M(t, E).$$

Hence

$$\big| F(t; x,E) - \Lambda(E) \big| \leqq M(t, E) - m(t, E) \leqq h e^{-kt}$$

and thus the integral

(5.2) $$s(x, E) = \int_0^\infty [F(t; x, E) - \Lambda(E)]dt$$

is absolutely convergent.

We now consider the dispersion of the integral $(1/T)\int_0^T \lambda[X(t)]dt$ (in the mean's square sense) i.e. the quantity:

$$\sigma^2(T) = E\left\{\left[\frac{1}{T}\int_0^T \lambda[X(t)]dt - E\left(\frac{1}{T}\int_0^T \lambda[X(t)]dt\right)\right]^2\right\},$$

(5.3)

$$= E\left\{\left[\frac{1}{T}\int_0^T \lambda[X(t)]dt - \mu\right]^2\right\} - \left[E\left\{\frac{1}{T}\int_0^T \lambda[X(t)]dt\right\} - \mu\right]^2$$

where $\mu = \langle \Lambda, \lambda \rangle$. We shall now prove the following theorem:

THEOREM IV. *Under the conditions of Theorem I of §3, $T\sigma^2(T)$ tends, as $T \to \infty$, towards a finite positive limit D^2 independent of the initial distribution and given by*

$$D^2 = \int_{-\infty}^{+\infty} \Lambda(dx)[\lambda(x) - \mu]^2$$

$$+ 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Lambda(dx)s(x, dy)[\lambda(x) - \mu][\lambda(y) - \mu].$$

In other words, if T is large, $\sigma^2(T)$ is of the order $1/T$ i.e.

$$\sigma^2(T) \sim D^2/T,$$

or

$$\sigma(T) \sim D/T^{1/2}.$$

Proof of the theorem. We shall give it in several steps.

(i) Consider the limit, as $T \to \infty$, of the expression

$$T\left[E\left\{\frac{1}{T}\int_0^T \lambda[X(t)]dt\right\} - \mu\right] = \int_0^T [E\{\lambda[X(t)]\} - \mu]\, dt$$

$$= \int_0^T dt\left[\int_{-\infty}^{+\infty} [\phi(t, dx) - \Lambda(dx)]\lambda(x)\right]$$

$$= \int_{-\infty}^{+\infty}\left[\int_0^T [\phi(t, dx) - \Lambda(dx)]dt\right]\lambda(x).$$

Letting $T \to \infty$ this expression tends towards

$$\int_{-\infty}^{+\infty} s(dx)\lambda(x) = \langle s, \quad \lambda \rangle < \infty.$$

Thus

$$T^2\left[E\left\{\frac{1}{T}\int_0^T \lambda[X(t)]dt\right\} - \mu\right]^2 \to [\langle s, \lambda\rangle]^2 < \infty$$

and

$$T\left[E\left\{\frac{1}{T}\int_0^T \lambda[X(t)]dt\right\} - \mu\right]^2$$

tends towards zero in the same manner as $1/T$. Thus the second term on the right of (6.4) contributes nothing to the limit of $T\sigma^2(T)$.

(ii) Let us study the contribution of the first term, i.e. of

$$TE\left\{\left[\frac{1}{T}\int_0^T \lambda[X(t)]dt - \mu\right]^2\right\} = TE\left\{\left[\frac{1}{T}\int_0^T (\lambda[X(t)] - \mu)dt\right]^2\right\}$$

$$= \frac{1}{T}\int_0^T \int_0^T E\{[\lambda[X(u)] - \mu][\lambda[X(v)] - \mu]\}\, du\, dv$$

$$= \frac{1}{T}\int_0^T E\{[\lambda[X(t)] - \mu]^2\}\, dt$$

$$+ \frac{2}{T}\int_0^T du \int_u^T dv E\{[\lambda[X(u)] - \mu][\lambda[X(v)] - \mu]\}.$$

(a) The first term on the right is equal to

$$\frac{1}{T}\int_0^T dt \int_{-\infty}^{+\infty} \phi(t, dx)[\lambda(x) - \mu]^2 \to \int_{-\infty}^{+\infty} \Lambda(dx)[\lambda(x) - \mu]^2.$$

(b) The second term can be written, setting $v - u = w$,

$$\frac{2}{T} \int_0^T du \int_0^{T-u} dw E\{ [\lambda[X(u)] - \mu][\lambda[X(u+w)] - \mu] \}$$

$$= \frac{2}{T} \int_0^T du \int_0^{T-u} dw \left[\int_x \int_y \phi(u, dx) F(w; x, dy) [\lambda(x) - \mu][\lambda(y) - \mu] \right]$$

$$= \frac{2}{T} \int_x \int_y \left[\int_0^T du \int_0^{T-u} dw \phi(u, dx) F(w; x, dy) [\lambda(x) - \mu][\lambda(y) - \mu] \right]$$

$$= \frac{2}{T} \int_x \int_y \int_0^T du \phi(u, dx) \left[\int_0^{T-u} dw [F(w; x, dy) - \Lambda(dy)] \right] [\lambda(x) - \mu][\lambda(y) - \mu]$$

$$+ 2 \frac{T-u}{T} \int_x \int_y \int_0^T du \phi(u, dx) \Lambda(dy) [\lambda(x) - \mu][\lambda(y) - \mu].$$

But for fixed T the last term on the right vanishes identically because it contains the factor

$$\int_{-\infty}^{+\infty} \Lambda(dy) [\lambda(y) - \mu] = \int_{-\infty}^{+\infty} \Lambda(dy) \lambda(y) - \mu \equiv 0.$$

Hence, as $T \to \infty$ this quantity tends towards the limit

$$2 \int_x \int_y \Lambda(dx) s(x, dy) [\lambda(x) - \mu][\lambda(y) - \mu].$$

Finally, as $T \to \infty$, $T \sigma^2(T)$ tends towards the finite limit

$$D^2 = \int_{-\infty}^{+\infty} \Lambda(dx) [\lambda(x) - \mu]^2 + 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Lambda(dx) s(x, dy) [\lambda(x) - \mu][\lambda(y) - \mu]$$

independent of the initial distribution, and the theorem is completely proved. Note that a similar problem was treated in the finite case by M. Fréchet [2].

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