ADDENDUM TO THE PAPER ON PARTIALLY STABLE ALGEBRAS

BY

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I regret to announce that there is a serious error in my paper in these Transactions, volume 84, pp. 430–443. The error was discovered by Louis Kokoris who found that on line 8 of page 434 the expression given as $4[g(bz)](az)$ should have been $4[g(az)](bz)$. As a consequence the computation of $P(z, g, az, b)$ *yields nothing*, the *proof* of formula (30) *is not valid*, and the important *Lemma* 9 *is not proved*. Thus the paper does not give a proof of its major result stated as Theorem 1.

Nevertheless, the theorems of the paper are all correct and we shall provide a revision of the proof here. This revised proof has been checked by Louis Kokoris to whom the author wishes to express his great thanks.

We observe first that the equation

$$gS_{ab} = gS_aS_b - (wa)[z\phi_g(b)] \tag{29}$$

occurs before the error and *is correct*. The results of §4 on pages 434 and 435 depend on Lemma 9, and their use must be postponed until we apply them in a new Lemma 25. Equation (41) is correct as proved and will become a part of a corrected version of Lemma 17. The following result will be used here and is proved correctly in the paper.

LEMMA 14. *Let a and b be in* $\mathfrak{B}$ *and g be in* $\mathfrak{G}$. *Then*

$$g[(wa)(bz)] = b\phi_g(a) - \phi_{gS_b}(a). \tag{42}$$

Lemma 17 uses Theorem 1 and so needs revision. The computation of $P(wa, bz, c, w)$ of its proof actually yields $4a(bc) - 3b(ca) - c(ab) = \phi_h(a) - 4\phi_g(c)$. The interchange of a and c implies the interchange of g and h and results in the formula $4c(ba) - 3b(ca) - a(cb) = \phi_g(c) - 4\phi_h(a)$. The quantity $\phi_h(a)$ can now be eliminated and we obtain the second equation in a relation which we shall number (41R). The proof of the first relation is exactly as in the paper.

LEMMA 17R. *Let* $g = (wa)(bz)$, *where a and b are in* $\mathfrak{B}$. *Then*

$$f_g(c) = 0, \qquad \phi_g(c) = (ac)b - a(cb) \tag{41R}$$

for every c of $\mathfrak{B}$.

Since Theorem 1 is *still not valid* at this point we shall require the following result.

Received by the editors June 26, 1957.

LEMMA A. *The algebra* $\mathfrak{B}$ *is a special Jordan algebra such that* $\mathfrak{B}=e\mathfrak{F}+\mathfrak{N}$, *where e is the unity element of the algebra* $\mathfrak{A}$, *and* $\mathfrak{N}$ *is the radical of* $\mathfrak{B}$.

For Lemma 2 states that the algebra $\mathfrak{B}$ is isomorphic to the algebra $\mathfrak{A}_u(1)=\mathfrak{B}u$. It is also known[1] that the mapping $a_1\to S_{1/2}(a_1)$, defined by the multiplication $x_{1/2}a_1=x_{1/2}S_{1/2}(a_1)$ in our u-stable case, is a homomorphic mapping of $\mathfrak{B}u$ onto the special Jordan algebra of all of the linear transformations $S_{1/2}(a_1)$ where a_1 ranges over all elements of $\mathfrak{B}u$. These elements all have the form $a_1=au$ for a in $\mathfrak{B}$, and if $S_{1/2}(a_1)=0$ then $wS_{1/2}(a_1)=w(au)=0$. However, Lemma 3 states that $w(au)=2^{-1}(wa)$. Hence the kernel of our homomorphism consists of all elements au such that $wa=0$, $w(wa)=a=0$. It follows immediately that $\mathfrak{B}$ is a special Jordan algebra. The basic assumption that $\mathfrak{A}$ has degree two implies that the only idempotent of $\mathfrak{B}u$ is u, the only idempotent of $\mathfrak{B}$ is its unity element e. It is then known[2] that $\mathfrak{B}=e\mathfrak{F}+\mathfrak{N}$ as desired.

We are now able to use Lemma 17R, Lemma A, and Lemma 4 to derive the following result.

LEMMA B. *The inclusion relation* $[(w\mathfrak{B})(\mathfrak{B}z)](\mathfrak{B}z)\subseteq w\mathfrak{N}$ *holds.*

For, we can always write every element a of $\mathfrak{B}$ in the form $a=\alpha e+c$ where α is in $\mathfrak{F}$ and c is in $\mathfrak{N}$. Since $w(bz)=(wa)z=0$ for every a and b of $\mathfrak{B}$ we also write b in the form $b=\beta e+d$ with d in $\mathfrak{N}$ and have $(wa)(bz)=(wc)(dz)$, that is,

$$(w\mathfrak{B})(\mathfrak{B}z) = (w\mathfrak{N})(\mathfrak{N}z). \tag{1A}$$

But if $g=(wa)(bz)$ we use (41R) to see that $[(wa)(bz)](tz)=-w\phi_g(t)$ where $\phi_g(t)=(at)b-a(tb)$. By (1A) we can take both a and b to be in $\mathfrak{N}$, $\phi_g(t)$ is in $\mathfrak{N}$ and our proof is complete.

Lemmas 18 and 19 of our original version state that $[w(bc)](az)=b[(wc)(az)]+c[(wb)(az)]$ and that consequently

$$f_g(a)[(wa)(bz)] = 0 \tag{47}$$

for every a and b of $\mathfrak{B}$. However, we cannot use Lemma 16 to conclude that $(w\mathfrak{B})(\mathfrak{B}z)=0$ and we shall, in fact, be forced to delete Lemma 16. The proof of the relation

$$[wf_g(a)](zb) + [wf_g(b)](za) = 0 \tag{45}$$

is correct and this relation is used later. We now use our results to obtain a corrected proof of the following lemma.

LEMMA 20. *Let there exist a nonsingular element* $c=f_g(a)$ *for some g in* $\mathfrak{G}$ *and a in* $\mathfrak{B}$. *Then* $(wB)(Bz)=0$.

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(1) See Lemma 1 of *A theory of power-associative commutative algebras*, these Transactions, vol. 69 (1950) pp. 503–527.

(2) Ibid., Theorem 2.

For (45) is equivalent to $(wc)(az)=0$ if $c=f_g(a)$. Lemma 18 then implies that in this case $[w(bc)](az)=c[(wb)(az)]$. However, (10) states that $(wb)(az)=-(wa)(bz)$ and so (47) implies that $[w(bc)](az)=0$. If c is any nonsingular element of the special Jordan algebra $\mathfrak{B}$ we can write $c=\gamma e+d$ where d is in $\mathfrak{N}$ and γ is in $\mathfrak{F}$. It is known(3) that the corresponding right multiplication R_d of the algebra $\mathfrak{B}$ is nilpotent and it follows that the right multiplication $R_c=\gamma I+R_d$ is nonsingular. Hence the equation $bc=t$ has a solution b in $\mathfrak{B}$ for every t of $\mathfrak{B}$. It follows that our assumption that $c=f_g(a)$ is nonsingular implies that $(wt)(az)=0$ for every t of $\mathfrak{B}$ and every a of $\mathfrak{B}$ such that $f_g(a)$ is nonsingular. It remains to consider the singular elements $f_g(m)$. The sum $f_g(m)+f_g(a)=f_g(a+m)$ is then nonsingular, that is, is not in the radical $\mathfrak{N}$. Then our proof implies that $(wt)[(a+m)z]=0=(wt)(az)+(wt)(mz)=(wt)(mz)$ as desired, and we have completed a proof of the relation $(w\mathfrak{B})(\mathfrak{B}z)=0$ in the case of a nonsingular value of the function $f_g(a)$.

We are now able to combine our results so as to restore the validity of the first of our basic theorems.

LEMMA C. *Let a simple commutative power-associative algebra $\mathfrak{A}$ of degree two contain an element g of $\mathfrak{G}$ and an element a of $\mathfrak{B}$ such that $f_g(a)$ is nonsingular. Then Lemmas* 9, 10, 11, 12, 13, *and* 15 *are all correct and Theorem* 1 *is correct.*

For Lemma 20 implies that the relation (29) reduces to $S_{ab}=S_aS_b=S_bS_a$ and this and Lemma 20 give Lemma 9 completely. Thus all of the results of §4 are now correct.

Lemma C reduces our study to what we have called the "*singular case*," and to the possible case where every $f_g(a)$ is singular but some $\phi_g(a)$ is nonsingular. We shall actually be able to show that the latter case cannot occur and indeed that every $\phi_g(a)$ is in $\mathfrak{N}$. Our first new result in this direction follows.

LEMMA D. *Let a, b, and c be in $\mathfrak{B}$ so that $g=(wa)(bz)$ and $h=(wa)(cz)$ are in G. Then gh is in $\mathfrak{N}$.*

By (1A) we can assume that a, b, and c are all in $\mathfrak{N}$. Compute $P(wa, wa, bz, cz)$ to see that $8gh+4a^2(bc)=2(wa)[g(cz)+h(bz)+(wa)(bc)]+(bz)[a^2(cz)+2h(wa)]+(cz)[a^2(bz)+2g(wa)]$. Then

$$8gh = -4a^2(bc)+2(wa)[-w\phi_g(c)-w\phi_h(b)+w\cdot a(bc)]+b(a^2c)$$
$$+2b[\phi_h(a)+zf_h(a)]+c(a^2b)+2c[\phi_g(a)+zf_g(a)].$$

Since a, b, and c are all in the radical $\mathfrak{N}$ of $\mathfrak{B}$ all terms on the right side of this equation are in $\mathfrak{N}+\mathfrak{N}z$. But gh is in $\mathfrak{B}$ and hence is in $\mathfrak{N}$.

(3) See Theorem 1 of the author's *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

We now use the relation (42) in the form

$$h[(wa)(tz)] = t\phi_h(a) - \phi_{hS_t}(a), \tag{2A}$$

where h is any element of $\mathfrak{G}$, and a and t are arbitrary elements of $\mathfrak{B}$. Assume that

$$h = (wa)(dz) \tag{3A}$$

for the same a and some d in $\mathfrak{B}$ and use Lemma B to see that $\phi_h(a)$ is in $\mathfrak{N}$ and so $t\phi_h(a)$ is in $\mathfrak{N}$. By Lemma D we know that the left member of (2A) is in $\mathfrak{N}$. Thus (2A) implies the following result.

LEMMA E. *Let a, d, and t be any elements of $\mathfrak{N}$ and $h=(wa)(dz)$. Then $\phi_{hS_t}(a)$ is in $\mathfrak{N}$.*

There is a minor error in the statement of Lemma 21 which actually does not affect the argument but which should be corrected as follows. We compute $P(wa, z, g, g)$ and see that

$$\begin{aligned} 0 &= 2g[(wa)g\cdot z] + (wa)(g^2z) = 2g[f_g(a)\cdot z + \phi_g(a)] + (wa)(g^2z) \\ &= -2w\{f_g[\phi_g(a)] + \phi_g[f_g(a)]\} + 2gS_c + (wa)(g^2z) \end{aligned}$$

where $c=\phi_g(a)$. This yields the corrected version of Lemma 21 which we now state.

LEMMA 21R. *Let a be in $\mathfrak{B}$, g be in $\mathfrak{G}$, and $c=\phi_g(a)$. Then*

$$2gS_c = -(wa)(g^2z), \qquad f_g[\phi_g(a)] + \phi_g[f_g(a)] = 0. \tag{49R}$$

We have now obtained the key elements in a proof of the singularity of the function $\phi_g(a)$.

LEMMA F. *The elements $\phi_g(a)$ are in $\mathfrak{N}$ for every g of $\mathfrak{G}$ and a of $\mathfrak{B}$.*

For assume that there does exist an element g of $\mathfrak{G}$ and an element a of $\mathfrak{B}$ such that $c=\phi_g(a)$ is nonsingular. By (49R) we have

$$h = gS_c = (wa)(dz) \tag{4A}$$

where $d=-1/2g^2$ is in $\mathfrak{B}$. We now use (29) to see that if $t=c^{-1}$ then $gS_{ct} = gS_e = g$, and so

$$gS_cS_t = hS_t = g - (wc)[z\phi_g(t)]. \tag{5A}$$

Hence

$$g = hS_t + k \tag{6A}$$

where k is in $(w\mathfrak{B})(\mathfrak{B}z)$. By (4A) and Lemma E we know that $\phi_{hS_t}(a)$ is in $\mathfrak{N}$. By Lemma B we know that $\phi_k(a)$ is in $\mathfrak{N}$. But then (6A) implies that $\phi_g(a) = \phi_{hS_t}(a)+\phi_k(a)=c$ is in $\mathfrak{N}$, contrary to hypothesis.

We are now ready to turn to the discussion of the singular case of Section 6. The proof there makes some slight and unnecessary use of Lemma 9 and we correct the argument as follows. Note that Lemma 9 cannot be used since we have only proved its validity in the nonsingular case. We observe that

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{B}z + w\mathfrak{B} + \mathfrak{G} \tag{7A}$$

and we now write

$$\mathfrak{M} = \mathfrak{N} + \mathfrak{N}z, \tag{8A}$$

so that $\mathfrak{M}$ is the radical of the Jordan algebra $\mathfrak{C} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) = \mathfrak{B} + \mathfrak{B}z$. Then the combination of the hypothesis that every $f_g(a)$ is in $\mathfrak{N}$ and Lemma F is equivalent to the inclusion relations

$$(w\mathfrak{B})\mathfrak{G} \subseteq \mathfrak{M}, \qquad \mathfrak{G}(\mathfrak{B}z) \subseteq w\mathfrak{N}. \tag{8A}$$

Define $\mathfrak{G}^*$ to be the vector spanned by all elements of the form gS_a for g in $\mathfrak{G}$ and a in $\mathfrak{N}$. Then Lemma F implies that

$$\mathfrak{G}\mathfrak{M} \subseteq \mathfrak{G}^* + w\mathfrak{N}, \qquad \mathfrak{G}^*\mathfrak{B} = \mathfrak{G}^* + \mathfrak{G}^*\mathfrak{N} \subseteq \mathfrak{G}^*. \tag{9A}$$

Also Lemma F and (42) imply that

$$\mathfrak{G}[(w\mathfrak{B})(\mathfrak{B}z)] \subseteq \mathfrak{N}. \tag{10A}$$

Then (29) implies that, if we write $x \equiv y$ for x and y in $\mathfrak{B}$ whenever $x - y$ is in $\mathfrak{N}$, we have the property

$$(gS_{ab})h \equiv (gS_aS_b)h \tag{11A}$$

for every g and h of $\mathfrak{G}$ and a and b of $\mathfrak{N}$. We now have all of the results of (55)–(60), the argument which forms the last paragraph of page 440 is valid, and the relation

$$\mathfrak{G}\mathfrak{G}^* \subseteq \mathfrak{N} \tag{12A}$$

is correct.

We are now able to prove that the space

$$\mathfrak{S} = w\mathfrak{N} + \mathfrak{G}^* + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{M} \tag{13A}$$

is an ideal of $\mathfrak{A}$. Indeed $\mathfrak{A}\mathfrak{M} = \mathfrak{M}\mathfrak{M} + (w\mathfrak{B})\mathfrak{M} + \mathfrak{G}\mathfrak{M} \subseteq \mathfrak{M} + w\mathfrak{N} + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{G}^* \subseteq \mathfrak{S}$. Also $\mathfrak{A}(w\mathfrak{N}) = \mathfrak{B}(w\mathfrak{N}) + (\mathfrak{B}z)(w\mathfrak{N}) + \mathfrak{G}(w\mathfrak{N}) + (w\mathfrak{B})(w\mathfrak{N}) \subseteq w\mathfrak{N} + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{M} \subseteq \mathfrak{S}$. We see next that $\mathfrak{A}\mathfrak{G}^* = \mathfrak{B}\mathfrak{G}^* + (\mathfrak{B}z)\mathfrak{G}^* + (w\mathfrak{B})\mathfrak{G}^* + \mathfrak{G}\mathfrak{G}^* \subseteq \mathfrak{G}^* + w\mathfrak{N} + \mathfrak{M} \subseteq \mathfrak{S}$, and that $\mathfrak{A}[(w\mathfrak{B})(\mathfrak{B}z)] = (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{N}[(w\mathfrak{B})(\mathfrak{B}z)] + (\mathfrak{B}z)[(w\mathfrak{B})(\mathfrak{B}z)] + (w\mathfrak{B})[(w\mathfrak{B})(\mathfrak{B}z)] + (w\mathfrak{B})\mathfrak{G} \subseteq (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{G}^* + w\mathfrak{N} + w\mathfrak{N} + \mathfrak{M} \subseteq \mathfrak{S}$ and our proof that $\mathfrak{S}$ is an ideal of $\mathfrak{A}$ is complete. Since $\mathfrak{S}$ does not contain the idempotent e the hypothesis that $\mathfrak{A}$ is simple implies that $\mathfrak{M} = 0$, $\mathfrak{B} = e\mathfrak{F}$, and so $\mathfrak{A}$ is actually a Jordan algebra in this case.

This completes our revision but we take this opportunity to correct the following misprints:

p. 431, line −4. Should read $w[(ab)g - w\phi_g(a)\cdot(bz) - w\phi_g(b)\cdot(az)]$.
p. 432, line 3. Delete $-4af_g(b)$ at end of sentence.
p. 432, formula (13). Replace $\phi_g(a)$ by $\phi_g(ab)$.
p. 432, line 3 below (13). Replace first $-$ by $+$.
p. 432, line 4 below (13). Replace first $-$ by $+$.
p. 433, line 11. Replace last gS_aS_b by gS_bS_a.
p. 435, Lemma 13. Replace M by $\mathfrak{M}$.
p. 436, line below (44). Replace $\mathfrak{A}$ by $\mathfrak{H}$.
p. 436, line 7 below (44). Add $+(ab)(zc)$ at end of line.
p. 436, line 8 below (44). Add $+(ab)(zc)$ before "Since."
p. 436, line 10 below (44). Add $+(ab)(zc)$ before "is in."
p. 440, formula (54). Replace $\subseteq\mathfrak{G}$ by $\subseteq\mathfrak{M}$.

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[4] The work on this paper was sponsored by the Office of Ordnance Research, United States Army, under Contract DA-19-059-ORD-2329.