A CLASS OF NONCOMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS[1]

BY

ROBERT H. OEHMKE

1. **Introduction.** Let A be a nonassociative algebra over a field F. If for each x in A the subalgebra $F[x]$ of all polynomials in x over F is an associative algebra we call A a *power-associative* algebra. In particular the elements of A must satisfy the identity

$$(1) \qquad\qquad x \cdot (x \cdot x) = (x \cdot x) \cdot x.$$

To any algebra A over a field F of characteristic not 2 we can attach a commutative algebra $A^{(+)}$ over F which is the same vector space as A and which has a product (x, y) expressible in terms of the product $x \cdot y$ of A by

$$(2) \qquad\qquad (x, y) = \frac{1}{2} (x \cdot y + y \cdot x).$$

Powers of elements in $A^{(+)}$ agree with their powers in A. Thus the power-associativity of A implies the power-associativity of $A^{(+)}$.

This paper is chiefly concerned with the class of nonassociative algebras satisfying (1) and having $A^{(+)}$ a separable Jordan algebra. We shall denote this class by the letters SJ.

Suppose that G is a nonassociative algebra over a field F of characteristic not 2 and G contains a subspace S closed under the operation (x, y) defined by (2). The set of all finite linear combinations of elements in G of the form $xy - yx$ for x and y in S form a subspace $U(S)$ of G. If T is any linear mapping on $U(S)$ into S we call the mapping T a *bonding mapping* of G [2][2]. Every bonding mapping determines an algebra $B(G, S, T)$ which is the same vector space as S and is defined by the product

$$(3) \qquad\qquad x \cdot y = \frac{1}{2} (xy + yx) + (xy - yx) T$$

for all x and y in S, where xy is the product in G. We say that $B(G, S, T)$ is *bonded* to G. We note that

226

(4) $$x \cdot y + y \cdot x = xy + yx.$$

Albert has shown [2] that any algebra of characteristic not 2, 3 or 5, satisfying (1) and having $A^{(+)}$ a *central simple* Jordan algebra of degree $t > 2$ has a scalar extension A_K which is obtained from a bonding mapping. Here the degree t is defined as the maximum number of primitive orthogonal idempotents e_i in the expression of the unity quantity $e = e_1 + \cdots + e_t$ for any scalar extension of the algebra. We shall extend this result and show that for all algebras A of class SJ there exists a direct sum G of simple associative algebras and 3 by 3 matric algebras over a Cayley algebra such that $A = B(G, S, T)$.

A class of algebras A satisfying (1) such that $A^{(+)}$ is a semi-simple Jordan algebra with an inseparable component will be exhibited for which no bonding mapping exists.

2. **Bonding mappings.** A separable Jordan algebra is defined as a semi-simple Jordan algebra each of whose components has a separable center over its base field. Let A be an algebra of class SJ over the field F. The algebra $A^{(+)}$ is a semi-simple Jordan algebra over F and is therefore a direct sum of simple Jordan algebras.

Any simple Jordan algebra is isomorphic to a Jordan algebra of linear transformations or is an algebra of order 27 over its center. These algebras are called special and exceptional Jordan algebras respectively. The simple special Jordan algebras are the Jordan algebras associated with one of the following types under the multiplication defined in (2): (a) a simple associative algebra over F, (b) the set of symmetric elements of a simple associative algebra with an involution J of the first kind over F, (c) the set of symmetric elements of a simple associative algebra with an involution J of the second kind over F, (d) a Clifford system over a field K containing F [1; 3; 5].

THEOREM 1. *Let A be an algebra over a field F of class SJ and let $A^{(+)}$ be special. Then there exists an associative algebra G over F such that $A = B(G, S, T)$.*

We first prove the following elementary results.

LEMMA 1. *If for some scalar extension K of F we have A_K bonded to G_K the algebra A is bonded to G.*

Let T_0 be a linear mapping satisfying (3) for all x and y in A_K. Then in particular if x and y are in A we have $xy - yx$ in $U(A_K)$ and $(xy - yx)T_0 = x \cdot y - (xy + yx)/2$. Since $x \cdot y$ and $xy + yx = x \cdot y + y \cdot x$ are in A we also have $(xy - yx)T_0$ in A. The transformation T_0 cut down to A induces a linear mapping T over F such that $(xy - yx)T_0 = (xy - yx)T$ for all x and y in A. The mapping T is the desired mapping bonding A to G.

Now let $A = A_1 + \cdots + A_n$ where each $A_r^{(+)}$ is an ideal of $A^{(+)}$ and a simple Jordan algebra. Even though each $A_r^{(+)}$ is an ideal of $A^{(+)}$ it doesn't necessarily follow that A_r is an ideal of A or even a subalgebra of A under the

multiplication $x \cdot y$. However we do have the following:

LEMMA 2. *If x is in A_i, y is in A_j and $i \neq j$ we have $x \cdot y = 0$.*

By a linearization process and (4), (1) becomes

(5)
$$x \cdot (yz + zy) + y \cdot (xz + zx) + z \cdot (xy + yx)$$
$$= (yz + zy) \cdot x + (xz + zx) \cdot y + (xy + yx) \cdot z.$$

If we let $z = e_j$ in (5) where e_j is the unity element of A_j we obtain $x \cdot y = y \cdot x$. From (4) we have $2x \cdot y = xy + yx = 0$, so $x \cdot y = 0$.

We have seen that the only nonzero products in A arise from the products of two elements of the same component of $A^{(+)}$. Therefore we need only show the existence of an algebra G_r, a subspace S_r of G_r equal to the space A_r, and a transformation T_r mapping $U(S_r)$ into A and satisfying (3) for all elements x and y in A_r. The composition of all the G_r, S_r and T_r will be the desired G, S and T. The G_r, S_r and T_r will be defined separately for each of the types (a)–(d) above.

If $A_r^{(+)}$ is of type (a), it is the Jordan algebra associated with a simple separable associative algebra C over F. Since C is separable it has a splitting field K. The extension C_K is a direct sum of total matric algebras. By Lemma 1 and Lemma 2 we can assume that C is a total matric algebra and $A_r^{(+)}$ is the Jordan algebra associated with kt. The algebra has a matric basis $(\cdots e_{ij} \cdots)$ over F.

LEMMA 3. *If $A_r^{(+)}$ is the Jordan algebra associated with the total matric algebra C then $G_r = S_r = C$ and T_r is defined by*

(6)
$$(e_{ij}) T_r = e_{ii} \cdot e_{ij} - e_{ij}/2 \qquad \text{for } i \neq j,$$
$$(e_{ii}) T_r = - e_{1i} \cdot e_{i1} + e_{ii}/2.$$

Since (3) is linear over F we need only show that it is satisfied for the basal elements or that

(7)
$$e_{ij} \cdot e_{pq} = \frac{1}{2} e_{ij} \cdot e_{pq} + \frac{1}{2} e_{pq} \cdot e_{ij} + (e_{ij}e_{pq} - e_{pq}e_{ij}) T_r.$$

To prove that (7) holds under the definition in (6) we assume first that $i \neq j$, p, q and $j \neq p$, q. Put $x = e_{ij}$, $y = e_{pq}$ and $z = e_{pp}$ in (5). We obtain $e_{ij} \cdot e_{pq} = e_{pq} \cdot e_{ij}$. This identity together with (4) gives us $e_{ij} \cdot e_{pq} = 0$. Since the right hand member of (7) is 0 it has been verified in this case. Assume next that $i = p \neq q$, j. We put $x = e_{ij}$, $y = e_{ij}$, $z = e_{jq}$ in (5) and obtain $e_{ij} \cdot e_{iq} = 0$ and again (7) is satisfied. Now if $j = p$ and $i = q$ we must show that

(8)
$$e_{ij} \cdot e_{ji} = e_{ii} - e_{1i} \cdot e_{i1} + e_{1j} \cdot e_{j1}.$$

We use (5) with $x = e_{ij}$, $y = e_{j1}$ and $z = e_{1i}$, all indices distinct, and obtain (8). If $i = j$, (8) reduces to $e_{ii} \cdot e_{ii} = e_{ii}$ which is clear from (4). If $i = 1$ in (8) this

becomes $e_{ij} \cdot e_{j1} = e_{1j} \cdot e_{j1}$. If $i = j$ and $i \neq p = q$ in (7) the right hand member is zero. This identity is satisfied if $e_{ii} \cdot e_{pp}$ is also zero. The product $e_{ii} \cdot e_{pp}$ is seen to be 0 when we put $x = e_{ii}$, $z = y = e_{pp}$ in (5). The only remaining case to consider is $i = q$ and $j \neq p$. We need to show that $e_{ij} \cdot e_{pi} = e_{pj} - e_{pp} \cdot e_{pj}$ which is clear from (5) with $x = e_{ij}$, $y = e_{pi}$ and $z = e_{pp}$. Thus (3) is satisfied for x and y and A_r and T_r defined as above.

If $A_r^{(+)}$ is of type (b) it is the Jordan algebra associated with the set of symmetric elements of a simple separable associative algebra C over F with an involution J over F of the first kind. If K is a splitting field of C the involution J can be extended uniquely to an involution over K of C_K. The algebra C_K is a direct sum of total matric algebras with an involution J_0. The set of symmetric elements of C_K is $(A_r^{(+)})_K$. By Lemma 1 we may assume that $F = K$. We need the following two lemmas on involutions.

LEMMA 4. *If J is an involution of a direct sum of total matric algebras any component M is left fixed by J or is mapped into a component M^J of the same degree. The total matric algebra M^J will have a matric basis $(\cdots f_{ij} \cdots)$ where $f_{ij} = e_{ji}^J$ and $(\cdots e_{ji} \cdots)$ is a matric basis of M.*

We shall omit the proof of this lemma.

If M is a total matric algebra of degree $n = 2m$ then $M = M_m \times M_2$ (direct product) where $M_2 = (g_{11}, g_{12}, g_{21}, g_{22})$ and $M_m = (\cdots e_{ij} \cdots)$ for $i, j = 1, \cdots, m$. Every element of M is uniquely expressible in the form

$$(9) \qquad a = Ag_{11} + Bg_{12} + Cg_{21} + Dg_{22}$$

where A, B, C and D are elements of M_m. If we define

$$(10) \qquad a^J = D'g_{11} - B'g_{12} - C'g_{21} + A'g_{22}$$

where A', B', C' and D' are the transposes of A, B, C and D. J is an involution of M. The set of symmetric elements of M are of the form $a = Ag_{11} + (B - B')g_{12} + (C - C')g_{21} + A'g_{22}$ [4].

Two involutions J and J' of an algebra C are called cogredient if there exists an automorphism S over the center of C such that $J' = S^{-1}JS$.

LEMMA 5. *An involution J of a total matric algebra M is cogredient to transposition or to the involution described in (9)–(10).*

The proof of this lemma can be found in [4].

By these two lemmas we see that J_0 leaves M fixed and is cogredient to transposition, J_0 leaves M fixed and is cogredient to the involution defined in (9)–(10) or J_0 maps M into an isomorphic component. We shall treat each of these separately.

We first assume that J_0 is cogredient to transposition and leaves M fixed. We can choose a matric basis $(\cdots e_{ij} \cdots)$ on M such that $e_{ij}^{J_0} = e_{ji}$.

LEMMA 6. *If $A_r^{(+)}$ is the Jordan algebra associated with the set of symmetric*

elements under transposition of a total matric algebra M *then* $G_r = M$, $S_r = set$ *of symmetric elements and* T_r *is defined as*

(11)
$$(e_{ij} - e_{ji})T_r = - e_{jj} \cdot (e_{ij} + e_{ji}) + \frac{1}{2}(e_{ij} + e_{ji}) \qquad \text{for } i \neq j,$$

$$(e_{ii})T_r = \frac{1}{2} e_{ii}.$$

To prove that T_r satisfies (3) we need only show that it is satisfied for the elements e_{ii} and $e_{ij} + e_{ji}$ or that

(12)
$$(e_{ij} + e_{ji}) \cdot (e_{pq} + e_{qp}) = \frac{1}{2}(e_{ij} + e_{ji})(e_{pq} + e_{qp}) + \frac{1}{2}(e_{pq} + e_{qp})(e_{ij} + e_{ji})$$
$$+ [(e_{ij} + e_{ji})(e_{pq} + e_{qp}) - (e_{qp} + e_{pq})(e_{ij} + e_{ji})]T_r.$$

Assume first that $i \neq p, q, j$ and $j \neq p, q$. Put $x = e_{ij} + e_{ji}$, $y = e_{pq} + e_{qp}$ and $z = e_{pp}$ in (5). We obtain $(e_{ij} + e_{ji}) \cdot (e_{pq} + e_{qp}) = 0$ and since the right hand side of (12) is also 0 this identity is satisfied. Next if we put $x = e_{ij} + e_{ji}$, $y = e_{ip} + e_{pi}$ and $z = e_{pp}$ for $p \neq i, j$ we have $(e_{ij} + e_{ji}) \cdot (e_{ip} + e_{pi}) = (e_{jp} + e_{pj}) \cdot e_{pp} = -e_{pp}$ $\cdot(e_{jp} + e_{pj}) + (e_{pj} + e_{jp})$ when $i \neq j$ and $e_{ii} \cdot (e_{ip} + e_{pi}) = (e_{ip} + e_{pi}) \cdot e_{pp} = -e_{pp}$ $\cdot(e_{ip} + e_{pi}) + (e_{ip} + e_{pi})$ when $i = j$ and again (12) is satisfied. Now let $x = y = e_{ii}$, $z = e_{pq} + e_{qp}$ and i, p and q be distinct. Identity (5) gives us $e_{ii} \cdot (e_{pq} + e_{qp}) = 0$. If we put $i = p$ and $j = q$ in (12) it reduces to $(e_{ij} + e_{ji}) \cdot (e_{ij} + e_{ji}) = (e_{ij} + e_{ji})$ $\cdot(e_{ij} + e_{ji})$ and is certainly satisfied. It remains only to show that $e_{ii} \cdot e_{pp} = 0$ when $i \neq p$. This is obtained from (5) by putting $x = y = e_{ii}$ and $z = e_{pp}$.

We next consider the case where J_0 leaves M fixed and is cogredient to the involution described in (9)–(10). We shall use the following notation: $h_{ij} = e_{ij}g_{11} + e_{ji}g_{22}, f_{ij} = (e_{ij} - e_{ji})g_{12}, d_{ij} = (e_{ij} - e_{ji})g_{21}, \bar{h}_{ij} = e_{ij}g_{11} - e_{ji}g_{22}, \bar{f}_{ij} = (e_{ij} + e_{ji})g_{12}$ and $\bar{d}_{ij} = (e_{ij} + e_{ji})g_{21}$. The set S_r of all J_0-symmetric elements has a basis $(\cdots h_{ij} \cdots f_{mn} \cdots d_{st} \cdots)$. The set $U(S_r)$ has a basis $(\cdots \bar{h}_{ij} \cdots \bar{f}_{mn} \cdots \bar{d}_{st} \cdots)$.

LEMMA 7. *If* $A_r^{(+)}$ *is the Jordan algebra associated with the set of symmetric elements under the involution described in* (9)–(10) *of a total matric algebra* M *then* $G_r = M$, $S_r = set$ *of symmetric elements and* T_r *is defined by*

(13a)
$$(\bar{h}_{ij})T_r = - h_{jj} \cdot h_{ij} + \frac{1}{2} h_{ij},$$

(13b)
$$2(\bar{h}_{ii})T_r = f_{ij} \cdot d_{ji} + h_{ij} \cdot h_{ji} - h_{ii} - h_{jj},$$

(13c)
$$(\bar{f}_{ij})T_r = - h_{jj} \cdot f_{ij} + \frac{1}{2} f_{ij},$$

(13d)
$$(\bar{f}_{ii})T_r = f_{ji} \cdot h_{ij},$$

(13e)
$$(\bar{d}_{ii})T_r = - d_{ji} \cdot h_{ji},$$

$$(13f) \qquad (\bar{d}_{ij})T_r = h_{ii} \cdot d_{ij} - \frac{1}{2} d_{ij}$$

and $i \neq j$.

For a number of choices of x and y we have both $xy - yx$ and $xy + yx$ equal to 0. To show that (3) is satisfied in these cases we have merely to show that $x \cdot y = 0$. This can be done by using (5) with $z = h_{rr}$ and a proper choice of r. If both x and y are equal to h_{ii} we have $xy - yx = 0$ and $xy + yx = 2h_{ii}$, however it is easily demonstrated from (4) that $h_{ii} \cdot h_{ii} = h_{ii}$. To prove that (3) holds under the definition of T_r in (13a) we make use of the two identities $h_{ij} \cdot h_{jq} = h_{ii} \cdot h_{iq}$ and $h_{ii} \cdot h_{iq} = -h_{qq} \cdot h_{iq} + h_{iq}$ obtained from (5) by substituting $x = h_{ij}$, $y = h_{ji}$, $z = h_{iq}$ and $x = h_{iq}$, $y = h_{iq}$ and $z = h_{qi}$. The remaining products $x \cdot y$ with $xy - yx = \bar{h}_{ij}$ that we need to consider are $f_{ij} \cdot d_{jp}$, $f_{iq} \cdot d_{pq}$, $f_{ij} \cdot d_{pi}$ and $f_{ip} \cdot d_{iq}$. For each of these products we use (5) with $z = h_{pp}$.

In considering (13b) we must first show that the right hand member is independent of the choice of j. We let $x = f_{pj}$, $y = h_{ip}$ and $z = d_{ij}$ in (5) and get $f_{pj} \cdot d_{pj} - h_{pj} \cdot h_{jp} - f_{ij} \cdot d_{ij} + h_{ij} \cdot h_{ji} = h_{pi} \cdot h_{ip} + h_{jp} \cdot h_{pj} + h_{ij} \cdot h_{ji} - h_{ii} - h_{pp} - h_{jj} = 0$. Use (5) with $x = f_{ii}$, $y = h_{ij}$ and $z = d_{ii}$ to obtain $-f_{ij} \cdot d_{ij} = f_{ij} \cdot d_{ji}$. By setting $x = f_{ii}$, $y = h_{ji}$ and $z = d_{ij}$ we obtain a third identity $f_{ij} \cdot d_{ij} = f_{ij} \cdot d_{ji}$. These three identities combine to give us $f_{ij} \cdot d_{ji} + h_{ij} \cdot h_{ji} - f_{ip} \cdot d_{pi} - h_{ip} \cdot h_{pi} - h_{jj} + h_{pp} = 0$ and the right hand member of (13b) is independent of the choice of j. From these identities we can also obtain $(\bar{h}_{ii} - \bar{h}_{jj})T_r = 2^{-1}f_{ij} \cdot d_{ji} + 2^{-1}h_{ij} \cdot h_{ji} - 2^{-1}h_{ij} \cdot h_{ji} = h_{ij} \cdot h_{ji}$ and $(\bar{h}_{ii} + \bar{h}_{jj})T_r = 2^{-1}f_{ij} \cdot d_{ji} + 2^{-1}f_{ji} \cdot d_{ij} = f_{ij} \cdot d_{ji}$.

To prove that (3) holds with the definition of T_r given in (13c) we let $z = h_{pp}$ and x and y equal the two factors of the product $x \cdot y = f_{iq} \cdot h_{pq}$, $f_{pq} \cdot h_{qq}$, $f_{qi} \cdot h_{pi}$ in (5). This gives us three of the necessary four identities. The fourth identity, $f_{ij} \cdot h_{ii} = h_{jj} \cdot f_{ij}$ is obtained from (5) by letting $y = h_{ii}$, $x = f_{ij}$ and $z = h_{jj}$.

The right hand member of (13d) is independent of the choice of j, for if we let $x = f_{ji}$, $y = h_{pj}$ and $z = h_{ip}$ in (5) we have $f_{ji} \cdot h_{ij} = f_{pi} \cdot h_{ip}$. Terms of the form $\bar{f}_{ii}$ will arise only when we consider the products $f_{ij} \cdot h_{ji}$. Clearly these products satisfy (3) as T_r is defined in (13d) since $f_{ij} \cdot h_{ji} + h_{ji} \cdot f_{ij} = 0$.

We can show that (3) holds under the definitions (13e) and (13f) by using the identities proven for (13c) and (13d) with the subscripts 1 and 2 of g_{ij} interchanged and with the matric coefficients of g_{ij} replaced by their transposes. This leaves h_{ij} fixed and replaces f_{ij} by d_{ji}.

Finally if M is mapped onto the component M^{J_0} distinct from M by the involution J_0 the symmetric elements are linear combinations of elements $e_{ii} + f_{ii}$ and $e_{ij} + f_{ji}$ described in Lemma 4. We have

LEMMA 8. *If* $A_r^{(+)}$ *is the Jordan algebra associated with the set of symmetric elements under the involution* J_0 *of the direct sum of total matric algebras* M *and* M^{J_0} *then* $G_r = M \oplus M^{J_0}$, $S_r = $ *set of symmetric elements and* T_r *is defined by*

$$(e_{ij})T_r = \frac{1}{2}(e_{ii} + f_{ii}) \cdot (e_{ij} + f_{ji}) - \frac{1}{2}e_{ij},$$

$$(f_{ji})T_r = -\frac{1}{2}(e_{ii} + f_{ii}) \cdot (e_{ij} + f_{ji}) + \frac{1}{2}f_{ji},$$

(14)

$$(e_{ii})T_r = -\frac{1}{2}(e_{1i} + f_{i1}) \cdot (e_{i1} + f_{1i}) + \frac{1}{2}e_{ii},$$

$$(f_{ii})T_r = \frac{1}{2}(e_{1i} + f_{i1}) \cdot (e_{i1} + f_{1i}) - \frac{1}{2}f_{ii},$$

and $i \neq j$.

To show that (3) is satisfied under this definition of T_r for all products $x \cdot y$, except the product $(e_{ij}+f_{ji}) \cdot (e_{ji}+f_{ij})$, we use (4) and (5) with $z=e_{pp}+f_{pp}$ and the proper choice of p. When $1 \neq i, j$ we put $x=(e_{ij}+f_{ji})$, $y=(e_{ji}+f_{1j})$ and $z=(e_{1i}+f_{i1})$ in (5) to obtain $(e_{ij}+f_{ji}) \cdot (e_{ji}+f_{ij}) = (e_{1j}+f_{j1}) \cdot (e_{j1}+f_{1j}) - (e_{1i}+f_{i1})$ $\cdot (e_{i1}+f_{1i})+e_{ii}+f_{ii}$. When $i=1$ (3) reduces to $(e_{1j}+f_{j1}) \cdot (e_{j1}+f_{1j}) = (e_{1j}+f_{j1})$ $\cdot (e_{j1}+f_{1j})$. This completes the study of algebras of type (b).

If $A_r^{(+)}$ is a Jordan algebra of type (c) it is the algebra over F of all J-symmetric elements of a simple associative algebra C where J is an involution of the second kind. If K splits C over F then J can be extended uniquely to an involution of the second kind on C_K. The following lemma tells us the nature of this involution [3].

LEMMA 9. *Let M be a direct sum of total matric algebras over K. If M has an involution J of the second kind over K then M has $2t$ components and J leaves no component fixed.*

Since J carries each component into an isomorphic component we may apply Lemma 8.

If $A_r^{(+)}$ is a Jordan algebra of type (d) it is the Jordan algebra associated with a Clifford system over a field containing F. Again with the assumption of separability we can make a scalar extension by K that splits the center of $A_r^{(+)}$. Then $(A_r^{(+)})_K$ will be a direct sum of Jordan algebras associated with a Clifford system over K. We can assume that $F=K$ and $A_r^{(+)}$ is one of these components by Lemmas 1 and 2. $A_r^{(+)}$ is a subspace $(s_1, \cdots, s_n)$ of an associative algebra C. The multiplication of these basal elements in C is given by $s_i^2 = \alpha_i \neq 0$ in F and $s_i s_j = -s_j s_i$ for $i \neq j$.

LEMMA 10. *If $A_r^{(+)}$ is the Jordan algebra associated with the Clifford system $(s_1, \cdots, s_n)$ of an associative algebra C then we let $G_r=C$, $S_r=(s_1, \cdots, s_n)$ and T_r be defined as*

(15) $$2(s_i s_j)T_r = s_i \cdot s_j.$$

To show that T_r satisfies (3) we need only show that $s_i \cdot s_i = \alpha_i$ since $s_i \cdot s_j$

$= 2^{-1}s_is_j + 2^{-1}s_js_i + (s_is_j - s_js_i)T_r$ by the definition of T_r. Identity (4) gives us $s_i \cdot s_i + s_i \cdot s_i = 2\alpha_i$ so $s_i \cdot s_i = \alpha_i$. This completes the proof of Lemma 10 and Theorem 1.

3. **The exceptional simple Jordan algebra.** Let A be an algebra over F satisfying (1) such that $A^{(+)}$ is an exceptional simple Jordan algebra whose center is separable over F. The algebra A belongs to the class SJ. We can make a scalar extension K of F such that $(A^{(+)})_K$ will be a direct sum of exceptional central simple Jordan algebras over K. It has been shown [6] that any exceptional central simple Jordan algebra arises in the following manner. Let G be the algebra of all three-rowed matrices with elements in a Cayley algebra C over K. The algebra C has a basis $(e, u_2, \cdots, u_8)$ and multiplication is defined by $eu_i = u_ie = u_i$, $-u_ju_j = u_ju_i = \pm u_k$ for $i \neq j$, and $u_i^2 = \alpha_i \neq 0$ in K. The general element of C is $x = \beta_1 e + \beta_2 u_2 + \cdots + \beta_8 u_8$ and we can define an involution $x \rightarrow \bar{x}$ of C by $\bar{x} = \beta_1 e - (\beta_2 u_2 + \cdots + \beta_8 u_8)$. If $A = (x_{ij})$ is an element of G and P is a diagonal matrix of G with elements in K the mapping $A \rightarrow A^J = P(\bar{x}_{ij})'P^{-1}$ is an involution of G. The J-symmetric elements of G are closed under the multiplication $(A, B) = 2^{-1}(AB + BA)$ and form an exceptional central simple Jordan algebra under this multiplication.

If $P = (\pi_1, \pi_2, \pi_3)$ for π_i in K, the set of symmetric elements of G are finite linear combinations over K of the elements e_{ii}, $(\pi_i u)_{ji} - (\pi_j u)_{ij}$ and $(\pi_i e)_{ji} + (\pi_j e)_{ij}$ where u is any of the u_i above. We wish to determine the nature of the multiplication of A. From (4) we have that $e_{ii} \cdot e_{ii} = e_{ii}$. Also we obtain $e_{jj} \cdot e_{ii} = 0$ from (5) by putting $x = y = e_{ii}$ and $z = e_{jj}$. Identity (5) with $x = e_{ii} - e_{jj}$, $y = e_{ii}$, $z = (\pi_i u)_{ji} + (\pi_j u)_{ij}$ for any of the basal elements $e, u_2, \cdots, u_8$ gives us $2e_{jj} \cdot [(\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}] = (\pi_i u)_{ji} + (\pi_j \bar{u})_{ij} = e_{jj}[(\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}] + (\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}$. Now let both u and v be any of the eight basal elements of C. If we put $x = e_{jj}$, $y = (\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}$ and $z = (\pi_i v)_{ki} + (\pi_k \bar{v})_{ik}$ in (5), we have $[(\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}] \cdot [(\pi_i v)_{ki} + (\pi_k \bar{v})_{ik}] = 2^{-1}(\pi_i \pi_j \bar{u}v)_{kj} + 2^{-1}(\pi_i \pi_k \bar{v}u)_{jk}$. Next put $x = e_{ii}$, $y = e_{pp}$ and $z = (\pi_p u)_{qp} + (\pi_q \bar{u})_{pq}$ in (5) we get $e_{ii} \cdot [(\pi_p u)_{qp} + (\pi_q \bar{u})_{pq}] = 0$. Now put $x = (\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}$, $y = (\pi_p v)_{qp} + (\pi_q \bar{v})_{pq}$ and $z = e_{ii}$ in (5) and obtain $[(\pi_j u)_{ij} + (\pi_i \bar{u})_{ji}] \cdot [(\pi_p v)_{qp} + (\pi_q \bar{v})_{pq}] = 0$. We have shown that all products of basal elements in A agree with their products in $A^{(+)}$ except the product $[(\pi_i u)_{ji} + (\pi_j \bar{u})_{ij}] \cdot [(\pi_i v)_{ji} + (\pi_j \bar{v})_{ij}]$. We will now show that this product has the property. We first show this for $v = e$. Let $x = (\pi_j e)_{ij} + (\pi_i e)_{ji}$, $y = (\pi_i e)_{ki} + (\pi_k e)_{ik}$ and $z = (\pi_k \bar{u})_{jk}$ in (5) for $k \neq i, j$. We have $[(\pi_k \pi_j e)_{ij} + (\pi_k \pi_i e)_{ji}] \cdot [(\pi_j u)_{ij} + (\pi_i u)_{ji}] + [(\pi_j \pi_i e)_{ki} + (\pi_k \pi_j e)_{ik}] \cdot [(\pi_i u)_{ki} + (\pi_k \bar{u})_{ik}] = [(\pi_i \pi_j e)_{kj} + (\pi_i \pi_k e)_{jk}] \cdot [(\pi_j u)_{kj} + (\pi_k \bar{u})_{jk}]$. Letting $x = (\pi_j u)_{ij} + (\pi_i \bar{u})_{ji}$, $y = (\pi_i u)_{ki} + (\pi_k \bar{u})_{ik}$ and $z = (\pi_k u)_{jk} + (\pi_j \bar{u})_{kj}$ in (5) we have $\pi_j[(\pi_i e)_{ki} + (\pi_k e)_{ik}] \cdot [(\pi_i u)_{ki} + (\pi_k \bar{u})_{ik}] + \pi_k[(\pi_i e)_{ji} + (\pi_j e)_{ij}] \cdot [(\pi_j u)_{ij} + (\pi_i \bar{u})_{ji}] = \pi_i[(\pi_k e)_{jk} + (\pi_j e)_{kj}] \cdot [(\pi_k u)_{jk} + (\pi_j \bar{u})_{kj}]$. These two identities together give us $[(\pi_k e)_{jk} + (\pi_i e)_{ji}] \cdot [(\pi_k u)_{jk} + (\pi_j \bar{u})_{kj}] = 0$. Now if $u = v$ in our product we have the square of an element and since $x \cdot x + x \cdot x = xx + xx$, this product agrees with the product in $A^{(+)}$. The remaining case is where both u and v are distinct from e and from each other. Let w

be the unique basal element such that $uw=v$. Now let $x=(\pi_i u)_{ji}+(\pi_j\bar{u})_{ij}$, $y=(\pi_k w)_{ik}+(\pi_i\bar{w})_{ik}$ and $z=(\pi_k u)_{jk}+(\pi_j\bar{u})_{kj}$ in (5) and we have $[(\pi_i u)_{ji}+(\pi_j\bar{u})_{ij}]$ $\cdot[(\pi_k\pi_j v)_{ij}+(\pi_i\pi_k\bar{v})_{ji}]+[(\pi_k u)_{jk}+(\pi_j\bar{u})_{kj}]\cdot\pi_i[(\pi_k v)_{jk}+(\pi_j\bar{v})_{kj}]=0$. Now letting $x=(\pi_i u)_{ji}+(\pi_j\bar{u})_{ij}$, $y=(\pi_j v)_{kj}+(\pi_k\bar{v})_{jk}$ and $z=(\pi_j e)_{kj}+(\pi_k e)_{jk}$ in (5), we obtain $[(\pi_i u)_{ji}+(\pi_j\bar{u})_{ij}]\cdot[(\pi_k\pi_j v)_{ij}+(\pi_k\pi_i\bar{v})_{ji}]=[(\pi_k\pi_i u)_{jk}+(\pi_j\pi_i\bar{u})_{kj}]$ $\cdot[(\pi_k v)_{jk}+(\pi_j\bar{v})_{kj}]=0$. These two identities give us $[(\pi_k u)_{jk}+(\pi_j\bar{u})_{kj}]$ $\cdot[(\pi_k v)_{jk}+(\pi_j\bar{v})_{kj}]=0$. We can summarize the above in the following theorem.

THEOREM 2. *Let A be an algebra over a field F such that $A^{(+)}$ has an ideal I that is a simple, separable, exceptional Jordan algebra. Then I is an ideal of A and the multiplication of I in A agrees with the multiplication of I in $A^{(+)}$.*

This theorem together with Theorem 1 gives us:

THEOREM 3. *Let A be an algebra of class SJ. Then there is an algebra G that is a direct sum of a semi-simple associative algebra and algebras of three-rowed matrices over a Cayley algebra such that $A=B(G, S, T)$.*

For each component of $A^{(+)}$ that is an exceptional Jordan algebra we define $T_r \equiv 0$. This together with the mapping of Theorem 1 defined on the components of $A^{(+)}$ that are special gives us the desired mapping T.

4. **An algebra not of the form** $B(G, S, T)$. Let F be a field of characteristic $p \neq 2$, 3 with two elements α and β that are algebraically independent and transcendental over the prime field of F. Let a and b be roots of the irreducible equations $x^p - \alpha = 0$ and $x^p - \beta = 0$ over F respectively. The field $K = F(a, b)$ is a nonsimple extension of F. The field K is a Jordan algebra under the multiplication xy of the field. We shall now define a second product $x \cdot y$ such that the vector space K is a noncommutative algebra A over F satisfying (1) and having $A^{(+)} = K$. This algebra has a basis consisting of all elements $a^s b^t$ for $s, t = 0, 1, \cdots, p-1$. We define

(16)
$$a^s b^t \cdot a^m b^n = a^{s+m} b^{t+n} + (sn - tm)(\alpha - \beta)$$
$$= \alpha^i \beta^q a^r b^i + (sn - tm)(\alpha - \beta),$$

where $s+m = r+pq$, $t+n = i+pj$ and $0 \leq r$, $i \leq p-1$. Since $a^p = \alpha$, $b^p = \beta$ and the characteristic of K is p we see immediately that this definition holds for all values of s, t, m and n. Now let $x = \sum c_i a^{s_i} b^{t_i}$, $y = \sum d_j a^{m_j} b^{n_j}$. Then $x \cdot y + y \cdot x = xy + yx + \sum c_i d_j (s_i n_j - t_i m_j)(\alpha - \beta) + \sum c_i d_j (m_j t_i - n_j s_i)(\alpha - \beta) = 2xy$ and $A^{(+)} = K$. However A is not commutative since $ab \cdot ab^2 - ab^2 \cdot ab = 2(\alpha - \beta)$ $\neq 0$. Again let $x = \sum c_i a^{s_i} b^{t_i}$. Then $(x \cdot x) \cdot x = (xx) \cdot x = (xx)x + \sum c_i c_j c_k (s_i t_k + s_j t_k - t_i s_k - t_j s_k)(\alpha - \beta) = (xx)x$ and

$$x \cdot (x \cdot x) = x(xx) + \sum c_i c_j c_k (s_k t_i + s_k t_j - t_k s_i - t_k s_j)(\alpha - \beta) = x(xx) = (xx)x = (x \cdot x) \cdot x.$$

Therefore Identity (1) holds for the elements of A.

Assume that there exists an algebra G that is a simple associative algebra or an algebra of three-rowed matrices over a Cayley-Dickson algebra such that $A = B(G, S, T)$. If G is of the latter type we see that T must be identically zero and A is commutative. Therefore we shall assume that G is associative. We denote its product by $x \circ y = xy$ for all elements of K we have $x \circ y = y \circ x$ and $x \cdot y = 2^{-1}(x \circ y + y \circ x) + (x \circ y - y \circ x)T = xy$ and there is no transformation that bonds G to A.

We shall now show that the vector space K is a commutative subalgebra of G under the product $x \circ y$ and therefore A cannot be bonded to G by the above. Recall that $x \circ y + y \circ x = 2xy$ for all x and y in K. In particular $x \circ x = xx$ so no confusion should arise from the symbol x^2. Now $a^2 \circ b = a \circ (a \circ b) = -(a \circ b) \circ a + 2a(a \circ b) = b \circ a^2 + 2a(a \circ b) - 2a(b \circ a) = -a^2 \circ b + 4a(a \circ b) - 2a^2 b$. So $a^2 \circ b = 2a(a \circ b) - a^2 b$. But $a^2 \circ b = -a \circ b \circ a + 2a(a \circ b)$; therefore $(a \circ b \circ a) = a^2 b$ and $(a \circ b)^2 = (a \circ b \circ a \circ b) = a^2 b \circ b$ and $(b \circ a)^2 = b \circ a^2 b$. We also have $2ab \circ (a \circ b) = a \circ b \circ a \circ b + b \circ a \circ a \circ b = -a \circ b \circ b \circ a + 2a \circ b \circ b \circ a + a^2 b^2 = 2(a \circ b) \circ ab$. Therefore $ab \circ (a \circ b) = (a \circ b) \circ ab = (a \circ b)ab = ab(a \circ b)$. But $ab(a \circ b) = ab(-b \circ a + 2ab) = -ab(b \circ a) + 2a^2 b^2 = -(b \circ ab) \circ a + 2a^2 b^2 = ab \circ b \circ a - 2a^2 b \circ a + 2a^2 b^2$, so $(a \circ b)$ satisfies the identity

$$(17) \qquad\qquad x^2 + (ab)x - 2a^2 b^2 = 0.$$

Now $(a \circ b)^2 = (-b \circ a + 2ab)^2 = (b \circ a)^2 - 4ab(b \circ a) + 4a^2 b^2 = -(a \circ b)^2 + 4ab(a \circ b) - 2a^2 b^2$ or $(a \circ b)$ satisfies the identity $x^2 - 2(ab)x + a^2 b^2 = 0$. This together with (17) gives us $(ab)(a \circ b) - a^2 b^2 = 0$. Multiplying by $a^{-1} b^{-1}$ we have $(a^{-1} b^{-1}) \circ (ab(a \circ b)) - ab = a^{-1} b^{-1} \circ ab \circ (a \circ b) - ab = a \circ b - ab = 0$. Therefore $a \circ b = ab$ and G is commutative.

5. **Subalgebras of** $B(G, S, T)$. Let A be an algebra $B(G, S, T)$ where G is an associative algebra.

THEOREM 4. *Let G' be a commutative subalgebra of G. Then $G' \cap A$ is an associative, commutative subalgebra of A.*

For if G' is a commutative subalgebra of G and x and y are in $G' \cap A$ we have $xy - yx = 0$. Expressing the product $x \cdot y$ in terms of the mapping T we have $2x \cdot y = xy + yx + 2(xy - yx)T = 2xy$. Therefore products in A of the elements of $G' \cap A$ agree with their products in G.

THEOREM 5. *Let A be an algebra of class SJ and let S be an associative commutative ideal of $A^{(+)}$. Then S is an associative commutative ideal of A.*

The components of G corresponding to S can be taken as the associative algebra S since $S = S^{(+)}$. By Theorem 4, S is an associative, commutative subalgebra of A. Since the components of $A^{(+)}$ form orthogonal subspaces of A by Lemma 1 we have that S is an ideal of A.

The following theorem is a conjecture of R. Schafer [7].

THEOREM 6. *Let A be a finite power-associative ring without elements of additive order 2, 3 or 5. If every element a of A satisfies an equation of the form $a^{n(a)} = a$, $n(a)$ an integer greater than 1, then A is a vector space sum $A = A_1 + A_2$. The subspace A_1 is an ideal of A and a direct sum of finite fields. The subspace A_2 is bonded to A and $A_2^{(+)}$ is a direct sum of classical 3-dimensional central simple Jordan algebras without nilpotent elements $\neq 0$.*

Schafer has proved [7] that under these hypotheses $A^{(+)}$ is a direct sum of finite fields and such 3-dimensional Jordan algebras. Since the center of each component of $A^{(+)}$ is a finite field, $A^{(+)}$ is a separable semi-simple Jordan algebra; hence it is of class SJ. By Theorem 3 there is an algebra G such that $A = B(G, S, T)$. Write $A = A_1 + A_2$ where $A_1^{(+)}$ is a direct sum of finite fields and $A_2^{(+)}$ is a direct sum of 3-dimensional Jordan algebras. By Theorem 5, $A_1^{(+)} = A_1$ is a direct sum of finite fields.

BIBLIOGRAPHY

1. A. A. Albert, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 503–527.

2. ———, *New simple power-associative algebras*, Summa Brasiliensis Mathematicas vol. 2 (1951) pp. 1–12.

3. F. D. Jacobson and N. Jacobson, *Classification and representation of semi-simple Jordan algebras*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169.

4. G. K. Kalisch, *On special Jordan algebras*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 482–494.

5. L. A. Kokoris, *New results on power-associative algebras*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 363–373.

6. R. D. Schafer, *The exceptional simple Jordan algebras*, Amer. J. Math. vol. 70 (1948) pp. 82–94.

7. ———, *A generalization of a theorem of Albert*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 452–455.

MICHIGAN STATE UNIVERSITY,
    EAST LANSING, MICHIGAN
UNIVERSITY OF CHICAGO,
    CHICAGO, ILL.