PROOF OF AN AXIOM OF LUKASIEWICZ(1)

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In this note we prove that axiom A5 (cf. §10 of the preceding paper by Rose and Rosser [1]) is derivable from the axioms A1–A4. We use the notation of [1] and the formulas of [1] are referred to by their numbers. In order to make some of the longer formulas more easily readable, we shall separate blocks of letters by a space. We shall use formulas only from the first three sections, which depend only on axioms A1 to A4.

It easily follows from (1.6), (3.5), (1.8), (3.4), and (1.9), in that order,

\[ \vdash ARS = CCRSS \]
\[ = CNSNCRS \]
\[ = BSNCRS \]
\[ = BSNCRNNS \]
\[ = BSLRNS. \]

Hence, by the commutativity of \( L \), expressed in (3.10),

(1) \[ \vdash ARS = BSLNSR. \]

Now, by (1) and the commutativity of \( A \), expressed in (2.2),

(2) \[ \vdash BRLNRS = BSLNSR. \]

(3.1) and (1.8) give

\[ \vdash BNPP. \]

By repeated applications of (3.32) we get

(3) \[ \vdash BBBBBNPP NBPNQ LNBPNQNP LLBPNQPNQ. \]

By (3) and the commutative and associative laws for \( B \), expressed by (3.11) and (3.29),

(4) \[ \vdash BBBBBNBPQNQP LLBPNQPNQ LNBPNQNP P. \]

By de Morgan's law, derived from (3.9) and (3.4),

\[ \vdash BNBPQNP = NLBPNQNP. \]

So by (4),

Received by the editors August 20, 1956.

(1) The preparation of this paper was supported in part by the United States Navy under Contract No. NONR 401(20)-NR 043-167 monitored by the Office of Naval Research; reproduction in whole or in part is permitted for any purpose of the United States government.
In (2) take $R$ to be $NLBPNQP$ and take $S$ to be $NQ$ and use (3.4). (5) now leads to

\[ \vdash BBNQ \ LQNLBPNQP \ LNBPNQNP \ P, \]

By the associative and commutative laws for $B$, (6) leads to

\[ \vdash BBNPQ \ LNBPNQNP \ LQNLBPNQP. \]

In (2) take $R$ to be $BPNQ$ and $S$ to be $NP$ and use (3.4). (7) now gives

\[ \vdash BBNP \ LBPBNQ \ LQNLBPNQP. \]

By the associative and commutative laws for $B$ and (8),

\[ \vdash BBLBPBNQ \ LQNLBPNQP \ NP. \]

By the commutativity of $L$, and (9),

\[ \vdash BBLBPBNQP \ LNLBPNQPQ \ NP. \]

In (2) take $R$ to be $LBPBNQP$ and $S$ to be $Q$. This gives by (10),

\[ \vdash BQ \ LNQLBPNQP \ NP. \]

From (11) and the commutativity and associativity of $L$, we get

\[ \vdash BQ \ LLNQPBPBNQ \ NP. \]

By (1.9) and (1.8)

\[ \vdash LNQP = NCNQNP = NBQNP. \]

So by (12),

\[ \vdash BQ \ LNBQNPBPBNQ \ NP. \]

Using (13) and the commutativity and associativity of $B$,

\[ \vdash BBQNP \ LNBQNPBPBNQ. \]

So by (1) and (14),

\[ \vdash ABPNQBQNP. \]

By (1.8), (3.5), and the commutativity of $A$, this gives

\[ \vdash ACPQCQP. \]

References


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