PROOF OF AN AXIOM OF LUKASIEWICZ[1]

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In this note we prove that axiom A5 (cf. §10 of the preceding paper by Rose and Rosser [1]) is derivable from the axioms A1–A4. We use the notation of [1] and the formulas of [1] are referred to by their numbers. In order to make some of the longer formulas more easily readable, we shall separate blocks of letters by a space. We shall use formulas only from the first three sections, which depend only on axioms A1 to A4.

It easily follows from (1.6), (3.5), (1.8), (3.4), and (1.9), in that order,

$$\begin{aligned} \vdash ARS &\equiv CCRSS \\ &\equiv CNSNCRS \\ &\equiv BSNCRS \\ &\equiv BSNCRNNS \\ &\equiv BSLRNS. \end{aligned}$$

Hence, by the commutativity of L, expressed in (3.10),

(1) $$\vdash ARS \equiv BSLNSR.$$

Now, by (1) and the commutativity of A, expressed in (2.2),

(2) $$\vdash BRLNRS \equiv BSLNSR.$$

(3.1) and (1.8) give

$$\vdash BNPP.$$

By repeated applications of (3.32) we get

(3) $$\vdash BBBBNPP\ NBPNQ\ LNBPNQNP\ LLBPNQPNQ.$$

By (3) and the commutative and associative laws for B, expressed by (3.11) and (3.29),

(4) $$\vdash BBBBNBPNQNP\ LLBPNQPNQ\ LNBPNQNP\ P.$$

By de Morgan's law, derived from (3.9) and (3.4),

$$\vdash BNBPNQNP \equiv NLBPNQP.$$

So by (4),

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(5) $$\vdash BBBNLBPNQP\ LLBPNQPNQ\ LNBPNQNP\ P.$$

In (2) take R to be $NLBPNQP$ and take S to be NQ and use (3.4). (5) now leads to

(6) $$\vdash BBBNQ\ LQNLBPNQP\ LNBPNQNP\ P.$$

By the associative and commutative laws for B, (6) leads to

(7) $$\vdash BBBPNQ\ LNBPNQNP\ LQNLBPNQP.$$

In (2) take R to be $BPNQ$ and S to be NP and use (3.4). (7) now gives

(8) $$\vdash BBNP\ LPBPNQ\ LQNLBPNQP.$$

By the associative and commutative laws for B and (8),

(9) $$\vdash BBLPBPNQ\ LQNLBPNQP\ NP.$$

By the commutativity of L, and (9),

(10) $$\vdash BBLBPNQP\ LNLBPNQPQ\ NP.$$

In (2) take R to be $LBPNQP$ and S to be Q. This gives by (10),

(11) $$\vdash BBQ\ LNQLBPNQP\ NP.$$

From (11) and the commutativity and associativity of L, we get

(12) $$\vdash BBQ\ LLNQPBPNQ\ NP.$$

By (1.9) and (1.8)

$$\vdash LNQP \equiv NCNQNP \equiv NBQNP.$$

So by (12),

(13) $$\vdash BBQ\ LNBQNPBPNQ\ NP.$$

Using (13) and the commutativity and associativity of B,

(14) $$\vdash BBQNP\ LNBQNPBPNQ.$$

So by (1) and (14),

$$\vdash ABPNQBQNP.$$

By (1.8), (3.5), and the commutativity of A, this gives

A5. $$\vdash ACPQCQP.$$

References

1. Alan Rose and J. Barkley Rosser, *Fragments of many-valued statement calculi*, Trans. Amer. Math. Soc. vol. 87 (1957) pp. 1-53.

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