THE FOURIER COEFFICIENTS OF THE INVARIANTS $j(2^{1/2};\tau)$ AND $j(3^{1/2};\tau)$

BY

JOHN RALEIGH

Introduction

1. In [4](1), E. Hecke initiated investigations on a class of properly discontinuous groups, here denoted by $\{G(\lambda_q)\}$. Any $G(\lambda_q)$ is generated by the two substitutions $S(\tau)=\tau+\lambda_q$, $T(\tau)=-1/\tau$, where $\lambda_q=2\cos(\pi/q)$, q integer $\geqq 3$.

When $q=3$, $G(\lambda_3)\equiv G(1)$ is the modular group with invariant

$$j(1;\tau) = 12^3 J(1;\tau) = x^{-1} + 2^3\cdot 3\cdot 31 + 2^2\cdot 3^3\cdot 1823x + \cdots,$$

where $x=\exp(2\pi i\tau)$. This invariant is usually denoted briefly by $j(\tau)$ in the literature.

For $q=4$, the group $G(\lambda_4)\equiv G(2^{1/2})$ was investigated by J. W. Young, [13], who obtained the invariant, here denoted by $j(2^{1/2};\tau)$, as a quotient of theta-null series. A relationship between $j(2^{1/2};\tau)$ and $j(\tau)$ is discussed and utilized by R. Fricke, [3, vol. II, 1er Abs., 5es Kap., §2]. The above paper by J. W. Young is not mentioned in the work of Fricke.

If $q=6$, the group $G(\lambda_6)\equiv G(3^{1/2})$ is the subject of a paper by J. I. Hutchinson, [5], who obtains the invariant, here denoted by $j(3^{1/2};\tau)$ also as quotient of theta-null series.

In [7], H. Rademacher obtains convergent series for the Fourier coefficients of $j(1;\tau)$ and in [11] W. H. Simons uses methods similar to those of [7] for the determination of the Fourier coefficients of the invariants $\lambda(\tau)$ and $1/\lambda(\tau)$, algebraically related to $j(1;\tau)$.

The purpose of this paper is to extend the method of [7] to obtain convergent series for the Fourier coefficients of $j(2^{1/2};\tau)$ and $j(3^{1/2};\tau)$.

In §2, two relations between $j(2^{1/2};\tau)$ and $j(\tau)$ are established directly for a convenient calculation in closed form of the first few coefficients in terms of those of $j(\tau)$. This supplies a partial check on our main theorem.

The corresponding Formulae (2.04) and (2.05) may be determined by other methods. However, to the best of our knowledge, they do not appear explicitly in the literature. In §3 the modified Farey dissection of the circle is discussed. In §§4, 5, 6, 7, we establish the main result, explicit in Formula (7.7). In the second part of §5, (Formulae (5.1)′ and (5.3)′), a sum of exponen-

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(1) Numbers in brackets refer to the bibliography.

tials, $B_k(n)$, appears, and we show that $B_k(n)$ is also reducible to a Kloosterman sum.

In §9 we establish also two relations between $j(3^{1/2}; \tau)$ and $j(\tau)$ explicitly in Formulae (9.04) and (9.05). These relations are new. The remainder of §9 indicates how the same methods, used for $j(2^{1/2}; \tau)$, have been carried out to obtain convergent series for the Fourier coefficients of $j(3^{1/2}; \tau)$.

Details of estimates are partially omitted, since they would reproduce, except for multiplicative constants, those of H. Rademacher, [7].

I. The Fourier coefficients of the invariant $j(2^{1/2}; \tau)$ and its relation to $j(1; \tau)$

2. **The group $G(2^{1/2})$ and its invariant.** The group $G(2^{1/2})$, whose generators are $S(\tau)=\tau+2^{1/2}$, $T(\tau)=-1/\tau$, has a fundamental region defined by the inequalities:

$$-\frac{2^{1/2}}{2} \leqq \Re(\tau) < \frac{2^{1/2}}{2}, \qquad \Im(\tau) > 0,$$

$$|\tau| \geqq 1 \text{ if } \Re(\tau) \leqq 0, \qquad |\tau| > 1 \text{ if } \Re(\tau) > 0.$$

(cf. [4] or [13]).

Utilizing a remark by Rausenberger, [8], we let

$$\phi(\tau) = j(2^{1/2}\tau) + j(\tau/2^{1/2}) = x^{-2} + x^{-1} + 2c_0 + \cdots,$$

where $x=\exp(2\pi i\tau/2^{1/2})$ and $j(\omega)\equiv j(1; \omega)=12^3 J(1; \omega)$. Then $\phi(\tau)$ is invariant under $S(\tau)$ and $T(\tau)$, hence under all transformations of $G(2^{1/2})$, but has a double pole in x at $\tau=i\infty$, or $x=0$. Let us denote by $J(2^{1/2}; \tau)$ the invariant belonging to $G(2^{1/2})$. Such an invariant is a simple automorphic function for $G(2^{1/2})$ in the sense of Ford, [1, Ch. IV], and hence has a pole in x at the parabolic point $\tau=i\infty$. Moreover, the fundamental region being schlicht and simply-connected, $J(2^{1/2}; \tau)$ must be a schlicht function, hence the pole is simple (cf. [1, p. 91]).

From the above considerations we infer that there exists a relation of the form

$$A\cdot J^2(2^{1/2}; \tau) + B\cdot J(2^{1/2}; \tau) + C = \phi(\tau) \tag{2.001}$$

with A, B, C to be determined.

Similarly, considering the function $\psi(\tau)=j(2^{1/2}\tau)\cdot j(\tau/2^{1/2})$, also invariant under the transformations of $G(2^{1/2})$, with a pole of third order in x, we infer that:

$$A'\cdot J^3(2^{1/2}; \tau) + B'\cdot J^2(2^{1/2}; \tau) + C'\cdot J(2^{1/2}; \tau) + D' = \psi(\tau). \tag{2.001$'$}$$

We preassign the boundary correspondence as follows:

$$J(2^{1/2}; \rho_2) = 0, \qquad J(2^{1/2}; i) = 1, \qquad J(2^{1/2}; i\infty) = \infty$$

where $\rho_2 = -2^{1/2}(1-i)/2$. Since these conditions are not sufficient to determine the constants A, B, C, A', B', C', D', we exploit the transformation equation of rank 2, (cf. [2, vol. II, p. 371], or [3, vol. III, p. 395]). Such an equation is the result of elimination of σ between the three relations:

$$(2.01)\quad j(\omega) = 2^6\frac{(4\sigma-1)^3}{\sigma},\quad j'(\omega) \equiv j\left(\frac{\omega}{2}\right) = 2^6\frac{(4\sigma'-1)^3}{\sigma'},\quad \sigma\sigma' = 1,$$

where $\sigma(\omega)$ is a one-valued function of the transformation polygon T_2, (cf. [3, vol. III, pp. 297 ff., pp. 390 ff.]). For our purpose we simply interpret $\sigma(\omega)$ as a parameter. We obtain, from (2.01),

$$(2.02)\quad \begin{aligned} j(\omega) + j'(\omega) &= 2^6(2^6t^2 - 7^2t - 2^3\cdot 13),\\ j(\omega)\cdot j'(\omega) &= 2^{12}(17-4t)^3, \end{aligned}$$

where $t=\sigma+\sigma'$. Now, comparing (2.001) and (2.001)$'$ with (2.02), we infer that:

$$(2.03)\quad J(2^{1/2};\tau) = \frac{\alpha t+\beta}{\delta},\qquad \alpha,\ \beta,\ \delta \text{ constants.}$$

But, from (2.01) when $\sigma=\sigma'=-1$, $t=-2$,

$$j(\omega) = 2^6\cdot 5^3 = j'(\omega) = j(\omega/2).$$

On the other hand, from the Classinvariant theory, (cf. [3, vol. III, p. 394]), we have:

$$j(i2^{1/2}) = 2^6\cdot 5^3 = j\left(-\frac{1}{i2^{1/2}}\right) = j\left(\frac{i}{2^{1/2}}\right);$$

i.e. such a value is assumed by $j(\omega)$ only at points homologous to $\omega = i2^{1/2}$, and if $\omega = 2^{1/2}\tau$, this happens when $\tau = i$. However we have preassigned that $J(2^{1/2};\, i)=1$, hence, from (2.03)

$$(2.031)\quad \frac{-2\alpha+\beta}{\delta} = 1.$$

Similarly, when $\sigma=\sigma'=1$, $t=2$,

$$j(\omega) = 2^6\cdot 3^3 = j'(\omega) = j\left(\frac{\omega}{2}\right),\qquad \text{by (2.01).}$$

Again,

$$j(2^{1/2}\rho_2) = j\left\{2^{1/2}\left(-\frac{2^{1/2}}{2}\right)(1-i)\right\} = j(-1+i) = 2^6\cdot 3^3,$$

$$j'(2^{1/2}\rho_2) = j\left(\frac{\rho_2}{2^{1/2}}\right) = j\left(-\frac{1-i}{2}\right) = j\left(\frac{2}{1-i}\right) = j(1+i) = 2^6\cdot 3^3.$$

This value is assumed by $j(\omega)$ only at points homologous to $\omega=2^{1/2}\rho_2$, hence if we let $\omega=2^{1/2}\tau$, this happens when $\tau=\rho_2$. By the preassigned correspondence $J(2^{1/2};\,\rho_2)=0$. Hence, for $t=2$, (2.03) supplies the equation:

$$\frac{2\alpha+\beta}{\delta}=0. \tag{2.032}$$

Solving (2.031) and (2.032), we obtain $\beta=-2\alpha$, $\delta=-4\alpha$; i.e. the linear relation(2):

$$J(2^{1/2};\,\tau)=\frac{t-2}{-4}\cdot \tag{2.033}$$

Either by substitution in (2.02) or by a process of differentiation, the constants A, B, C, A', B', C', D' are readily calculated. By setting

$$j(2^{1/2};\,\tau)=2^8\cdot J(2^{1/2};\,\tau),$$

we obtain the following equations, relating $j(2^{1/2};\tau)$ with $j(2^{1/2}\tau)$ and $j(\tau/2^{1/2})$:

$$j^2(2^{1/2};\,\tau)-3^3\cdot 23j(2^{1/2};\,\tau)+2^7\cdot 3^3=j(2^{1/2}\tau)+j(\tau/2^{1/2}), \tag{2.04}$$

$$\left\{j(2^{1/2};\,\tau)+2^4\cdot 3^2\right\}^3=j(2^{1/2}\tau)\cdot j(\tau/2^{1/2}). \tag{2.05}$$

Since $j^{1/3}(\omega)$, and hence $j^{1/3}(2\omega)$, have series expansions with integral coefficients, (cf. Fricke [2, vol. 2, p. 344] or Petersson [6, p. 56]), Equation (2.05) shows that the Fourier coefficients of $j(2^{1/2};\,\tau)$ are rational integers. Moreover, Formulae (2.04) or (2.05) permit the calculation of the first few coefficients in terms of the coefficients C_ν of the modular invariant $j(\omega)$, the C_ν's being known in closed form up to C_{100}, (cf. [12]):

$$\begin{aligned} j(2^{1/2};\,\tau) &= x^{-1}+2^3\cdot 13+2^2\cdot 1093x+2^{11}\cdot 47x^2+\cdots \\ &= f(x) = x^{-1}+\sum_{n=0}^{\infty} c_n x^n, \end{aligned} \tag{2.1}$$

where $x=\exp\,(2\pi i\tau/2^{1/2})$.

The substitutions of $G(2^{1/2})$ can be separated into two classes, as follows: (cf. [13; 9]),

$$V=\begin{pmatrix} a & b2^{1/2} \\ c2^{1/2} & d \end{pmatrix}, \qquad ad-2bc=1, \tag{2.2}$$

$$V'=\begin{pmatrix} a'2^{1/2} & b' \\ c' & d'2^{1/2} \end{pmatrix}, \qquad 2a'd'-b'c'=1, \tag{2.3}$$

where a, b, c, d, a', b', c', d' are rational integers and

(2) Obtained by other methods, a linear relation such as (2.033) appears in [3, vol. II, p. 104].

$$(2.4) \qquad j(2^{1/2}; V(\tau)) = j(2^{1/2}; V'(\tau)) = j(2^{1/2}; \tau).$$

Accordingly we write:

CASE I.

$$V = \begin{pmatrix} h' & -(hh'+1)2^{1/2}/k \\ k2^{1/2}/2 & -h \end{pmatrix},$$

where

$$(2.5) \qquad hh' \equiv -1 \pmod{k}, \qquad (h, k) = 1, \qquad k \equiv 0 \pmod{2}.$$

CASE II.

$$V' = \begin{pmatrix} h^*2^{1/2} & -(2hh^*+1)/k \\ k & -h2^{1/2} \end{pmatrix},$$

where

$$(2.5)' \qquad 2hh^* \equiv -1 \pmod{k}, \qquad (2h, k) = 1, \qquad k \equiv 1 \pmod{2}.$$

Setting

$$\tau = iz/k + h2^{1/2}/k, \qquad \Re(z) > 0,$$

the invariance property (2.4), with the notation of (2.1), becomes:

$$(2.6) \qquad f\left(\exp\left\{-\frac{2\pi z}{2^{1/2}k} + 2\pi i\,\frac{h}{k}\right\}\right) = f\left(\exp\left\{-\frac{4\pi}{2^{1/2}kz} + 2\pi i\,\frac{h'}{k}\right\}\right)$$

for Case I; and

$$(2.6)' \qquad f\left(\exp\left\{-\frac{2\pi z}{2^{1/2}k} + 2\pi i\,\frac{h}{k}\right\}\right) = f\left(\exp\left\{-\frac{2\pi}{2^{1/2}kz} + 2\pi i\,\frac{h^*}{k}\right\}\right)$$

in Case II.

3. With the notation of (2.1), by Cauchy's theorem,

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}}\,dx,$$

where C denotes the circle defined by $|x| = \exp(-2^{1/2}\pi N^{-2})$. Using the Farey dissection of order N of the circle C, by means of arcs $\xi_{h,k}$, we may write

$$c_n = \sum_{h,k;0\leqq h<k\leqq N}{}' \frac{1}{2\pi i}\int_{\xi_{h,k}} \frac{f(x)}{x^{n+1}}\,dx.$$

Here and in the following, $\sum'$ denotes summation running over all h, k with $(h, k) = 1$. On the arcs $\xi_{h,k}$ we put

$$x = \exp\left(-\frac{2\pi}{2^{1/2}}N^{-2} + 2\pi i\frac{h}{k} + \frac{2\pi}{2^{1/2}}i\phi\right);$$

then

$$(3.1)\qquad \begin{aligned} c_n = \frac{1}{2^{1/2}} &\sum_{h,k;0\leqq h<k\leqq N}{}' \exp(-2\pi inh/k) \\ &\cdot \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{2\pi i(h/k)-2\pi(N^{-2}-i\phi)/2^{1/2}})e^{2\pi n(N^{-2}-i\phi)/2^{1/2}}d\phi. \end{aligned}$$

From (2.4) it appears that the points $h2^{1/2}/k$ have the same singular character as $\tau = i\infty$, hence the Farey arcs, of order N, will be defined as follows:

To every term $h2^{1/2}/k$ in the Farey series of order N, $(0\leqq h<k\leqq N$, $(h, k)=1)$ corresponds an arc

$$\left(\frac{h}{k}2^{1/2} - \vartheta'_{h,k}, \frac{h}{k}2^{1/2} + \vartheta''_{h,k}\right)$$

with

$$\frac{h}{k}2^{1/2} - \vartheta'_{h,k} = \frac{h+h_1}{k+k_1}2^{1/2}, \quad \frac{h}{k}2^{1/2} + \vartheta''_{h,k} = \frac{h+h_2}{k+k_2}2^{1/2},$$

where

$$(3.2)\qquad \frac{h_1}{k_1}2^{1/2} < \frac{h}{k}2^{1/2} < \frac{h_2}{k_2}2^{1/2},$$

$$k, k_1, k_2 \leqq N.$$

Here $h_1 2^{1/2}/k_1$ and $h_2 2^{1/2}/k_2$ are the terms adjacent to $h2^{1/2}/k$ in the Farey series of order N.

The end points of the arc in question are the "mediants" lying between the terms (3.2). Since

$$hk_1 - h_1 k = 1, \qquad h_2 k - hk_2 = 1$$

or

$$hk_1 \equiv 1 \pmod{k}, \qquad hk_2 \equiv -1 \pmod{k},$$

we have, from (2.5) and (2.5)′

$$(3.3)\qquad k_1 \equiv -h' \pmod{k}, \qquad k_2 \equiv h' \pmod{k} \qquad \text{when } k \equiv 0 \pmod{2};$$

and

$$(3.3)'\qquad k_1 \equiv -2h^* \pmod{k}, \qquad k_2 \equiv 2h^* \pmod{k} \qquad \text{when } k \equiv 1 \pmod{2}.$$

The mediants do not belong to the Farey series of order N, hence

$$k_1 + k > N, \qquad k_2 + k > N$$

or, by (3.2)

$$N - k < k_1 \leqq N, \qquad N - k < k_2 \leqq N. \tag{3.4}$$

The integers k_1 and k_2 are thus uniquely determined. We have

$$\vartheta'_{h,k} = \frac{2^{1/2}}{k(k_1 + k)}, \qquad \vartheta''_{h,k} = \frac{2^{1/2}}{k(k_2 + k)}. \tag{3.5}$$

4. If we introduce the abbreviation

$$w = N^{-2} - i\phi, \tag{4.01}$$

we may write:

$$c_n = \frac{1}{2^{1/2}} \sum_{h,k;0\leqq h<k\leqq N}' e^{-2\pi inh/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{-2\pi in(h/k)-(2\pi/2^{1/2})\cdot(kw/k)}) e^{2\pi nw/2^{1/2}} d\phi$$

$$= \frac{1}{2^{1/2}} \left\{ \sum_2{}' + \sum_1{}' \right\},$$

where $\sum_1'$ means summation over h, k with k odd, $0 \leqq h < k \leqq N$, $(h, k) = 1$; and $\sum_2'$ means summation over corresponding h, k with k even.

If we make use of formulae (2.6) and (2.6)$'$, we get:

$$\begin{aligned} c_n = {} & \frac{1}{2^{1/2}} \sum_2{}' e^{-2\pi inh/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{2\pi ih'/k - 4\pi/2^{1/2}k^2 w}) e^{2\pi nw/2^{1/2}} d\phi \\ & + \frac{1}{2^{1/2}} \sum_1{}' e^{-2\pi inh/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{2\pi ih^*/k - 2\pi/2^{1/2}k^2 w}) e^{2\pi nw/2^{1/2}} d\phi. \end{aligned} \tag{4.1}$$

Let

$$f(x) = x^{-1} + D(x) \tag{4.2}$$

$$D(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{4.3}$$

Then the expression (4.1) can be split into four parts:

$$c_n = Q^{(e)}(n) + R^{(e)}(n) + Q^{(0)}(n) + R^{(0)}(n), \tag{4.4}$$

with

$$Q^{(e)}(n) = 2^{-1/2} \sum_2{}' e^{-2\pi i(nh+h')/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{4\pi/(2^{1/2}k^2 w) + 2\pi nw/2^{1/2}} d\phi, \tag{4.41}$$

$$R^{(e)}(n) = 2^{-1/2} \sum_2{}' e^{-2\pi inh/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} D(e^{2\pi ih/k - 4\pi/(2^{1/2}k^2 w)}) e^{2\pi nw/2^{1/2}} d\phi, \tag{4.42}$$

$$(4.41)' \quad Q^{(0)}(n) = 2^{-1/2} \sum_1{}' e^{-2\pi i(nh+h^*)/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi/(2^{1/2}k^2w)+2\pi nw/2^{1/2}} d\phi,$$

$$(4.42)' \quad R^{(0)}(n) = 2^{-1/2} \sum_1{}' e^{-2\pi inh/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} D(e^{2\pi ih^*/k-2\pi/(2^{1/2}k^2w)}) e^{2\pi nw/2^{1/2}} d\phi.$$

In order to estimate $Q^{(e)}(n)$ and $Q^{(0)}(n)$ we divide the intervals of integration into three parts as follows:

$$-\vartheta'_{h,k} = -\frac{2^{1/2}}{k(k_1+k)} \leqq -\frac{2^{1/2}}{k(N+k)} < \frac{2^{1/2}}{k(N+k)} \leqq \frac{2^{1/2}}{k(k_2+k)} = \vartheta''_{h,k}.$$

For k even, $k=2p$, $p=1, 2, \cdots$, we have:

$$(4.5) \quad \begin{aligned} Q^{(e)}(n) &= 2^{-1/2} \sum_{p=1}^{[N/2]} \sum_{h \bmod 2p}{}' e^{-2\pi i(nh+h')/2p} \int_{-2^{1/2}/2p(N+2p)}^{2^{1/2}/2p(N+2p)} \\ &+ 2^{-1/2} \sum_{p=1}^{[N/2]} \sum_{h \bmod 2p}{}' e^{-2\pi i(nh+h')/2p} \int_{-2^{1/2}/2p(k_1+2p)}^{-2^{1/2}2p(N+2p)} \\ &+ 2^{-1/2} \sum_{p=1}^{[N/2]} \sum_{h \bmod 2p}{}' e^{-2\pi i(nh+h')/2p} \int_{2^{1/2}/2p(N+2p)}^{2^{1/2}/2p(k_2+2p)} \\ &= Q_0^{(e)}(n) + Q_1^{(e)}(n) + Q_2^{(e)}(n). \end{aligned}$$

In all three integrals the integrand is

$$e^{4\pi/(2^{1/2}\cdot 4p^2w)+2\pi n/(2^{1/2}w)} d\phi = e^{\pi/(2^{1/2}p^2w)+2\pi n/(2^{1/2}w)} d\phi.$$

If k is odd, $k=2p-1$, $p=1, 2, \cdots,$

$$(4.5)' \quad \begin{aligned} Q^{(0)}(n) &= 2^{-1/2} \sum_{p=1}^{[(N+1)/2]} \sum_{h \bmod (2p-1)}{}' e^{-2\pi i(nh+h^*)/(2p-1)} \int_{-2^{1/2}/(2p-1)(N+2p-1)}^{2^{1/2}/(2p-1)(N+2p-1)} \\ &+ 2^{-1/2} \sum_{p=1}^{[(N+1)/2]} \sum_{h \bmod (2p-1)}{}' e^{-2\pi i(nh+h^*)/(2p-1)} \int_{-2^{1/2}/(2p-1)(k_1+2p-1)}^{-2^{1/2}/(2p-1)(N+2p-1)} \\ &+ 2^{-1/2} \sum_{p=1}^{[(N+1)/2]} \sum_{h \bmod (2p-1)}{}' e^{-2\pi i(nh+h^*)/(2p-1)} \int_{2^{1/2}/(2p-1)(N+2p-1)}^{2^{1/2}/(2p-1)(k_2+2p-1)} \\ &= Q_0^{(0)}(n) + Q_1^{(0)}(n) + Q_2^{(0)}(n). \end{aligned}$$

In these three integrals the integrand is

$$e^{2\pi/(2^{1/2}(2p-1)^2w)+2\pi n/(2^{1/2}w)} d\phi.$$

5. Let us set

$$(5.1) \quad A_{2p}(n) = \sum_{h \bmod 2p}{}' e^{-2\pi i\cdot(nh+h')/2p}.$$

Then

$$Q_0^{(e)}(n) = 2^{-1/2} \sum_{p=1}^{[N/2]} A_{2p}(n) \cdot \int_{-2^{1/2}/2p(N+2p)}^{2^{1/2}/2p(N+2p)} e^{\pi(/2^{1/2}p^2w)+2\pi nw/2^{1/2}} d\phi$$

and with

$$\phi = (N^{-2} - w)/i,$$

$$(5.2)\quad Q_0^{(e)}(n) = 2^{-1/2} \sum_{p=1}^{[N/2]} A_{2p}(n) \cdot \frac{1}{i} \int_{N^{-2}-i2^{1/2}/2p(N+2p)}^{N^{-2}+i2^{1/2}/2p(N+2p)} e^{\pi/(2^{1/2}p^2w)+2\pi nw/2^{1/2}} dw$$

where $A_{2p}(n)$ is a Kloosterman sum for which we have the estimate (Rademacher [7] and references there given):

$$(5.3)\qquad |A_{2p}(n)| < C(2p)^{2/3+\epsilon}(2p, n)^{1/3}.$$

In the w-plane we take as a contour the rectangle R_1 of vertices

$$\pm N^{-2} \pm \frac{i}{2^{1/2}p(N + 2p)}\cdot$$

Then

$$(5.4)\quad \begin{aligned} Q_0^{(e)}(n) &= \frac{2\pi}{2^{1/2}} \sum_{p=1}^{[N/2]} A_{2p}(n) \int_{R_1} e^{\pi/(2^{1/2}p^2w)+2\pi nw/2^{1/2}} dw \\ &\quad - \frac{1}{i2^{1/2}} \sum_{p=1}^{[N/2]} A_{2p}(n) \left\{ \int_{N^{-2}+iM_1}^{-N^{-2}+iM_1} + \int_{-N^{-2}+iM}^{-N^{-2}-iM_1} + \int_{-N^{-2}-iM_1}^{N^{-2}-iM_1} \right\}, \end{aligned}$$

with $M_1 = 1/2^{1/2}p(N+2p)$. Consequently

$$Q_0^{(e)}(n) = \frac{2\pi}{2^{1/2}} \sum_{p=1}^{[N/2]} A_{2p}(n) \cdot L_{2p}(n) - \frac{1}{i2^{1/2}} \sum_{p=1}^{[N/2]} A_{2p}(n) \{J_1 + J_2 + J_3\},$$

where the four integrals $L_{2p}(n)$, J_1, J_2, J_3 have the same integrand.

Estimating J_1 and J_3, we have:

$$\begin{aligned} w &= u \pm \frac{i}{2^{1/2}p(N + 2p)}, \\ -N^{-2} &\leqq u \leqq N^{-2}, \\ \mathfrak{R}(w) &= u \leqq N^{-2}, \\ \mathfrak{R}\left(\frac{1}{w}\right) &= \frac{u}{u^2 + 1/2p^2(N + 2p)^2} < N^{-2} \cdot 2p^2(N + 2p)^2 \\ &= 2p^2\left(1 + \frac{2p}{N}\right)^2 \leqq 8p^2. \end{aligned}$$

Hence $\left| e^{\pi/2^{1/2}p^2w+2\pi nw/2^{1/2}} \right| \leqq e^{8\pi/2^{1/2}+2\pi nN^{-2}/2^{1/2}}$, and

$$\begin{matrix} |J_1| \\ |J_3| \end{matrix} \leqq 2N^{-2}e^{2\pi(4+nN^{-2})/2^{1/2}} \leqq C_1p^{-1}N^{-1}e^{2\pi nN^{-2}/2^{1/2}}. \tag{5.5}$$

We pass on to the estimation of J_2. In J_2,

$$w = -N^{-2} + iv, \qquad -\frac{1}{2^{1/2}p(N+2p)} \leqq v \leqq \frac{1}{2^{1/2}p(N+2p)},$$

$$\Re(w) = -N^{-2} < 0, \qquad \Re\left(\frac{1}{w}\right) = \frac{-N^{-2}}{N^{-4}+v^2} < 0;$$

hence

$$\left| e^{\pi/2^{1/2}p^2w+2\pi nw/2^{1/2}} \right| < 1$$

and

$$|J_2| < \frac{2}{2^{1/2}p(N+2p)} < 2p^{-1}N^{-1}. \tag{5.6}$$

By the results (5.3), (5.5) and (5.6) we have:

$$\left| \sum_{p=1}^{[N/2]} A_{2p}(n)\{J_1+J_2+J_3\} \right| \leqq C_2 \sum_{p=1}^{[N/2]} (2p)^{2/3+\epsilon}(2p,\, n)^{1/3}p^{-1}N^{-1}e^{2\pi nN^{-2}/2^{1/2}}$$

If we assume $n \geqq 1$, then $(2p, n) \leqq n$, and hence

$$\sum_{p=1}^{[N/2]} A_{2p}(n)\{J_1+J_2+J_3\} = O(e^{2\pi nN^{-2}/2^{1/2}}n^{1/3}N^{-1/3+\epsilon}). \tag{5.7}$$

Now we consider the integral denoted by $L_{2p}(n)$ in (5.4)

$$\begin{aligned} L_{2p}(n) &= \frac{1}{2\pi i}\int_{R_1} e^{\pi/(2^{1/2}p^2w)+2\pi nw/2^{1/2}}dw \\ &= \frac{1}{2\pi i}\int_{R_1} \sum_{\mu=0}^{\infty} \frac{(\pi/2^{1/2}p^2w)^\mu}{\mu!} \sum_{\nu=0}^{\infty} \frac{(2\pi nw/2^{1/2})^\nu}{\nu!}\, dw \\ &= \operatorname*{Residue}_{w=0}\, e^{\pi/(2^{1/2}p^2w)+2\pi nw/2^{1/2}} \\ &= \sum_{\nu=0}^{\infty} \left(\frac{\pi}{2^{1/2}p^2}\right)^{\nu+1} \frac{1}{(\nu+1)!}\left(\frac{2\pi n}{2^{1/2}}\right)^\nu \frac{1}{\nu!} \\ &= \frac{2^{1/2}}{2pn^{1/2}} \sum_{\nu=0}^{\infty} \left(\frac{\pi n^{1/2}}{p}\right)^{2\nu+1} \frac{1}{\nu!(\nu+1)!}, \end{aligned}$$

or

$$(5.8) \qquad L_{2p}(n) = \frac{2^{1/2}}{2pn^{1/2}} I_1\left(\frac{4\pi n^{1/2}}{2p}\right), \qquad p = 1, 2, \cdots,$$

where $I_1(z)$ is the Bessel function of first order with purely imaginary argument. From (5.4), (5.7), (5.8) we have

$$(5.9) \quad Q_0^{(e)}(n) = \frac{2\pi}{n^{1/2}} \sum_{p=1}^{[N/2]} \frac{A_{2p}(n)}{2p} I_1\left(\frac{4\pi n^{1/2}}{2p}\right) + O(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}).$$

Passing to the case of k odd, let us write

$$(5.1)' \qquad B_{2p-1}(n) = \sum_{h \bmod (2p-1)}{}' e^{-2\pi i(nh+h^*)/(2p-1)},$$

or

$$B_k(n) = \sum_{h \bmod k; k \text{ odd}}{}' e^{-2\pi i(nh+h^*)/k},$$

with $2hh^* \equiv -1 \pmod{k}$. The sum $B_k(n)$ is also a Kloosterman sum. Indeed, if $k=1$, $B_1(n) = A_1(n) = 1$ with

$$A_k(n) = \sum_{h \bmod k}{}' e^{2\pi i(-nh-h')/k}, \qquad hh' \equiv -1 \pmod{k}.$$

We may write

$$B_k(n) = \sum_{h \bmod k;\, k \text{ odd}}{}' e^{-2\pi i(nh+h'/2)/k}, \qquad k \geqq 3.$$

Since k is odd, we have $(k, 2) = 1$ and therefore the congruence

$$2 \cdot x \equiv 1 \pmod{k}$$

has the unique solution

$$x \equiv 2^{\phi(k)-1} \pmod{k},$$

i.e.

$$(5.101)' \qquad \frac{1}{2} \equiv 2^{\phi(k)-1} \pmod{k},$$

where $\phi(k)$ denotes the Euler function(3). Hence

$$B_k(n) = \sum_{h \bmod k;\, k \text{ odd}}{}' e^{-2\pi i/k(nh+2^{\phi(k)-1}\cdot h')} = S(-n, 2^{\phi(k)-1}; k)$$

in the Salié [10] notation. By the properties, (cf. [10]), of Kloosterman sums,

$$S(-n, 2^{\phi(k)-1}; k) = S(1, -n\cdot 2^{\phi(k)-1}; k) = S(-n\cdot 2^{\phi(k)-1}, 1; k),$$

(3) From (5.101)′ follows that $n\cdot 2^{\phi(k)-1} \equiv n\cdot 2^{-1} \pmod{k} \equiv n(1-k)/2 \pmod{k}$, k odd.

since we have $(k, 2^{\phi(k)-1})=1$. Hence, returning to our notation,

$$B_k(n) = A_k(n\cdot 2^{\phi(k)-1}) = A_k\left(\frac{1-k}{2}\, n\right), \ k \text{ odd}.$$

Accordingly

$$(5.3)' \qquad |B_{2p-1}(n)| < C^*(2p-1)^{2/3+\epsilon}(2p-1, n)^{1/3}, \qquad p \neq 1.$$

From (4.5)'

$$Q_0^{(0)}(n) = 2^{-1/2} \sum_{p=1}^{[(N+1)/2]} B_{2p-1}(n)\cdot \int_{-2^{1/2}/(2p-1)(N+2p-1)}^{2^{1/2}/(2p-1)(N+2p-1)} e^{2\pi/2^{1/2}(2p-1)^2 w + 2\pi n w/2^{1/2}} d\phi,$$

or, with $w=N^{-2}-i\phi$,

$$(5.2)' \qquad \begin{aligned} Q_0^{(0)}(n) = 2^{-1/2} &\sum_{p=1}^{[(N+1)/2]} B_{2p-1}(n) \\ &\cdot \frac{1}{i}\int_{N^{-2}-i2^{1/2}/(2p-1)(N+2p-1)}^{N^{-2}+i2^{1/2}/(2p-1)(N+2p-1)} e^{2\pi/2^{1/2}(2p-1)^2 w + 2\pi n w/2^{1/2}} dw. \end{aligned}$$

Taking as a contour the rectangle R_2 of vertices

$$\pm N^{-2} \pm \frac{i2^{1/2}}{(2p-1)(N+2p-1)},$$

$Q_0^{(0)}(n)$ is split up into four integrals in the same manner as $Q_0^{(e)}(n)$:

$$(5.4)' \qquad \begin{aligned} Q_0^{(0)}(n) = \frac{2\pi}{2^{1/2}} &\sum_{p=1}^{[(N+1)/2]} B_{2p-1}(n)\cdot M_{2p-1}(n) \\ &- \frac{1}{i2^{1/2}} \sum_{p=1}^{[(N+1)/2]} B_{2p-1}(n)\{K_1+K_2+K_3\}. \end{aligned}$$

We obtain the estimates

$$(5.5)' \qquad \begin{matrix} |K_1| \\ |K_3| \end{matrix} \leqq 2N^{-2}e^{2\pi(2+nN^{-2})/2^{1/2}} \leqq C_1^* p^{-1}N^{-1}e^{2\pi n N^{-2}/2^{1/2}},$$

$$(5.6)' \qquad \begin{aligned} |K_2| &< \frac{2(2^{1/2})}{(2p-1)(N+2p-1)} \\ &< 4p^{-1}N^{-1}, \end{aligned}$$

$$(5.7)' \qquad \sum_{p=1}^{[(N+1)/2]} B_{2p-1}(n)\{K_1+K_2+K_3\} = O(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}).$$

Also

$$
\begin{aligned}
M_{2p-1}(n) &= \frac{1}{2\pi i}\int_{R_2} e^{2\pi/2^{1/2}(2p-1)^2 w+2\pi n w/2^{1/2}}dw \\
&= \int_{R_2}\sum_{\mu=0}^{\infty}\frac{(2\pi/2^{1/2}(2p-1)^2 w)^{\mu}}{\mu!}\sum_{\nu=0}^{\infty}\frac{(2\pi n w/2^{1/2})^{\nu}}{\nu!}\,dw \\
&= \sum_{\nu=0}^{\infty}\left(\frac{2^{1/2}\pi}{(2p-1)^2}\right)^{\nu+1}\frac{1}{(\nu+1)!}\left(\frac{2\pi n}{2^{1/2}}\right)^{\nu}\frac{1}{\nu!} \\
&= \frac{1}{(2p-1)n^{1/2}}\sum_{\nu=0}^{\infty}\left(\frac{\pi(2n)^{1/2}}{2p-1}\right)^{2\nu+1}\frac{1}{\nu!(\nu+1)!},
\end{aligned}
$$

i.e.

$$
\text{(5.8)}' \qquad
\begin{aligned}
M_{2p-1}(n) &= \frac{1}{(2p-1)n^{1/2}} I_1\left(\frac{4\pi n^{1/2}}{2^{1/2}(2p-1)}\right) \\
&= \frac{2^{1/2}}{(2p-1)(2n)^{1/2}} I_1\left(\frac{2\pi(2n)^{1/2}}{2p-1}\right).
\end{aligned}
$$

From (5.4)′, (5.7)′, (5.8)′,

$$
\text{(5.9)}' \qquad
\begin{aligned}
Q_0^{(0)}(n) = \frac{2\pi}{(2n)^{1/2}}\sum_{p=1}^{[(N+1)/2]}\frac{B_{2p-1}(n)}{2p-1} I_1\left(\frac{2\pi(2n)^{1/2}}{2p-1}\right) \\
+ O\left(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}\right),
\end{aligned}
$$

where

$$
B_{2p-1}(n) = A_{2p-1}((1-p)n).
$$

6. At this stage we return to formulae (4.5) and (4.5)′ and we show that

$$
\text{(6.1)} \qquad
\begin{matrix} Q_1^{(e)}(n) \\ Q_2^{(e)}(n) \end{matrix} = O\left(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}\right),
$$

$$
\text{(6.1)}' \qquad
\begin{matrix} Q_1^{(0)}(n) \\ Q_2^{(0)}(n) \end{matrix} = O\left(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}\right).
$$

These estimations follow closely the method used by Rademacher [7], hence details are omitted.

7. Referring to formula (4.4), $R^{(e)}(n)$ and $R^{(0)}(n)$ can also be broken into three sums:

$$
\text{(7.1)} \qquad R^{(e)}(n) = S_1^{(e)} + S_2^{(e)} + S_3^{(e)},
$$

$$
\text{(7.1)}' \qquad R^{(0)}(n) = S_1^{(0)} + S_2^{(0)} + S_3^{(0)},
$$

by using the same decomposition of the Farey arc

$$-\vartheta'_{h,k} \leqq \phi \leqq \vartheta''_{h,k}.$$

We find, again omitting details,

$$R^{(e)}(n) = O(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}), \tag{7.5}$$

$$R^{(0)}(n) = O(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}). \tag{7.5$'$}$$

Finally, putting together the results of (5.9), (5.9)′, (6.1) (6.1)′, (7.5), (7.5)′ we obtain

$$\begin{aligned} c_n = {} & \frac{2\pi}{n^{1/2}} \sum_{\nu=1}^{[N/2]} \frac{A_{2\nu}(n)}{2\nu} I_1\left(\frac{4\pi n^{1/2}}{2\nu}\right) \\ & + \frac{2\pi}{(2n)^{1/2}} \sum_{\nu=1}^{[(N+1)/2]} \frac{A_{2\nu-1}((1-\nu)n)}{2\nu-1} I_1\left(\frac{2\pi(2n)^{1/2}}{2\nu-1}\right) \\ & + O(e^{2\pi n N^{-2}/2^{1/2}} n^{1/3} N^{-1/3+\epsilon}). \end{aligned} \tag{7.6}$$

For every fixed $n \geqq 1$, we let $N \to \infty$ and we have the

THEOREM. *The Fourier coefficients of the invariant*

$$j(2^{1/2}; \tau) = e^{-2\pi i\tau/2^{1/2}} + c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau/2^{1/2}}$$

are determined by the sum of two convergent series

$$\begin{aligned} c_n = \frac{2\pi}{n^{1/2}} \Bigg\{ & \sum_{\nu=1}^{\infty} \frac{A_{2\nu}(n)}{2\nu} I_1\left(\frac{2\pi n^{1/2}}{\nu}\right) \\ & + \frac{1}{2^{1/2}} \sum_{\nu=1}^{\infty} \frac{A_{2\nu-1}((1-\nu)n)}{2\nu-1} I_1\left(\frac{2\pi(2n)^{1/2}}{2\nu-1}\right) \Bigg\}, \end{aligned} \tag{7.7}$$

where $n \geqq 1$, *and*

$$A_k(m) = \sum_{h \bmod k}{}' e^{-2\pi i(mh+h')/k}, \qquad hh' \equiv -1 \pmod{k}.$$

8. The same remark has to be made here, as by Rademacher [7], concerning the exclusion of the value $n=0$. The coefficient c_0 has been obtained in (2.1) as $c_0 = 2^3 \cdot 13$.

Using formula (7.7), calculations have been made with only four terms of each series, obtaining, e.g.:

$$c_1 = 4371.46, \qquad \text{exact value: } c_1 = 4372,$$

$$c_2 = 96{,}255.92, \qquad \text{exact value: } c_2 = 96{,}256.$$

II. The Fourier coefficients of the invariant $j(3^{1/2};\tau)$ and its relation to $j(1;\tau)$

9. Let $J(3^{1/2};\tau)$ be the invariant belonging to $G(3^{1/2})$, with the properties:

$$J(3^{1/2};\rho_3) = 0, \qquad J(3^{1/2};i) = 1, \qquad J(3^{1/2};i\infty) = \infty,$$

where

$$\rho_3 = \frac{-3^{1/2}+i}{2}.$$

As in §2, here we exploit the transformation equation of rank 3, (cf. [2, vol. II, p. 385] or [3, vol. III, p. 395]). Such an equation is the result of elimination of σ between the three relations:

$$(9.01)\qquad \begin{aligned} j(\omega) &= \frac{3^3(\sigma+1)(9\sigma+1)^3}{\sigma}, \\ j'(\omega) &= j\left(\frac{\omega}{3}\right) = \frac{3^3(\sigma'+1)(9\sigma'+1)^3}{\sigma'}, \qquad \sigma\sigma' = 1, \end{aligned}$$

where $\sigma(\omega)$ is a one-valued function of the transformation polygon T_3, (cf. [3, vol. III, pp. 297 ff. and pp. 390 ff.]). For our purpose we simply interpret $\sigma(\omega)$ as a parameter. We obtain

$$(9.02)\qquad \begin{aligned} j(\omega) + j'(\omega) &= 3^3(3^6t^3 + 2^2\cdot 3^5t^2 - 2^2\cdot 479t - 2^5\cdot 59), \\ j(\omega)\cdot j'(\omega) &= 3^6(t+2)\cdot(2\cdot 41 + 3^2t)^3, \end{aligned}$$

where $t=\sigma+\sigma'$. The functions

$$\phi(\tau) = j(3^{1/2}\tau) + j(\tau/3^{1/2}), \qquad \psi(\tau) = j(3^{1/2}\tau)\cdot j(\tau/3^{1/2})$$

are invariant under the transformations

$$S(\tau) = \tau + 3^{1/2}, \qquad T(\tau) = -1/\tau,$$

$\phi(\tau)$ with a pole of order 3 in x, $\psi(\tau)$ with a pole of order 4 in x, $x=e^{2\pi i\tau/3^{1/2}}$. We infer that

$$A\cdot J^3(3^{1/2};\tau) + B\cdot J^2(3^{1/2};\tau) + C\cdot J(3^{1/2};\tau) + D = \phi(\tau)$$

and

$$A'\cdot J^4(3^{1/2};\tau) + B'\cdot J^3(3^{1/2};\tau) + C'\cdot J^2(3^{1/2};\tau) + D'\cdot J(3^{1/2};\tau) + E' = \psi(\tau).$$

Now, comparing these two last equations with equations (9.02), we infer that:

$$(9.03)\qquad J(3^{1/2};\tau) = \frac{\alpha t+\beta}{\gamma}.$$

But, from (9.01), when $\sigma=\sigma'=1$, $t=2$,

$$j(\omega) = 2^4 3^3 5^3 = j'(\omega) = j(\omega/3).$$

On the other hand, from the Classinvariant theory, (cf. Fricke, [3, vol. III, p. 395]), we have:

$$j(i3^{1/2}) = 2^4 3^3 5^3 = j(-1/i3^{1/2}) = j(i/3^{1/2});$$

i.e. such value is assumed by $j(\omega)$ only at points homologous to $\omega = i3^{1/2}$, and if $\omega = 3^{1/2}\tau$, this happens when $\tau = i$. However we have preassigned that $J(3^{1/2}; i) = 1$; hence, from (9.03) we obtain:

$$(9.031) \qquad (2\alpha + \beta)/\delta = 1.$$

Similarly, when $\sigma = \sigma' = -1$, $t = -2$,

$$j(\omega) = j'(\omega) = 0, \qquad \text{by (9.01).}$$

Again

$$j(3^{1/2}\rho_3) = j\left\{3^{1/2}\left(\frac{-3^{1/2} + i}{2}\right)\right\} = j\left(\frac{-1 + i3^{1/2}}{2}\right) = 0,$$

$$j'(3^{1/2}\rho_3) \equiv j\left(\frac{\rho_3}{3^{1/2}}\right) = j\left(\frac{-3^{1/2} + i}{2\cdot 3^{1/2}}\right) = j\left(\frac{2\cdot 3^{1/2}}{3^{1/2} - i}\right) = 0.$$

By the preassigned correspondence, $J(3^{1/2}; \rho_3) = 0$, hence (9.03) supplies the equation:

$$(9.032) \qquad (-2\alpha + \beta)/\delta = 0.$$

Solving (9.031) and (9.032), we obtain $\beta = 2\alpha$, $\delta = 4\alpha$, i.e.

$$(9.033) \qquad J(3^{1/2}; \tau) = (t + 2)/4.$$

Either by substitution in (9.02) or by a process of differentiation, the constants A, B, C, D and A', B', C', D', E' are readily calculated. By setting

$$j(3^{1/2}; \tau) = 2^2 3^3 J(3^{1/2}; \tau),$$

we obtain the following equations, relating $j(3^{1/2}; \tau)$ with $j(3^{1/2}\tau)$ and $j(\tau/3^{1/2})$:

$$(9.04) \quad j^3(3^{1/2}; \tau) - 2\cdot 3^2\cdot 7 j^2(3^{1/2}; \tau) + 2^7\cdot 23 j(3^{1/2}; \tau) = j(3^{1/2}\tau) + j(\tau/3^{1/2}),$$

$$(9.05) \qquad j(3^{1/2}; \tau)\cdot\{2^6\cdot 3 + j(3^{1/2}; \tau)\}^3 = j(3^{1/2}\tau)\cdot j(\tau/3^{1/2}).$$

Using (9.04) or (9.05), the first few coefficients of $j(3^{1/2}; \tau)$ may be computed:

$$(9.06) \quad (j3^{1/2}; \tau) = x^{-1} + 2\cdot 3\cdot 7 + 3^3\cdot 29x + 2^5\cdot 271x^2 + \cdots = f(x),$$

where $x = \exp(2\pi i\tau/3^{1/2})$. Also from (9.04) and (9.05) we obtain a quadratic equation in $j(3^{1/2}; \tau)$, whose coefficients involve $\{j(3^{1/2}\tau) + j(\tau/3^{1/2})\}$ and $\{j(3^{1/2}\tau)\cdot j(\tau/3^{1/2})\}$ linearly.

Although this group has been known for a long time, (cf. [5] and references there given), relations (9.04) and (9.05) are new. To the best of our knowledge, they do not appear anywhere in the literature.

As shown by J. I. Hutchinson, [5], the substitutions of $G(3^{1/2})$ form two classes:

(I) $$V = \begin{pmatrix} a & b3^{1/2} \\ c3^{1/2} & d \end{pmatrix}, \qquad ad - 3bc = 1,$$

(II) $$V' = \begin{pmatrix} a'3^{1/2} & b' \\ c' & d'3^{1/2} \end{pmatrix}, \qquad 3a'd' - b'c' = 1,$$

where $a, b, c, d, a', b', c', d'$ are rational integers. We write:

$$V = \begin{Bmatrix} h' & -\dfrac{hh'+1}{k}\,3^{1/2} \\ \dfrac{k3^{1/2}}{3} & -h \end{Bmatrix}$$

with $hh' \equiv -1 \pmod{k}$, $(h, k) = 1$, when $k \equiv 0 \pmod{3}$;

$$V' = \begin{Bmatrix} h^*3^{1/2} & -\dfrac{3hh^*+1}{k} \\ k & -h3^{1/2} \end{Bmatrix}$$

with $3hh^* \equiv -1 \pmod{k}$, $(h, k) = 1$ when $k \equiv 1 \pmod{3}$ or $k \equiv 2 \pmod{3}$. We have

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}},$$

where

$$f(x) = x^{-1} + \sum_{n=0}^{\infty} a_n x^n = j(3^{1/2}; \tau),$$

and C is the circle $|x| = e^{-2\pi N^{-2}/3^{1/2}}$.

The Farey arcs are taken around the point $(h/k)3^{1/2}$, and hence we proceed as in the case of $G(2^{1/2})$. We obtain

$$\begin{aligned} a_n = {} & \frac{2\pi}{n^{1/2}} \sum_{\nu=1}^{[N/3]} \frac{A_{3\nu}(n)}{3\nu} I_1\left(\frac{4\pi n^{1/2}}{3\nu}\right) \\ & + \frac{2\pi}{(3n)^{1/2}} \sum_{\nu=1}^{[(N+1)/3]} \frac{A_{3\nu-1}(\nu n)}{3\nu-1} I_1\left(\frac{4\pi(3n)^{1/2}}{3(3\nu-1)}\right) \\ & + \frac{2\pi}{(3n)^{1/2}} \sum_{\nu=1}^{[(N+2)/3]} \frac{A_{3\nu-2}((1-\nu)n)}{3\nu-2} I_1\left(\frac{4\pi(3n)^{1/2}}{3(3\nu-2)}\right) \\ & + O(e^{2\pi n N^{-2}/3^{1/2}} n^{1/3} N^{-1/3+\epsilon}), \end{aligned} \tag{9.6}$$

where $n \geqq 1$, $A_k(m)$ being defined as in (7.7), whence a Theorem which supplies the Fourier coefficients of $j(3^{1/2}; \tau)$ by means of convergent series, by letting $N \to \infty$, n fixed.

10. In the class of groups $G(\lambda_q)$, with $1 \leqq \lambda_q = 2 \cos \pi/q$, described in §1, $G(1)$, $G(2^{1/2})$ and $G(3^{1/2})$ are the only groups whose arithmetical character is known, i.e. we know the coefficients of the substitutions. Hence the Farey dissection may be determined, and the Hardy-Littlewood method can be applied to obtain the Fourier coefficients of the invariants.

In a long series of papers (cf. [6] and references there given), H. Petersson has found other methods for the determination of the F. C. of certain classes of automorphic forms. In ([6, p. 13]), Petersson announces his intentions to extend these results to certain classes of automorphic functions. These methods presuppose, in general, the full knowledge of the group.

Although an important contribution toward the arithmetical character of the groups $G(\lambda_q)$ is due to D. Rosen [9] even for the next in order particular case of $q=5$, $\lambda_5 = (1+5^{1/2})/2$, we were not able to take advantage, to this date, of such results, in order to determine the Group $G(\lambda_5)$ nor to devise the proper Farey dissection.

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University of Pennsylvania,
Philadelphia, Pa.