REGULAR CURVES ON RIEMANNIAN MANIFOLDS(1)

BY

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Introduction. A regular curve on a Riemannian manifold is a curve with a continuously turning nontrivial tangent vector.(2) A regular homotopy is a homotopy which at every stage is a regular curve, keeps end points and directions fixed and such that the tangent vector moves continuously with the homotopy. A regular curve is closed if its initial point and tangent coincides with its end point and tangent. In 1937 Hassler Whitney [17] classified the closed regular curves in the plane according to equivalence under regular homotopy. The main goal of this work is to extend this result to regular curves on Riemannian manifolds.

THEOREM A. *Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. Then there is a* 1-1 *correspondence between the set π_0 of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_0 and $\pi_1(T, x_0)$.*

This correspondence may be described as follows. If $\bar{f}\in\pi_0$ let f be a representative of $\bar{f}$ and let Φf at t be the vector of T whose base point is $f(t)$ and whose direction is defined by $f'(t)$, the derivative of f at t. Then Φf is a curve on T which represents an element of $\pi_1(T, x_0)$. The correspondence of Theorem A is that induced by Φ.

If f is a closed regular curve in the plane then its rotation number $\gamma(f)$ is the total angle which $f'(t)$ turns as t traverses I. The Whitney-Graustein Theorem says that two closed regular curves on the plane are regularly homotopic if and only if they have the same rotation number. Using the fact that the unit tangent bundle of the plane is $E^2\times S^1$, this theorem follows from Theorem A.

Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. The space of all regular curves on M starting at the point and direction determined by x_0 is denoted by E. A map π from E onto T is defined by sending a curve into the tangent of its endpoint at its endpoint. The following can be considered as the fundamental theorem of this work.

THEOREM B. *The triple (E, π, T) has the covering homotopy property for polyhedra.*

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(1) The material in this paper is essentially a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan, 1956.

(2) These definitions will be made precise in the body of the work. Also, theorems stated will be proved later.

Let Γ be the fiber over x_0 of (E, π, T) and let Ω be the ordinary loop space of T at x_0.

THEOREM C. *The map Φ is a weak homotopy equivalence between Γ and Ω.*

Theorem B is used to obtain Theorem C and, in turn, Theorem A follows from Theorem C.

If f is a regular curve on M, Φf is a curve, as we shall say, a lifted curve, on $T=T(M)$. Clearly, not all curves on T are lifted curves. In particular, every lifted curve must be an integral curve of a certain 1-form ω_0 on T. If the integral curves of ω_0 were exactly the lifted curves, Theorems A, B, and C could be proved by proving theorems on integral curves. Unfortunately, however, ω_0 admits as integral curves some non-lifted curves. Nevertheless, these considerations raise questions concerning the loop space of integral curves of ω_0 on T.

THEOREM D. *Let ω be a 1-form of Class A on a three dimensional manifold M such that $\omega \wedge d\omega \neq 0$ on M and let $x_0 \in M$. Denote by Ω_ω the loop space at x_0 of piecewise regular curves on M which are integral curves of ω and by Ω the ordinary loop space of M at x_0. Then the inclusion $i: \Omega_\omega \to \Omega$ is a weak homotopy equivalence.*

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1. **Fiber spaces.** A *triple* (E, p, B) will consist of two arcwise connected spaces E, B and a map p from E into B. A triple will be said to have the CHP if it has the covering homotopy property for polyhedrons [11].

If g is a map from a space X into a space Y, then the restriction of g to a subset A of X will be denoted by $g_{|A}$ or sometimes just g. P will always denote a polyhedron. The set $\{t \text{ real} \mid a \leqq t \leqq b\}$ is denoted by $[a, b]$. A *cube* I^k is the Cartesian product of k copies of I, the closed unit interval.

The following proposition is well-known. It is a special case of a theorem proved in [6, p. 136].

PROPOSITION 1.1. *Let (E, p, B) be a triple which has the covering homotopy property for cubes. Then it also has the CHP.*

LEMMA 1.2. *Let a triple (E, p, B) have the CHP. Let σ be a simplex of some dimension n and let $g: \sigma \times I \to B$ be given. Suppose $\dot\sigma$ is the (pointset) boundary of σ, $A = \dot\sigma \times I \cup \sigma \times 0 \subset \sigma \times I$ and $f: A \to E$ covers $g_{|A}$. Then there exists an extension F of f to all of $\sigma \times I$ covering g.*

The proof is immediate. There is a homeomorphism from $\sigma \times I$ onto $I^n \times I$ which sends A homeomorphically onto $I^n \times 0$. Then the application of the CHP yields the desired map F.

The statement of the following theorem is quite similar to what Hurewicz calls the *Uniformization Theorem* in [5]. It will be found useful in proving the CHP for a triple whose base space is a manifold.

PROPOSITION 1.3. *Suppose a triple* (E, p, B) *has the CHP locally; that is, for each point* $x \in B$, *there exists a neighborhood* V *of* x *such that* $(p^{-1}(V), p, V)$ *has the CHP. Then* (E, p, B) *has the CHP.*

Proof. Let $H: P \times I \to B$ be a given homotopy and $h: P \times 0 \to E$ a covering of $H_{|P\times 0}$. We will define a covering homotopy $\overline{H}: P \times I \to E$. For each $y \in H(P \times I)$ let V_y be a neighborhood of y so that $(p^{-1}(V_y), p, V_y)$ has the CHP. Assume $P \times I$ has been given some definite metric. Denote by δ the Lebesgue number of the covering $\{U_y = H^{-1}(V_y) | y \in H(P \times I)\}$ of $P \times I$. Put $I_0 = [0, \delta/3]$. It is sufficient to define $\overline{H}$ on $P \times I_0$ for then iteration will yield a full covering homotopy.

Take a simplicial complex K such that $|K| = P$. Let $\mathrm{Sd}(K)$ be a subdivision of K such that the diameter of any simplex of $\mathrm{Sd}(K)$ is less than $\delta/3$, and let $\mathrm{Sd}(K)^r$ be the r-skeleton of $\mathrm{Sd}(K)$. If v is a vertex of $\mathrm{Sd}(K)$ the choice of I_0 yields that $H(v \times I_0)$ is contained in some neighborhood V where $(p^{-1}(V), p, V)$ has the CHP. This fact immediately gives a definition of $\overline{H}$ on $|\mathrm{Sd}(K)^0| \times I_0$. Proceeding by induction suppose $\overline{H}$ has been defined on $|\mathrm{Sd}(K)^{r-1}| \times I_0$, and σ^r is an r-simplex of $\mathrm{Sd}(K)$. From the choices of $\mathrm{Sd}(K)$ and I_0 it follows that $\sigma^r \times I_0$ has diameter less than δ. Then $H(\sigma^r \times I_0)$ is contained in some V such that $(p^{-1}(V), p, V)$ has the CHP. Already $\overline{H}$ has been defined on $\dot{\sigma}^r \times I_0 \cup \sigma^r \times 0$. Then application of Lemma 1.2 yields $\overline{H}$ on $\sigma^r \times I_0$. In this manner $\overline{H}$ is defined on all of $|\mathrm{Sd}(K)^r| \times I_0$ and then by induction on all of $|\mathrm{Sd}(K)| \times I_0$. This proves 1.3.

2. **Regular curves.** By a manifold we shall mean a connected Riemannian manifold of class 3 and dimension greater than 1. There are no assumptions such as completeness or compactness. If M is a manifold, $T_0(M)$ or often just T_0 will be the space (or bundle) of all tangent vectors of M. By $T(M)$ or T is meant the sub-bundle of T_0 which consists of the unit tangent vectors of M.

As usual, a curve on a space is a map of I into the space. Let M be a manifold and let f be a curve on M whose derivative [1, p. 46] exists at the value $t_0 \in I$. This derivative is an element $v(t_0)$ of $M_{f(t_0)}$, the tangent vector space of M at $f(t_0)$. By convention in this work the *derivative* $f'(t_0)$ of f at t_0 will be the pair $(f(t_0), v(t_0))$. Thus, $f'(t_0)$ will be an element of $T_0(M)$. By the magnitude of $f'(t_0)$ or $|f'(t_0)|$ we mean the magnitude of $v(t_0)$.

A *parametrized regular* curve on a manifold M is a curve f on M such that $f'(t)$ exists, is continuous and has positive magnitude for each $t \in I$. Two parametrized regular curves f and g will be called *equivalent* if there exists a homeomorphism h of I onto itself such that for all $t \in I$, $h'(t)$ exists, is continuous and positive, and $f(t) = g(h(t))$. It is an easy matter to check that this is a true equivalence. A *regular curve* is an equivalence class of parametrized regular curves.

The arc-length of a regular curve g, defined in the usual way, exists, and is independent of its representative. It will be denoted by $L(g)$. Implicit use of the following proposition will be made throughout this work.

PROPOSITION 2.1. *If g is a regular curve on a manifold then there exists a unique representative of g still denoted by g such that $|g'(t)| = L(g)$ for all $t \in I$. $L(g)$ is the only possible constant value here.*

The proof for the plane is in [17]. The same proof holds for the general case of a manifold. The representative of a regular curve given by 2.1 will be called *distinguished*. A distinguished representative is just a parametrization proportional to arc-length. *Unless we note otherwise, a regular curve will be identified with its distinguished representative.*

Let M be a manifold and x_0 a fixed point of $T(M)$. We denote by $E(M)$ or sometimes simply E, the space of regular curves on M whose normalized initial tangents are x_0; in other words,

$$E(M) = \left\{ g \text{ is a regular curve on } M \,\middle|\, \frac{g'(0)}{|g'(0)|} = x_0 \right\}.$$

Let $\bar{d}: T_0 \times T_0 \to R^+$ be any metric on T_0 (R^+ is the space of non-negative real numbers). Then for f and $g \in E$, let

$$d(f, g) = \max \left\{ \bar{d}[f'(t), g'(t)] \mid t \in I \right\}.$$

From the fact that $\bar{d}$ is a metric, it follows easily that d is a metric on E. We will suppose E to have the topology induced by d.

Let f_i be a sequence of points of E converging to a point f of E. Then for each $t \in I$, $f_i'(t)$ converges to $f'(t)$. If a sequence x_n of points of T_0 converges to x_0 then from the topology of T_0 it follows that the base points of x_n in M converge to the base point of x_0. Thus $f_i(t)$ converges to $f(t)$ for each $t \in I$.

The map $\pi: E \to T$ of Theorem B may be defined by $\pi(f) = f'(1)/|f'(1)|$. To speak of (E, π, T) as a triple, E must be arcwise connected. This is proved later (Lemma 6.2).

3. **The reduction of the proof of Theorem B to** 3.1 **and** 3.2. The proof of Theorem B depends essentially on Propositions 3.1 and 3.2.

PROPOSITION 3.1. *Let M be a manifold and $p: E(M) \to M$ be the map $p(g) = g(1)$. Then (E, p, M) has the CHP.*

Suppose M is a manifold. Let $\pi_1: T_0(M) \to M$ be the map which sends a tangent vector into its base point. A homotopy $f_v: P \to T_0(M)$ will be called *vertical* if for all $p \in P$ and $v \in I$, $\pi_1 f_v(p) = \pi_1 f_0(p)$. A homotopy $f_v: P \to X$ (X any space) is said to be *stationary* on a subpolyhedron A of P if $f_v(p) = f_0(p)$ for all $p \in A$ and $v \in I$.

PROPOSITION 3.2. *Let M be a manifold and $f_v: P \to T(M)$ be a given vertical homotopy with $\bar{f}: P \to E(M)$ covering f_0. Then there exists a covering homotopy $\bar{f}_v: P \to E$. Furthermore if f_v is stationary on a subpolyhedron A of P then $\bar{f}_v$ will also be.*

Propositions 3.1 and 3.2 will be proved in the following sections. Now we will show how Theorem B follows from 3.1 and 3.2.

LEMMA 3.3. *Let P be a polyhedron and A be a subpolyhedron which is a strong deformation retract [2] of P. Let g and h be maps of P into a space X which agree on A. Then there exists a homotopy $H: P \times I \to X$ between g and h which is stationary on A.*

Proof. Since A is a strong deformation retract of P there is a homotopy $K: P \times I \to P$ such that $K(p, 0) = p$, $K(p, 1) \in A$, and if $p \in A$, $K(p, t) = p$. The desired homotopy $H: P \times I \to X$ may be defined as follows:

$$H(p, t) = gK(p, 2t) \qquad 0 \leqq t \leqq 1/2,$$
$$H(p, t) = hK(p, 2 - 2t) \qquad 1/2 \leqq t \leqq 1.$$

The k-sphere is denoted by S^k.

LEMMA 3.4. *Let P and A be as above with P contractible, and M be an n dimensional manifold. Let $F: P \to M$ be given and $g: P \to T = T(M)$, $h: P \to T$ be two covering maps of F which agree on A. Then there exists a homotopy $h_v: P \to T$ between g and h such that h_v is stationary on A and for each $v \in I$, h_v covers F.*

Proof. Let E' be the induced bundle $F^{-1}(T)$ [13, p. 47].

$$\begin{array}{ccc} E' & \xrightarrow{q_2} & T \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \pi_1} \\ P & \xrightarrow{F} & M \end{array}$$

By definition:

$$E' = \{(p, t) \in P \times T \mid F(p) = \pi_1(t)\}.$$
$$q_2(p, t) = t, \qquad q_1(p, t) = p.$$

Since P is contractible, E' is a product $P \times S^{n-1}$ [13, p. 53] with q_1 being the projection of E' onto P. Let $\pi': E' \to S^{n-1}$ be the other projection.

Define $\bar{g}: P \to E' \subset P \times T$ by $\bar{g}(p) = (p, g(p))$ and let $\bar{g}^*: P \to S^{n-1}$ be the composition $\pi'\bar{g}$. Similarly, define $\bar{h}$ and $\bar{h}^*$ from h.

Apply the previous lemma to obtain a homotopy $\bar{h}_v^*: P \to S^{n-1}$ between $\bar{g}^*$ and $\bar{h}^*$ which is stationary on A. Define $\bar{h}_v: P \to E' = P \times S^{n-1}$ by $\bar{h}_v(p) = (p, \bar{h}_v^*(p))$ and $h_v: P \to T$ by $h_v = q_2\bar{h}_v$. It can be quickly checked that h_v

satisfies the lemma. q.e.d.

To prove Theorem B it is sufficient by Proposition 1.1 to show that (E, π, T) has the covering homotopy property for cubes. Suppose, then, we are given a homotopy $f_v: P \to T$ and a map $\bar{f}: P \to E$ covering f_0 where P is a cube. We will construct a covering homotopy $\bar{f}_v: P \to E$.

Application of Proposition 3.1 yields a covering map $\bar{h}: P \times I \to E$ of $\pi_1 f_v$ such that $\bar{h}(p, 0) = \bar{f}(p)$. Lemma 3.4 then yields a homotopy $H_u: P \times I \to T$ such that (1) $H_0(p, v) = \pi \bar{h}(p, v)$, (2) $H_1(p, v) = f_v(p)$, (3) $H_u(p, v)$ covers $\pi_1 f_v(p)$ for each $u \in I$, and (4) H_u is stationary on $P \times 0$. By (3) H_u is a vertical homotopy so Proposition 3.2 applies to yield a homotopy $\overline{H}_u: P \times I \to E$ of $\bar{h}$ which covers H_u.

We assert that $\overline{H}_1(p, v)$ can be taken as the desired covering homotopy $\bar{f}_v(p)$. From $H_1(p, v) = f_v(p)$ it follows that $\overline{H}_1(v, p)$ covers $f_v(p)$. Since H_u is stationary on $P \times 0$, $\overline{H}_u$ is also. Then $\overline{H}_1(p, 0) = \overline{H}_0(p, 0) = \bar{h}(p, 0) = \bar{f}(p)$ or $\overline{H}_1(p, v)$ is a homotopy of $\bar{f}(p)$. This shows that Theorem B follows from Propositions 3.1 and 3.2.

4. **Proof of Proposition** 3.1. We will need two lemmas. By $v \perp w$ it is meant that the vectors v and w are perpendicular.

LEMMA 4.1. *Let $n > 1$ and S^{n-1} be the unit vectors of Euclidean n-space E^n considered as a vector space. Suppose P is a cube and a map $w: P \to S^{n-1}$ is given. Then there exists a map $u: P \to S^{n-1}$ such that for all $p \in P$, $u(p) \perp w(p)$.*

This lemma is not true for a general polyhedron. In particular, if $P = S^{n-1}$ and w is the identity, the existence of such a map u implies the existence of a unit vector field on S^{n-1}. It is well known that this is impossible for odd n.

Proof of 4.1. Let $V_{n,2}$ be the Stiefel manifold [13, p. 33] of ordered orthogonal unit 2-frames in E^n. With a projection p_1 sending a 2-frame into its first vector, $V_{n,2}$ becomes an $(n-2)$-sphere bundle over S^{n-1}. Let E' be the induced bundle $w^{-1}(V_{n,2})$.

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d] & V_{n,2} \arrow[d, "p_1"] \\ P \arrow[r, "w"] & S^{n-1} \end{tikzcd}$$

Since P is contractible E' is a product. Let $s: P \to E'$ be any cross-section, and let $p_2: V_{n,2} \to S^{n-1}$ send a 2-frame into its second vector. Then the composition $u = p_2 f s$ has the desired property. q.e.d.

LEMMA 4.2. *Given y_0, $0 < y_0 \leqq 1/2$, there exists a real continuous differentiable function $\beta(y)$ defined on I such that* (1) $\beta(0) = \beta(1) = 0$, (2) $\beta'(0) = 0$, *and* (3) *for* $y_0 \leqq y \leqq 1$, $\beta'(y) \geqq 4$.

Proof. Consider the function:

$$r(y) = 0 \qquad 0 \leqq y \leqq y_0/2,$$

$$r(y) = -\frac{8}{y_0}(1 - y_0)y + r(1 - y_0) \qquad y_0/2 \leqq y \leqq y_0,$$

$$r(y) = 4y - 4 \qquad y_0 \leqq y \leqq 1.$$

Note that $r(y)$ could be taken as $\beta(y)$ except for the fact that it has corners at $y = y_0/2$ and $y = y_0$. By "rounding off the corners" of $r(y)$ the desired function can be obtained.

In order to prove 3.1 it is sufficient by Proposition 1.3 to show that $(\bar{p}^{-1}(U_0), \bar{p}, U_0)$ has the CHP where U_0 is a coordinate neighborhood of M. Since U_0 is homeomorphic to E^n we can identify the two spaces under this homeomorphism. Thus $U_0 = E^n \subset M$. $T_0(U_0)$ is a product space $E^n \times E^n$ where the first factor comes from the base point and the second from the direction and magnitude of a vector. We identify each of the two factors of $T_0(U_0)$ with a single E^n whose elements we consider as vectors. If S^{n-1} is the space of unit vectors of E^n, $T(U_0) = E^n \times S^{n-1}$. The magnitude of a vector v of E^n is written $|v|$.

For convenience the following new convention is used in this section and the next. The derivative of a regular curve at a point of U_0 will not carry the base point. That is, it is now the projection of the old derivative onto the second factor of $T_0(U_0)$. This is possible since most of the analysis in these sections is concerned with $T_0(U_0)$ and U_0.

By Proposition 1.1 it is enough to show that $(\bar{p}^{-1}(U_0), \bar{p}, U_0)$ has the covering homotopy property for cubes. Let $h_v: P \to U_0$ be a given homotopy with $\bar{h}: P \to \bar{p}^{-1}(U_0)$ covering h_0 where P is a cube. We will construct a covering homotopy $\bar{h}_v: P \to \bar{p}^{-1}(U_0)$.

Choose J with $0 \leqq J < 1$ such that for all $p \in P$ and $t \in [J, 1]$, $\bar{h}(p)(t) \in U_0$. Then choose J_0 with $J \leqq J_0 < 1$ so that for all $p \in P$ and $t \in [J_0, 1]$,

$$|\bar{h}'(p)(t) - \bar{h}'(p)(1)| < \frac{|\bar{h}'(p)(1)|}{10}.$$

The following choices are motivated by the need to insure the regularity of the covering homotopy curves we are constructing. Let

$$K = \max \{ |h_v(p) - h_0(p)| \mid v \in I, p \in P \}.$$

If $K = 0$, $y_0 = 1/2$. Otherwise let

$$y_0 = \min \left\{ \frac{1}{2}, \frac{|\bar{h}'(p)(1)|(1 - J_0)}{6K} \middle| p \in P \right\}.$$

The compactness of P yields that $y_0 > 0$.

Taking y_0 as above, let $\beta(y)$ be the function given by Lemma 4.2.

By taking $w(p) = \bar{h}'(p)(1)/|\bar{h}'(p)(1)|$, Lemma 4.1 yields a map $u: P \to S^{n-1}$

such that $u(p) \perp \bar{h}'(p)(1)$.

We define the desired covering homotopy $\bar{h}_v: P \to \bar{p}^{-1}(U_0)$ as follows. For $0 \leqq t \leqq J_0$ set $\bar{h}_v(p)(t) = \bar{h}(p)(t)$. For $J_0 \leqq t \leqq 1$ let $s = s(t) = (t - J_0)/(1 - J_0)$; then set

$$\bar{h}_v(p)(t) = \bar{h}(p)(t) + s^2[h_v(p) - h_0(p)] + \beta(s) \mid h_v(p) - h_0(p) \mid u(p).$$

Here $\bar{h}(p)$ is to be taken distinguished (see §2), but $\bar{h}_v(p)$, in general will not be. Note that for $t \geqq J_0$ all the terms used to define $\bar{h}_v(p)$ lie in U_0 and hence the additions make sense.

The following properties of $\bar{h}_v$ can be readily checked:

(1) $\bar{h}_v(p)(t)$ is continuous in v, p, and t.

(2) $\bar{h}_v(p)(1) = h_v(p)$.

(3) $\bar{h}_0(p) = \bar{h}(p)$.

The derivative of $\bar{h}_v(p)(t)$ for $t \geqq J_0$ can be computed to be:

$$\bar{h}_v'(p)(t) = \bar{h}'(p)(t) + s'2s[h_v(p) - h_0(p)] + s'\beta'(s) \mid h_v(p) - h_0(p) \mid u(p).$$

Then it can be seen:

(4) $\bar{h}_v(p)$ is differentiable.

(5) $\bar{h}_v'(p)(0)/|\bar{h}_v'(p)(0)| = x_0$. The derivative meant here is in the sense of §2.

The following requires proof:

(6) $\bar{h}_v(p)$ is a regular curve.

For this it is sufficient to show that $\bar{h}_v'(p)(t) \neq 0$ for $t \geqq J_0$. For such t we can write $\bar{h}_v'(p)(t) = A_1 + A_2$ where

$$A_1 = \bar{h}'(p)(t) + s'\beta'(s) \mid h_v(p) - h_0(p) \mid u(p)$$

and

$$A_2 = 2s's[h_v(p) - h_0(p)].$$

We will divide the proof into two parts.

Case I. $s \leqq y_0$: We claim $|A_1| \geqq (9/10)|\bar{h}'(p)(1)|$.

For a certain number Δ,

$$A_1 = \bar{h}'(p)(1) + \Delta u(p) - (\bar{h}'(p)(1) - \bar{h}'(p)(t))$$

and then by the triangle inequality

$$|A_1| \geqq |\bar{h}'(p)(1) + \Delta u(p)| - |\bar{h}'(p)(1) - \bar{h}'(p)(t)|.$$

By the choice of J_0 we obtain

$$|A_1| \geqq |\bar{h}'(p)(1) + u(p)| - \frac{1}{10} |\bar{h}'(p)(1)|.$$

Finally, since $u(p) \perp \bar{h}'(p)(1)$ we have $|A_1| \geqq (9/10)|\bar{h}'(p)(1)|$ as claimed.

On the other hand, by the choice of y_0 (since $s \leqq y_0$)

$$|A_2| \leqq s'(1/3)|\bar{h}'(p)(1)|(1-J_0) = (1/3)|\bar{h}'(p)(1)|.$$

From the triangle inequality it follows that $\bar{h}_v'(p)(t) \neq 0$.

CASE II. $s \geqq y_0$: We use a lemma.

LEMMA 4.3. *Let a, b, and c be vectors in E^n such that $|b| < (1/10)|a|$ and $c \perp a$. Let v be a scalar, $v > 4$. Then the inequality $|a+b+vc| > |3c|$ holds.*

Proof. Since

$$|a+b+vc| \geqq |a+vc| - |b|$$

it is sufficient to show

$$|a+vc|^2 \geqq (|3c| + |b|)^2$$

or using the fact that $c \perp a$

$$|a|^2 + |vc|^2 \geqq 9|c|^2 + 6|b||c| + |b|^2.$$

Since $v \geqq 4$ it is sufficient to show

$$|a|^2 \geqq -7|c|^2 + 6|b|c| + |b|^2.$$

This is easily checked considering separately the two cases

$$|c| \leqq |b| \quad \text{and} \quad |c| > |b|.$$

It follows from 4.3 that

$$|A_1| \geqq 3s'|h_v(p) - h_0(p)| \quad \text{taking } \bar{h}'(p)(1) = a,$$

$\bar{h}'(p)(1) - \bar{h}'(p)(t) = b$, $s'|h_v(p) - h_0(p)|u = c$, and $\beta'(s) = v$. By the choice of J_0, $|b| < 1/10|a|$ and since $s \geqq y_0$, $v \geqq 4$.

On the other hand $|A_2| \leqq 2s'|h_v(p) - h_0(p)|$. Then by the triangle inequality $\bar{h}_v'(p)(t) \neq 0$. This finishes the proof of (6).

Properties (1), (4), (5) and (6) imply that $\bar{h}_v(p)$ is really an element of E, (2) says that $\bar{h}_v$ covers h_v and (3) that $\bar{h}_v$ is a homotopy of $\bar{h}$. Therefore we have proved 3.1.

5. **Proof of Proposition** 3.2. Let U_0 be a coordinate neighborhood on M, and let $V = T(U_0)$. By the argument used to prove Proposition 1.3, it is sufficient to prove 3.2 for the case where $\{f_v(p) | v \in I, p \in P\} \subset V$. The notation and conventions of the last section will be continued.

Let $\pi_2: T_0(U_0) = E^n \times E^n \to E^n$ be the projection onto the second factor. Then $\pi_2(T(U_0)) \subset S^{n-1}$. The angle between two vectors of S^{n-1} is a continuous function of the vectors. This fact, together with the compactness of P, justifies the following choice. Pick $\epsilon > 0$ such that for all $p \in P$ and $|v - v'| \leq \epsilon$, the angle (measured in radians) between $\pi_2 f_v(p)$ and $\pi_2 f_{v'}(p)$ is less than $1/10$.

For $v \leq \epsilon$ let $\alpha_v(p)$ be the oriented angle from $\pi_2 f_0(p)$ to $\pi_2 f_v(p)$. By our choice of ϵ, $\alpha_v(p) < 1/10$.

For $v \leqq \epsilon$, and $t \in I$, we will define $Q_v(p, t)$ to be the following rotation of

E^n; that is, we are defining a map $Q_v: P\times I\to R_n$ where R_n is the rotation group of E^n. If $\pi_2 f_0(p)=\pi_2 f_v(p)$ let $Q_v(p, t)=e$, the identity rotation. Otherwise let $Q_v(p, t)$ rotate V, the unique plane determined by $\pi_2 f_0(p)$ and $\pi_2 f_v(p)$, through the angle $t\alpha_v(p)$ and leave $V^\perp$, the orthogonal complement of V, fixed.

That $Q_v(p, t)$ is continuous in v, p, and t and has a continuous first derivative in t, $Q_{v'}(p, t)$, is easily seen. Later, in fact, we will have occasion to compute this derivative.

Choose J, $0\leqq J<1$, such that

$$\{\bar{f}(p)(t)\mid p\in P, t\in [J, 1]\}\subset U_0.$$

LEMMA 5.1. *There exists a J_0 with $J\leqq J_0<1$ such that for all $p\in P$ and $t\in[J_0, 1]$,*

$$\left|\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{J_0-1}\right|\leqq\frac{4}{3}\,|\bar{f}'(p)(1)|\,.$$

Proof. It follows from the definition that

$$\bar{f}'(p)(1)=\lim_{t\to 1}\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{t-1}$$

so by the compactness of P there exists a J_0 with $J\leqq J_0<1$ such that for $t\in[J_0, 1]$,

$$\left|\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{t-1}-\bar{f}'(p)(1)\right|\leqq\frac{1}{3}\,|\bar{f}'(p)(1)|\,.$$

Then by the triangle inequality

$$\left|\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{t-1}\right|\leqq\frac{4}{3}\,|\bar{f}'(p)(1)|\,.$$

Also clearly for $J_0\leqq t\leqq 1$,

$$\left|\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{J_0-1}\right|\leqq\left|\frac{\bar{f}(p)(t)-\bar{f}(p)(1)}{t-1}\right|.$$

These last two inequalities yield the lemma.

Choose J_0 by 5.1 and such that also

$$|\bar{f}'(p)(1)-\bar{f}'(p)(t)|\;<(1/10)\min\{|f'(p)(1)|\mid p\in P\}$$

holds for all $p\in P$ and $t\in[J_0, 1]$. Then for $v\leqq\epsilon$, $\bar{f}_v$ is defined as follows.

For $0\leqq t\leqq J_0$ set $\bar{f}_v(p)(t)=\bar{f}(p)(t)$.

For $J_0\leqq t\leqq 1$ let $s=s(t)=(t-J_0)/(1-J_0)$; then set

$$\bar{f}_v(p)(t)=[\bar{f}(p)(t)-\bar{f}(p)(1)]Q_v(p, s)[e+(s^2-s)Q_v'(p, 0)]+\bar{f}(p)(1).$$

Here $Q_v(p, s)$, $Q_v'(p, 0)$ and e are to be considered as transformations acting on the right. The curve $\bar{f}(p)$ is taken distinguished, but in general, $\bar{f}_v(p)$ will not be.

The following properties of the covering homotopy can be quickly checked.

(1) $\bar{f}_v(p)(t)$ is continuous in v, p, and t.

(2) $\bar{f}_0(p)=\bar{f}(p)$.

(3) $\bar{f}_v(p)$ is stationary on a subpolyhedron A of P if $f_v(p)$ is.

The first derivative of $\bar{f}_v(p)$ can be computed as follows for $t \geqq J_0$:

$$\begin{aligned}\bar{f}_v'(p)(t) &= \bar{f}'(p)(t)Q_v(p, s)[e + (s^2 - s)Q_v'(p, 0)] \\ &\quad + s'[\bar{f}(p)(t) - \bar{f}(p)(1)]\{Q_v'(p, s)[e + (s^2 - s)Q_v'(p, 0)] \\ &\quad + Q_v(p, s)(2s - 1)Q_v'(p, 0)\}.\end{aligned}$$

Using this it can be further checked that:

(4) $\bar{f}_v(p)(t)$ is differentiable.

(5) $\bar{f}_v'(p)(0)/|\bar{f}_v'(p)(0)| = x_0$ in the sense of §2.

(6) $\bar{f}_v'(p)(1)/|\bar{f}_v'(p)(1)| = f_v(p)$ again in the sense of §2.

The following requires proof:

(7) The curve $\bar{f}_v(p)(t)$ is regular.

For the proof of (7) we will use:

Lemma 5.2. *Let* $0 \leqq t \leqq 1$. *Then* (a) *the transformation* $Q_v'(p, t)$ *reduces the magnitude of a vector to less than* 1/10 *of its original magnitude and* (b) *the transformation* $e+(t^2-t)Q_v'(p, 0)$ *does not change the magnitude of a vector by a factor of more than* 1/10.

Proof. For given p and v let coordinates $x_1, \cdots, x^n$ of E^n be chosen so that V (from the definition $Q_v(p, t)$) is the x_1-x_2 plane and the direction of $\pi_2 f_0(p)$ coincides with the x_2 axis. Then with this system suitably oriented $Q_v(p, t)$ can be represented in the matrix form,

$$\begin{pmatrix} \cos[t\alpha_v(p)] & -\sin[t\alpha_v(p)] & 0 \cdots & 0 \\ \sin[t\alpha_v(p)] & \cos[t\alpha_v(p)] & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 1 \end{pmatrix}.$$

Then $Q_v'(p, t)$ will be of the form,

$$-\alpha v(p)\begin{pmatrix} \sin[t\alpha_v(p)] & \cos[t\alpha_v(p)] & 0 \cdots & 0 \\ -\cos[t\alpha_v(p)] & \sin[t\alpha_v(p)] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 0 \end{pmatrix}.$$

If $\beta_1, \beta_2, \cdots, \beta_n$ are the components of a vector β in the above system then

$$|\beta Q_v'(p, t)| = |\alpha_v(p)| [(\beta_1)^2 + (\beta_2)^2]^{1/2} \leqq |\alpha_v(p)| |\beta|.$$

This yields (a) since $|\alpha_v(p)| < 1/10$. (b) follows from (a) and 5.2 is proved.

To prove (7) it is clearly sufficient to show that $\bar{f}_v'(p)(t) \neq 0$ for $t \geqq J_0$.

Let $\bar{f}_v'(p)(t) = A_1 + A_2$ where

$$A_1 = \bar{f}'(p)(t)Q_v(p, s)[e + (s^2 - s)Q_v'(p, p)]$$

and

$$A_2 = \left[\frac{\bar{f}(p)(t) - \bar{f}(p)(1)}{J_0 - 1}\right]\{Q_v'(p, s)[e + (s^2 - s)Q_v'(p, 0)] + Q_v(p, s)(2s - 1)Q_v'(p, 0)\}.$$

From Lemma 5.2(b) and the fact that $Q_v(p, s)$ does not change the magnitude of a vector, it follows that $|A_1| \geqq (9/10)|\bar{f}'(p)(t)|$. Then by the choice of J_0, $|A_1| \geqq (8/10)|\bar{f}'(p)(1)|$.

On the other hand, by Lemma 5.2 one easily obtains

$$|A_2| \leqq 3/10 \left|\frac{\bar{f}(p)(t) - \bar{f}(p)(1)}{J_0 - 1}\right|.$$

Then from the choice of J_0 (see 5.1) it follows that $|A_2| \leqq (4/10)|\bar{f}'(p)(1)|$. By the triangle inequality the inequalities on $|A_1|$ and $|A_2|$ yield $\bar{f}_v'(p)(t) \neq 0$ and hence (7).

Properties (1), (4), (5) and (7) imply that $\bar{f}_v(p)$ really belongs to E, (2) says that $\bar{f}_v$ is a homotopy of $\bar{f}$ and (6) that $\bar{f}_v$ covers f_v. Lastly, (3) is the stationary property demanded by 3.2. Thus $\bar{f}_v$ is a satisfactory covering homotopy for $v \leqq \epsilon$.

The above construction may be repeated if $\epsilon < 1$ using $\bar{f}_\epsilon(p)$ instead of $\bar{f}(p)$ and using a new value for J_0, if necessary. This yields a covering homotopy for $v \leqq 2\epsilon$. Iteration yields a full $\bar{f}_v$ and the proposition is proved.

6. **On the topology of the fiber Γ.** We recall some definitions and theorems of [11]. Let X be a space and $x_0 \in X$. The *path space* of X written $E_{x_0}(X)$ or sometimes E_{x_0} is the space of all curves (or paths) on X which start at x_0, with the compact open topology. Define $p: E_{x_0} \rightarrow X$ by sending a path onto its endpoint, i.e., let $p(f) = f(1)$. The *loop space* of X at x_0, $p^{-1}(x_0)$ is denoted by $\Omega(X)$ or Ω. It is shown in [11, pp. 479–481] that (E_{x_0}, p, X) has the CHP and that E_{x_0} is contractible.

Using the notation of the previous sections let M be a manifold, $T = T(M)$, $E = E(M)$ and $\Gamma = \Gamma(M) = \pi^{-1}(x_0)$. Define a map $\phi: E \rightarrow E_{x_0}(T)$ by $\phi(g)(t) = g'(t)/|g'(t)|$ for $g \in E$. It can be seen that ϕ is continuous as follows. Let E' be the set $E_{x_0}(T)$ endowed with the metric topology

$$d^*(f, g) = \max\{d[f(t), g(t)] \mid t \in I\}.$$

Then ϕ can be factored through E' by maps ϕ_1: $E \rightarrow E'$ and ϕ_2: $E' \rightarrow E_{x_0}$, where ϕ_2 is the identity. It is well known that the metric topology and the compact open topology are equivalent on E_{x_0}. See for example [4, p. 55]. Hence ϕ_2 is continuous. From the topology of E it follows that ϕ_1 is continuous. Thus ϕ is continuous. Let $\bar{\phi}$ be ϕ restricted to Γ. Then $\bar{\phi}(\Gamma) \subset \Omega$. This map is the same as $\bar{\phi}$ of the Introduction.

A map between two spaces is a *weak homotopy equivalence* [after 10, p. 299] if it induces an isomorphism of the homotopy groups of the spaces. The following theorem is well-known. A proof can be found in [14, p. 113].

THEOREM 6.1. *A weak homotopy equivalence induces isomorphisms of the singular homology groups of the spaces involved.*

Theorem C and 6.1 yield that $\bar{\phi}$ induces isomorphisms of the singular homology groups of Γ and Ω. This is of interest because a certain amount of attention has been given to the problem of determining the singular homology of loop spaces. For example see [11] and [15].

The proof of Theorem C requires the following lemma.

LEMMA 6.2. *If M is a manifold the space $E(M)$ is homotopically trivial.*

Proof. Consider first the case where M is Euclidean n-space E^n. Assume $\pi_1 x_0$ to be the origin of a coordinate system of E^n and let $\pi_2 x_0 = \bar{x}_0$ where π_2 is the projection of $T = E^n \times S^{n-1}$ onto S^{n-1}.

For some $k \geqq 0$ let f: $S^k \rightarrow E$ be given. To prove the lemma for E^n it is sufficient to show that f is homotopic to a constant. Since S^k is compact we can choose $J > 0$ close enough to 0 so that for all $p \in S^k$ and $t \in [0, J]$,

$$|f'(p)(t) - f'(p)(0)| < |f'(p)(0)|.$$

Then for $v \in [0, 1/2]$ let $f_v(p)(t) = f(p)(t - 2(1-J)vt)$. The curve $f(p)$ is to be distinguished, but $f_v(p)$ will not be, in general. This homotopy merely contracts $f(p)$ into a curve whose tangent is fairly close to a constant.

Define $e(t)$ as the fixed path of E given by $\bar{x}_0 t$. Then for $v \in [1/2, 1]$ define

$$f_v(p)(t) = (2 - 2v)f_{1/2}(p)(t) + (2v - 1)e(t).$$

where $f_{1/2}(p)(t)$ is the nondistinguished curve given by the previous homotopy.

It can be checked that $f_v(p)$ is really contained in E, that $f_0(p) = f(p)$ and that $f_1(p) = e$. It is the selection of J that yields the necessary regularity of $f_v(p)$ for $v \geqq 1/2$.

We have proved the lemma for $M = E^n$. The proof for a general M goes as follows. As before, let f: $S^k \rightarrow E$. Now for $v \leqq 1/2$ let $f_v(p)$ be a "shortening" of $f(p)$ so that for all $p \in S^k$, $f_{1/2}(p)$ lies in a certain coordinate neighborhood about x_0. For $v \in [1/2, 1]$ the homotopy is the same as the total homotopy for E^n. q.e.d.

Theorem C is proved as follows.

From the definition of ϕ it follows easily that ϕ commutes with the identity of T, i.e., $p\phi=\pi$ or

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"] & E_{x_0} \arrow[d, "p"] \\ T \arrow[r, "1"] & T \end{tikzcd}$$

commutes. Then, ϕ induces a homomorphism of the homotopy sequence of E into that of E_{x_0}. We have the following commutative diagram with the horizontal sequences exact.

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(\Gamma) \arrow[r] \arrow[d, "\bar{\phi}_\#"] & \pi_k(E) \arrow[r] \arrow[d, "\phi_\#"] & \pi_k(T) \arrow[r] \arrow[d, "1_\#"] & \cdots \\ \cdots \arrow[r] & \pi_k((T)) \arrow[r] & \pi_k(E_{x_0}) \arrow[r] & \pi_k(T) \arrow[r] & \cdots \end{tikzcd}$$

From $\pi_k(E)=\pi_k(E_{x_0})=0$ for all k (using 6.2), it follows that $\bar{\phi}_\#$ is an isomorphism for all k. This proves Theorem C.

7. **Classes of regular curves on a manifold.** Two regular curves on a manifold M are said to be *regularly homotopic* if they are homotopic and the homotopy $g_v\colon I\to M$ can be chosen such that for each $v\in I$, g_v is a regular curve, $g_v'(0)=g_0'(0)$, $g_v'(1)=g_0'(1)$, and $g_v'(t)$ depends continuously on v. A regular curve g on M will be called closed if $g'(0)=g'(1)$. It will be said to be at a point y_0 of T if $g'(0)/|g'(0)|=y_0$. Two closed regular curves on M are *freely regularly homotopic* if they are homotopic and the homotopy $g_v\colon I\to M$ can be chosen so that for each $v\in I$, g_v is a regular closed curve. Regular homotopy (free regular homotopy) is an equivalence relation and a *class* (*free class*) of regular curves on M will mean an equivalence class with respect to this relation.

M. Morse has investigated the behavior of locally simple sensed closed curves (or L-S-curves) under L-S-deformations. For definitions and discussion see [7; 8; 9]. In these articles he classified L-S-curves on closed 2-manifolds and E^2 into equivalence classes under L-S deformations. He has noted the similarity between this study and the classification of closed regular curves with free regular homotopies playing the role of L-S-deformations. The results of this section are parallel to Morse's.

From the definition of regular homotopy it follows that if M is a manifold two curves of $\Gamma(M)$ are regularly homotopic if and only if they lie in the same arcwise connected component of Γ, i.e., in the same element of $\pi_0(\Gamma)$. Using this fact Theorem A is the case $n=0$ of the following:

THEOREM 7.1. *If M is a manifold, there exists an isomorphism q from $\pi_n(\Gamma(M))$ to $\pi_{n+1}(T(M))$.*

Proof. Let $\Phi_\#: \pi_n(\Gamma(M)) \to \pi_n(\Omega(T))$ be the isomorphism of Theorem C, and $s: \pi_n(\Omega(T)) \to \pi_{n+1}(T)$ be the Hurewicz isomorphism [16, p. 210]. The composition of these two maps $q = s\Phi_\#$ gives the isomorphism demanded by the theorem. The rest of this section will be devoted to special cases of Theorem A.

THEOREM 7.2. *Let the dimension of a manifold M be greater than 2 and $x_0 \in T(M)$. Then two regular closed curves on M at x_0 are regularly homotopic if and only if they are homotopic with fixed end points.*

As we shall see, this theorem is far from true for 2-manifolds.

Proof of 7.2. The "only if" part is immediate from the definition of regular homotopy.

Consider the exact homotopy sequence of T. Then

$$\pi_1(S^{n-1}) \to \pi_1(T) \xrightarrow{\pi_{1\#}} \pi_1(M) \to \pi_0(S^{n-1})$$

is exact. Since $\pi_1(S^{n-1}) = \pi_0(S^{n-1}) = 0$ for $n > 2$, $\pi_{1\#}: \pi_1(T) \approx \pi_1(M)$. With this, Theorem A yields that $\pi_{1\#}q: \pi_0(\Gamma) \approx \pi_1(M)$. Moreover, from the definitions of π_1 and q we can consider $\pi_{1\#}q$ to be just the map induced by sending a curve into itself. Then 7.2 follows immediately from the definitions of $\pi_0(\Gamma)$ and $\pi_1(M)$.

Regular curve classes on some 2-manifolds will now be investigated.

(a) *The plane.* The following definition is due to Whitney [17]: If f is a regular closed curve in E^2, its rotation number $\gamma(f)$ is the total angle which $\pi_2 f'(t)$ turns as t traverses I. The function

$$f^*(t) = \pi_2 f'(t) / \left| \pi_2 f'(t) \right|$$

is a map of I into the unit circle. $\gamma(f)$ is 2π times the degree of this map.

THEOREM 7.3 (WHITNEY-GRAUSTEIN). *Two regular closed curves on the plane are freely regularly homotopic if and only if they have the same rotation number.*

Proof. Because a translation of a regular closed curve in the plane is a free regular homotopy and preserves its rotation number it is sufficient to consider curves of $\Gamma(E^2)$.

Consider the isomorphisms

$$\pi_0(\Gamma) \xrightarrow{q} \pi_1(T) \xrightarrow{\pi_{2\#}} \pi_1(S^1)$$

where π_2 is the projection of $T = E^2 \times S^1$ onto S^1. Let f be the element of Γ represented by $e^{2\pi i t}$ in complex coordinates such that the base point x_0 of Γ is the vector $2\pi i$ whose base point is the complex number 1. If $h \in \Gamma$, $\bar{h}$ will be the element of $\pi_0(\Gamma)$ containing h. Then $\pi_{2\#}q(\bar{f})$ will be a generator of

$\pi_1(S^1)$ say e. If g is any element of Γ, $\pi_{2\#}q(\bar{g})$ will be of the form me. From the definition of rotation number it follows that m is the rotation number of g. Since $\pi_{2\#}q$ is an isomorphism onto, this proves 7.3.

For the case of L-S-curves in the plane see [7].

(b) *The 2-sphere* S^2. It is known that $\pi_1(T(S^2))$ is cyclic of order 2; for example see [13]. Hence, by Theorem A we can put regular closed curves of S^2 at a point x_0 into two classes under regular homotopy equivalence. For the case of L-S-curves on S^2, see [8].

(c) *The torus* T^2. From the exact homotopy sequence of the tangent bundle $T(T^2)$, it can be deduced that $\pi_1(T(T^2))$ is $Z+Z+Z$ (Z is the infinite cyclic group). Then similar remarks to those of (b) apply.

(d) *The projective plane* P^2. It can be proved that $\pi_1(T(P^2))$ is cyclc of order four. Hence, there are four classes of regular closed curves at a point x_0 on P^2. For the case of L-S-curves see [9].

REMARK. By taking Γ as a fiber over a different point of T one can obtain results similar to those of Sections 6 and 7 for nonclosed curves.

8. **Regular curves perpendicular to a submanifold.** Let N be a regularly imbedded submanifold of a manifold M and let T be the unit tangent manifold of M. Let V be the normal bundle of N with respect to M; that is, V is the subspace of T which consists of all vectors which have their base points in N and are normal to N. Let x_0 be a point of V and Ω be the loop space of T at x_0. Denote by Ω_V the subspace of $E_{x_0}(T)$ which consists of the paths ending in V.

Let Γ_N be the subspace of $E(M)$ of curves whose final tangent is in V; i.e., $\Gamma_N=\pi^{-1}(V)$ where π is the map of Theorem B. Let π restricted to Γ_N be still denoted by π. Then we have:

THEOREM 8.1. *The triple* (Γ_N, π, V) *has the CHP.*

Proof. This theorem is an easy consequence of Theorem B. In fact, let the homotopy $h_v\colon P\to V$ be given with $\bar{h}\colon P\to\Gamma_N$ covering h_0. Theorem B yields a covering homotopy $\bar{h}_v\colon P\to E$. But since $\bar{h}_v$ covers h_v we have that $\bar{h}_v(p)\in\Gamma_N$ for all $v\in I$ and $p\in P$ and so $\bar{h}_v\colon P\to\Gamma_N$. This proves 8.1.

Similarly, (Ω_V, p, V) has the CHP. Let $\tilde{\phi}$ be the map ϕ of §6 restricted to Γ_N. Then

$$\begin{tikzcd} \Gamma_N \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"] & \Omega_V \arrow[d, "p"] \\ V \arrow[r, "1"] & V \end{tikzcd}$$

commutes so $\tilde{\phi}$ induces a homomorphism of the exact sequence of Γ_N into that of Ω_V. We have

$$\begin{array}{ccccccccc} \cdots \to & \pi_k(\Gamma) & \to & \pi_k(\Gamma_N) & \to & \pi_k(V) & \to \cdots \\ & \downarrow \bar{\phi}_\# & & \downarrow \tilde{\phi}_\# & & \downarrow 1_\# & \\ \cdots \to & \pi_k(\Omega) & \to & \pi_k(\Omega_V) & \to & \pi_k(V) & \to \cdots \end{array}$$

By Theorem C, $\bar{\phi}_\#$ is an isomorphism onto and, of course, $1_\#$ is the identity isomorphism. From this and the "Five" Lemma [2, p. 16], it follows that $\tilde{\phi}_\#$ is also an isomorphism onto. We have proved:

THEOREM 8.2. *The map $\tilde{\phi}$: $\Gamma_N \to \Omega_V$ defined above is a weak homotopy equivalence. Hence, by Theorem* 6.1 *it induces an isomorphism between the singular homology groups of* Γ_N *and* Ω_V.

9. **Integral curves of a 1-form.** Let $T(M)$ be the unit tangent bundle of a 2-manifold M and π: $T(M) \to M$ the projection. If $d\pi$ is the differential of π and $v \in M_m$, the tangent space of M at m, then $d\pi^{-1}_{(v,m)}(v)$ spans a two dimensional subspace containing the vertical. Hence there exists a 1-form $\omega_0 \neq 0$ on $T(M)$ annihilating this distribution of planes. If f is a regular curve on M then $\phi(f)$ (see the Introduction and Section 6) is an integral curve of ω_0. One might hope to get a characterization of regular curves this way. Unfortunately, however, ω_0 admits integral curves which are not the images under ϕ of regular curves. A curve lying in a single fiber of $T(M)$ is such an example. Thus it is not sufficient for the study of regular curves on a 2-manifold M to study integral curves of ω_0 on $T(M)$. However, there is still the question as to what can be said about integral curves of ω_0. This section was written as an attempt to answer this question.

Throughout the rest of §9 we will assume that M is a given manifold of dimension three. It seems very likely that the theory here generalizes to manifolds of higher dimension. However, because the treatment of 3-manifolds is so much simpler, we confine ourselves to this case.

A kind of curve essentially the same as the "stuckweise glatt" curves of [12] is considered here. A curve f on M is called a *parametrized piecewise regular curve* if there exist real numbers t_i for $i = 0, 1, \cdots, k$ with $t_0 = 0$, $t_k = 1$, and $t_i < t_{i+1}$ such that for each $i < k$, f restricted to $[t_i, t_{i+1}]$ is either constant or regular in the sense that $|f'(t)| \neq 0$ for $t \in [t_i, t_{i+1}]$. We say that such a curve is *distinguished* if its parameter is proportional to arc-length [12]. By changing the parameter one can associate to each parametrized piecewise regular curve a unique distinguished parametrized piecewise regular curve [see 12]. Two parametrized piecewise regular curves will be called *equivalent* if their associated distinguished curves are the same. A *piecewise regular curve* is an equivalence class of parametrized piecewise regular curves. Each such curve will have a unique distinguished representative. Oftentimes we will identify a piecewise regular curve with its distinguished representative.

As will be shown by an example at the end of the section, the theorems here hold only if some restriction is placed on the 1-forms.

Let q_0 be a fixed point of M, and let ω be a 1-form on M such that $\omega\wedge d\omega\neq 0$ on M. Denote by E_ω the space of all piecewise regular curves on M which start at q_0 and are integral curves of ω. Let E_ω be a metric space under the metric

$$d(f, g) = \max\left\{\bar{d}(f(t), g(t)) \mid t \in I\right\}$$

where $\bar{d}$ is any fixed metric on M. Define a map $p\colon E_\omega\to M$ by $p(g)=g(1)$. Let $p^{-1}(q_0)$ be denoted by Ω_ω.

THEOREM 9.1. *The triple* (E_ω, p, M) *has the CHP.*

Roughly speaking, Theorem 9.1 is proved as follows. First, by a classical theorem, there are local coordinates (x, y, z) about a point of M such that in them ω assumes an especially simple form. Here the fact that $\omega\wedge d_\omega\neq 0$ is used. By Proposition 1.3 we reduce the proof, in a sense, to this local situation. Then the local coordinates thus obtained are used to write down explicitly the desired covering homotopy equations.

We break the definition of the covering homotopy curves into four parts according to values of the parameter t. In general, this curve will turn out to have a corner at $t=1/4$, $t=1/2$, and $t=3/4$. The first part of the constructed curve is merely a reparametrization of the given covering curve. Then the construction is such that at $t=1/2$, the z coordinate of the covering homotopy has moved to a position over the z-coordinate of the given homotopy. At $t=3/4$ the x-coordinate has undergone a similar motion, and finally, at $t=1$, the y-coordinate of the covering homotopy projects into the y-coordinate of the given homotopy in M.

Proof of 9.1. Let $q\in M$. Take a coordinate neighborhood U of q with coordinates (x, y, z). In U we can write $\omega=Pdx+Qdy+Rdz$ where P, Q, and R are differentiable functions of x, y and z. Then

$$\omega \wedge d\omega = (PP' + QQ' + RR')dx \wedge dy \wedge dz$$

in U where

$$P' = R_y - Q_z, \qquad Q' = P_z - R_x, \qquad R' = Q_x - P_y.$$

Hence $PP'+QQ'+RR'\neq 0$ in U since $\omega\wedge d\omega\neq 0$. Then by a classical result of the theory of differential equations (see for example [3, p. 58]) there exist differentiable functions u, v and w of x, y, and z defined in a neighborhood of q such that $\omega=du+vdw$. Furthermore $0\neq\omega\wedge d\omega=du\wedge dv\wedge dw$ so that u, v and w form a coordinate system in a neighborhood say V of q. By 1.3 it is sufficient to prove the CHP for the triple $(p^{-1}(V), p, V)$. For convenience we will change the coordinates u, v, and w into x, y, and z respectively. So now x, y, and z are coordinates of V such that $\omega=dx+ydz$.

Let $h_v\colon P\to V$ be a given homotopy with $\bar{h}\colon P\to p^{-1}(V)$ covering h_0. We will construct a covering homotopy $\bar{h}_v\colon P\to p^{-1}(V)$. To describe these maps in the coordinate system (x, y, z) we use the following notation:

$$h_v(p) = (x_v(p), y_v(p), z_v(p)),$$
$$\bar{h}_v(p)(t) = (\bar{x}_v(p)(t), \bar{y}_v(p)(t), \bar{z}_v(p)(t)).$$

Then: for $0 \leqq t \leqq 1/4$ let

$$\bar{h}_v(p)(t) = \bar{h}(p)(4t)$$

for $1/4 \leqq t \leqq 1/2$ let $s = s(t) = 4t - 1$ and let

$$\bar{x}_v(p)(t) = x_0(p) - [z_v(p) - z_0(p)]y_0(p)s,$$
$$\bar{y}_v(p)(t) = y_0(p),$$
$$\bar{z}_v(p)(t) = z_0(p) + [z_v(p) - z_0(p)]s$$

for $1/2 \leqq t \leqq 3/4$ let $s = s(t) = 4t - 2$. Then let

$$\tilde{x}_v(p) = x_0(p) - y_0(p)[z_v(p) - z_0(p)],$$
$$\bar{x}_v(p)(t) = (36s^5 - 45s^4 - 20s^3 + 30s^2)\lfloor x_v(p) - \tilde{x}_v(p)\rfloor$$
$$+ 60\,|\,x_v(p) - \tilde{x}_v(p)\,|^{1/2} y_0(p)(s^3 - s) + \tilde{x}_v(p),$$
$$\bar{y}_v(p)(t) = \mathrm{Sg}\;|\,x_v(p) - \tilde{x}_v(p)\,|^{1/2}(s^2 - s) + y_0(p),$$
$$\mathrm{Sg} = \text{sign of } x_v(p) - \tilde{x}_v(p),$$
$$\bar{z}_v(p)(t) = -\,60\,|\,x_v(p) - \tilde{x}_v(p)\,|^{1/2}(s^3 - s) + z_v(p)$$

for $3/4 \leqq t \leqq 1$ let $s = s(t) = 4t - 3$ and

$$\bar{x}_v(p)(t) = x_v(p),$$
$$\bar{y}_v(p)(t) = [y_v(p) - y_0(p)]s + y_0(p),$$
$$\bar{z}_v(p)(t) = z_v(p).$$

In order to be sure that these equations define a satisfactory covering homotopy it must be checked that (1) $\bar{h}_v(p)$ is a curve in E_ω (for each p and v), (2) $\bar{h}_v(p)$ is a homotopy of $\bar{h}(p)$ or $\bar{h}_0(p) = \bar{h}(p)$, and (3) $\bar{h}_V(p)$ covers $h_V(p)$ or $\bar{h}_v(p)(1) = h_v(p)$.

We will check (1) first. It is easy to note that $\bar{h}_v(p)(0) = q_0$. Also, $\bar{h}_v(p)(t)$ is clearly continuous and piecewise regular between the values $t = 0$, $1/4$, $1/2$, $3/4$, and 1. It is necessary to check that $\bar{h}_v(p)(t)$ is well-defined at $t = 1/4$, $1/2$, and $3/4$ since at each of these values $\bar{h}_v(p)(t)$ is defined in two different ways. By substituting these values of t into the appropriate equations it can be seen that where the definitions overlap they agree. To complete the proof of (1) it needs to be shown that $\bar{h}_v(p)(t)$ satisfies $\omega = 0$ or, in other words,

$$\frac{d\bar{x}_v(p)(t)}{dt} + \bar{y}_v(p)(t)\,\frac{d\bar{z}_v(p)(t) = 0}{dt} = 0$$

identically for all v, p, and t. This is trivial for $t \leqq 1/4$. For $1/4 \leqq t \leqq 1$ to prove that this differential equation is satisfied it is sufficient to make three computations, one for $1/4 \leqq t \leqq 1/2$, one for $1/2 \leqq t \leqq 3/4$ and one for $3/4 \leqq t \leqq 1$. These are not difficult and will be left for the reader.

To show (2) we set $v=0$, getting

$$\begin{aligned} \bar{h}_0(p)(t) &= \bar{h}(p)(4t) && 0 \leqq t \leqq 1/4, \\ \bar{x}_0(p)(t) &= x_0(p), && \\ \bar{y}_0(p)(t) &= y_0(p) && 1/4 \leqq t \leqq 1, \\ \bar{z}_0(p)(t) &= z_0(p). && \end{aligned}$$

This is a parametrized piecewise regular curve whose associated distinguished curve is exactly the given curve $\bar{h}(p)(t)$.

It is trivially checked that (3) holds. q.e.d.

The space E_ω is contractible to a point. The deformation accomplishing this is $D: E_\omega \times I \to E_\omega$ defined by $D(f, v) = f(vt)$. Define a map $i: E_\omega \to E_{q_0}(M)$ (see §6) by letting $i(f)$ be the distinguished representative of f. Then i is continuous by the argument used to show that ϕ was continuous in §6. Let $\bar{\imath}$ be i restricted to Ω_ω where $\Omega_\omega = p^{-1}(q_0) \subset E_\omega$. The argument used in §6 to show that $\bar{\Phi}: \Gamma \to \Omega(T)$ was a weak homotopy equivalence may now be used to show that $\bar{\imath}: \Omega_\omega \to \Omega(M)$ is also a weak homotopy equivalence. This proves Theorem D.

The following example shows how Theorems 9.1 and D fail for a form ω_0 which is completely integrable. Let $M = E^3$ and $\omega_0 = xdx + ydy + zdz$ in a given Cartesian coordinate system (x, y, z) of E^3. Take for q_0 any point at distance $d > 0$ from the origin of E^3. Then any integral curve of ω_0 starting at q_0 stays on the surface of the 2-sphere $x^2 + y^2 + z^2 = d^2$. Clearly, the conclusions of Theorems 9.1 and D fail in this case. Actually, Theorem 9.1 is false for any 1-form which is completely integrable at a certain point $q_0 \in E^3$. For then short curves at q_0 must lie on a surface of E^3 and the covering homotopy property cannot possibly hold.

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