MEAN CONVERGENCE OF MARTINGALES(1)

BY

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1. **Introduction.** Let $(\Omega, \mathfrak{B}, \mu)$ denote a measure space consisting of a set Ω, a σ-algebra $\mathfrak{B}$ of subsets of Ω, and a measure μ defined on $\mathfrak{B}$ such that $\mu(\Omega)=1$. If $\mathfrak{B}_0$ is a σ-subalgebra of $\mathfrak{B}$, $L^p[\mathfrak{B}_0]$, $1 \leqq p < \infty$, will denote the totality of functions defined on Ω which are measurable relative to $\mathfrak{B}_0$ and pth-power summable with respect to μ. The norm of an element $x \in L^p[\mathfrak{B}]$ will be denoted by

$$\|x\|_p = \left(\int_\Omega |x|^p d\mu\right)^{1/p} < \infty.$$

In addition, $L^\infty[\mathfrak{B}_0]$ will denote the space of essentially bounded functions which are measurable relative to $\mathfrak{B}_0$. The norm of such a function will be denoted by $\|x\|_\infty = \inf\{k \mid |x| \leqq k \text{ a.e.}\} < \infty$. The conjugate space of $L^p[\mathfrak{B}_0]$, $1 \leqq p < \infty$, will be denoted by $L^q[\mathfrak{B}_0]$ where $1/p+1/q=1$ if $p>1$ and $q=\infty$ if $p=1$. If $x \in L^1[\mathfrak{B}]$ and $\mathfrak{B}_0$ is a σ-subalgebra of $\mathfrak{B}$, the conditional expectation of x relative to $\mathfrak{B}_0$ will be denoted by $E[x \mid \mathfrak{B}_0]$. The reader is referred to [1] for properties of the conditional expectation. The definition of a directed set will also be needed. A relation "$>$" defined for some pairs of a set A is said to direct A if the following three properties are satisfied:

(i) $a > a$ for all $a \in A$;

(ii) if $a > b$ and $b > c$, then $a > c$; and

(iii) if $a, b \in A$, then there exists a $c \in A$ such that $c > a$ and $c > b$.

A directed set is then a pair $(A, >)$ consisting of a set A and a relation "$>$" which directs A. A fixed directed set $(A, >)$ will be considered throughout this paper. It will be convenient to embed the directed set $(A, >)$ in a directed set $(A_\infty, >)$ which possesses a "last element"; that is, an ideal element "∞" is adjoined to A and the domain of the relation "$>$" is extended so that $\infty > a$ for all $a \in A_\infty = A \cup \{\infty\}$. A collection $\{x_a, a \in A\} \subset L^p[\mathfrak{B}]$ indexed by the set A will be called a net in $L^p[\mathfrak{B}]$. If $(B, \gg)$ is a directed set and there is a function g defined on B with values in A such that for each $a' \in A$ there is a $b' \in B$ such that $g(b) > a'$ whenever $b \gg b'$, then the net $\{x_{g(b)}, b \in B\}$ is called a subnet of the net $\{x_a, a \in A\}$. The reader is referred to [2] and [3] for properties of nets.

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A net $\{x_a, a\in A\}\subset L^1[\mathfrak{B}]$ is said to be a martingale if there is a net $\{\mathfrak{B}_a, a\in A\}$ of σ-subalgebras of $\mathfrak{B}$ such that

(i) $\mathfrak{B}_a\subset\mathfrak{B}_b$ whenever $a<b$;

(ii) x_a is measurable relative to $\mathfrak{B}_a$ or is equal a.e. to a function which is; and

(iii) $x_a=E[x_b|\mathfrak{B}_a]$ a.e. whenever $a<b$.

Following Doob, such a martingale will be denoted by $\{x_a, \mathfrak{B}_a, a\in A\}$ to emphasize the net of subalgebras.

It is possible to give a slightly more general definition of a martingale as follows. Suppose a net $\{x_b, b\in B\}\subset L^1[\mathfrak{B}]$ has the property that for every finite chain $B_f\subset B$, the net $\{x_b, b\in B_f\}$ is a martingale in the sense of the preceding paragraph. Such a net could also be called a martingale. It can be shown that these two definitions are equivalent if the indexing set is simply-ordered. The two definitions are not equivalent in general, however. Suppose $B=\{1, 1', 2, 3\}$ and $1<2$, $1'<2$, $2<3$, but 1 and $1'$ are not comparable. It is not difficult to construct a net $\{x_b, b\in B\}$ such that

$$\begin{aligned} E[x_2\mid x_1] &= x_1 \text{ a.e.},\\ E[x_2\mid x_{1'}] &= x_{1'} \text{ a.e.},\\ E[x_3\mid x_1, x_2] &= x_2 \text{ a.e.},\\ E[x_3\mid x_{1'}, x_2] &= x_2 \text{ a.e.},\\ E[x_3\mid x_1, x_{1'}, x_2] &\neq x_2 \text{ a.e.} \end{aligned}$$

Such a net cannot be a martingale in the sense of the preceding paragraph. Since all known nontrivial examples of martingales satisfy the first definition, only martingales in the sense of the preceding paragraph will be considered. Most of the results of this paper are also true if the second definition is used [4].

It will be shown below that the following four properties of a martingale $\{x_a, a\in A\}\subset L^p[\mathfrak{B}]$, $1\leqq p<\infty$, are equivalent:

P_1: The net $\{x_a, a\in A\}$ is uniformly integrable if $p=1$ or is strongly bounded if $1<p<\infty$.

P_2: The net $\{x_a, a\in A\}$ converges weakly in $L^p[\mathfrak{B}]$.

P_3: There exists an $x_\infty\in L^p[\mathfrak{B}]$ such that the net $\{x_a, a\in A_\infty\}$ is a martingale.

P_4: The net $\{x_a, a\in A\}$ converges strongly in $L^p[\mathfrak{B}]$.

Doob has shown that properties P_1, P_3, and P_4 are equivalent for martingales indexed by the positive integers [1]. In this case, P_1 implies convergence a.e. of the martingale and P_4 is simply a necessary and sufficient condition for uniform integrability of an almost everywhere convergent sequence of functions. For more general directed sets, however, P_1 need not imply convergence a.e. Dieudonné has constructed a martingale indexed by the finite subsets of the positive integers (directed by set inclusion) which possesses property P_1

but which does not converge a.e. [5]. Dunford and Tamarkin have shown that P_3 implies P_4 in connection with the theory of integration of functions of infinitely many variables [6].

2. **Convergence properties.** It will be convenient to introduce some notation at this point. If $\mathfrak{B}_0 \subset \mathfrak{B}$, $L[\mathfrak{B}_0]$ will denote the totality of finite linear combinations of characteristic functions of elements of $\mathfrak{B}_0$ and $S(\mathfrak{B}_0)$ will denote the smallest σ-algebra containing $\mathfrak{B}_0$. The following two lemmas will be needed.

LEMMA 1. *Let $\{x_a, \mathfrak{B}_a, a \in A\}$ be a martingale in $L^1[\mathfrak{B}]$ such that the set $\{x_a, a \in A\}$ is uniformly integrable. If $y \in L[S(\bigcup_A \mathfrak{B}_a)]$, then to every $\epsilon > 0$ there corresponds an $a_\epsilon \in A$ such that*

$$\left| \int_\Omega y(x_b - x_a)d\mu \right| < \epsilon \qquad \textit{for all } b > a > a_\epsilon.$$

Proof. Consider any $S \in S(\bigcup_A \mathfrak{B}_a)$. Since the set $\{x_a, a \in A\}$ is uniformly integrable, to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\int_R |x_a| \, d\mu < \epsilon/2 \qquad \text{for all } a \in A \text{ whenever } \mu(R) < \delta. \tag{1}$$

Since $\mathfrak{B}_a \subset \mathfrak{B}_b$ whenever $a < b$, $\bigcup_A \mathfrak{B}_a$ is an algebra of sets. Consequently, there is an $S_\delta \in \bigcup_A \mathfrak{B}_a$ such that $\mu(S \Delta S_\delta) < \delta$ [7, p. 56]. The assertion follows for the characteristic function χ_S by making use of the inequality

$$\left| \int_S (x_b - x_a)d\mu \right| \leqq \left| \int_{S_\delta} (x_b - x_a)d\mu \right| + \int_{S\Delta S_\delta} |x_a| \, d\mu + \int_{S\Delta S_\delta} |x_b| d\mu,$$

the definition of a martingale, and (1). The extension to finite linear combinations is immediate.

LEMMA 2. *Let $\{x_a, \mathfrak{B}_a, a \in A\}$ be a martingale in $L^p[\mathfrak{B}]$, $1 < p < \infty$. If $y \in L[\bigcup_A \mathfrak{B}_a]$, then there is an $a_y \in A$ such that*

$$\int_\Omega y(x_b - x_a)d\mu = 0 \qquad \textit{for all } b > a > a_y.$$

Proof. Consider any $S_0 \in \bigcup_A \mathfrak{B}_a$. Then there is some $a_0 \in A$ such that $S_0 \in \mathfrak{B}_{a_0}$. Consider any pair $a, b \in A$ such that $b > a > a_0$. Then $S_0 \in \mathfrak{B}_{a_0} \subset \mathfrak{B}_a$. Since $x_a = E[x_b | \mathfrak{B}_a]$ a.e., it follows from the definition of the conditional expectation that

$$\int_\Omega \chi_{S_0}(x_b - x_a)d\mu = 0.$$

The extension to finite linear combinations is immediate.

It will now be shown that P_1 implies P_2. Most of the difficulties involved in proving this implication are resolved by the following lemma which has nothing to do with martingales.

LEMMA 3. *If* $\{x_a, a\in A\}$ *is a net in* $L^p[\mathfrak{B}]$, $1\leqq p<\infty$, *which is uniformly integrable if* $p=1$ *or is a strongly bounded subset of* $L^p[\mathfrak{B}]$ *if* $1<p<\infty$, *then the net has a weak cluster point in* $L^p[\mathfrak{B}]$.

Proof. Let $\mathfrak{X}=\{x_a, a\in A\}$. Consider first the case $p=1$. By hypothesis, $\mathfrak{X}$ is a uniformly integrable subset of $L^1[\mathfrak{B}]$. Dunford has shown that a subset of $L^1[\mathfrak{B}]$ is uniformly integrable, if and only if it is conditionally weakly compact in $L^1[\mathfrak{B}]$ [8]. Consequently, the net $\{x_a, a\in A\}$ possesses a weak cluster point in $L^1[\mathfrak{B}]$ [2]. Consider the case $p>1$. By hypothesis, there is a number k such that $\mathfrak{X}\subset\{x\in L^p[\mathfrak{B}]\mid \|x\|_p\leqq k\}$. It is well known that the latter set is weakly compact in $L^p[\mathfrak{B}]$. Therefore, the net $\{x_a, a\in A\}$ possesses a weak cluster point in $L^p[\mathfrak{B}]$ [2].

The following theorem shows that the weak cluster point of the preceding lemma is a weak limit for martingales.

THEOREM 4. *Let* $\{x_a, \mathfrak{B}_a, a\in A\}$ *be a martingale in* $L^p[\mathfrak{B}]$, $1\leqq p<\infty$. *If* $p=1$ *and the set* $\{x_a, a\in A\}$ *is uniformly integrable, then the net* $\{x_a, a\in A\}$ *converges weakly in* $L^1[\mathfrak{B}]$. *If* $1<p<\infty$ *and the set* $\{x_a, a\in A\}$ *is strongly bounded, then the net* $\{x_a, a\in A\}$ *converges weakly in* $L^p[\mathfrak{B}]$.

Proof. It must be shown that there exists an $x_\infty\in L^p[\mathfrak{B}]$ such that

$$\lim_A \int_\Omega x_a z d\mu = \int_\Omega x_\infty z d\mu \qquad \text{for all } z\in L^q[\mathfrak{B}]. \tag{2}$$

It will first be shown that there is an $x_\infty\in L^p[\mathfrak{B}]$ such that (2) is true for all $z\in L[S(\bigcup_A \mathfrak{B}_a)]$ or for all $z\in L[\bigcup_A \mathfrak{B}_a]$ according as $p=1$ or $p>1$. Under either hypothesis, the net $\{x_a, a\in A\}$ has a weak cluster point $x_\infty\in L^p[\mathfrak{B}]$ by Lemma 3. Consequently, there is a subnet $\{x_{g(b)}, b\in B\}$ which converges weakly in $L^p[\mathfrak{B}]$ to x_∞ [2]. Therefore,

$$\lim_B \int_\Omega x_{g(b)} z d\mu = \int_\Omega x_\infty z d\mu \tag{3}$$

for all $z\in L[S(\bigcup_A \mathfrak{B}_a)]$ or all $z\in L[\bigcup_A \mathfrak{B}_a]$ according as $p=1$ or $p>1$. Consider any z in either of the latter sets of functions. In either case, to every $\epsilon>0$ there corresponds an $a_\epsilon\in A$ such that

$$\left|\int_\Omega z(x_b - x_a)d\mu\right| < \epsilon/2 \qquad \text{for all } b > a > a_\epsilon \tag{4}$$

according to Lemma 1 if $p=1$ or Lemma 2 if $1<p<\infty$. From the definition of a subnet, there is a $b'_\epsilon\in B$ such that $g(b)>a_\epsilon$ whenever $b\gg b'_\epsilon$. By (3) there, is a $b_\epsilon\in B$ (it may be assumed that $b_\epsilon\gg b'_\epsilon$) such that

$$\left|\int_\Omega x_{g(b)} z d\mu - \int_\Omega x_\infty z d\mu\right| < \epsilon/2 \text{ whenever } b \gg b_\epsilon. \tag{5}$$

Consider any $a' \in A$ such that $a' > g(b_\epsilon) > a_\epsilon$. By (4) and (5),

$$\left|\int_\Omega x_{a'} z d\mu - \int_\Omega x_\infty z d\mu\right| \leqq \left|\int_\Omega (x_{a'} - x_{g(b_\epsilon)}) z d\mu\right| + \left|\int_\Omega (x_{g(b_\epsilon)} - x_\infty) z d\mu\right| < \epsilon.$$

Therefore,

$$\lim_A \int_\Omega x_a z d\mu = \int_\Omega x_\infty z d\mu$$

for all $z \in L[S(\bigcup_A \mathfrak{B}_a)]$ or for all $z \in L[\bigcup_A \mathfrak{B}_a]$ according as $p=1$ or $p>1$. It will now be shown that (3) implies (2). Since a uniformly integrable subset of $L^1[\mathfrak{B}]$ is strongly bounded in $L^1[\mathfrak{B}]$, $\sup_A \|x_a\|_p = k < \infty$ under either hypothesis of the theorem. The trivial case where $k=0$ will not be considered. Since the sets $L[S(\bigcup_A \mathfrak{B}_a)]$ and $L[\bigcup_A \mathfrak{B}_a]$ are dense in $L^\infty[S(\bigcup_A \mathfrak{B}_a)]$ and $L^q[S(\bigcup_A \mathfrak{B}_a)]$, respectively, there exists, in either case, a z_ϵ in the respective dense set such that $\|z - z_\epsilon\|_q < \epsilon/2(k + \|x_\infty\|_p)$. It was shown above that there is an $a_\epsilon \in A$ such that

$$\left|\int_\Omega z_\epsilon (x_a - x_\infty) d\mu\right| < \epsilon/2 \qquad \text{for all } a > a_\epsilon. \tag{6}$$

By Hölder's inequality and (6), for all $a > a_\epsilon$,

$$\begin{aligned}
&\left|\int_\Omega z x_a d\mu - \int_\Omega z x_\infty d\mu\right| \\
&\quad \leqq \left|\int_\Omega (z - z_\epsilon) x_a d\mu\right| + \left|\int_\Omega z_\epsilon (x_a - x_\infty) d\mu\right| + \left|\int_\Omega (z_\epsilon - z) x_\infty d\mu\right| \\
&\quad \leqq \|z - z_\epsilon\|_q (\|x_a\|_p + \|x_\infty\|_p) + \left|\int_\Omega z_\epsilon (x_a - x_\infty) d\mu\right| \\
&\quad < \|z - z_\epsilon\|_q (k + \|x_\infty\|_p) + \epsilon/2 \\
&\quad < \epsilon.
\end{aligned}$$

Hence,

$$\lim_A \int_\Omega x_a z d\mu = \int_\Omega x_\infty z d\mu \qquad \text{for all } z \in L^q\left[S\left(\bigcup_A \mathfrak{B}_a\right)\right].$$

Now consider any $z \in L^q[\mathfrak{B}]$. Since $E[z | S(\bigcup_A \mathfrak{B}_a)] \in L^q[S(\bigcup_A \mathfrak{B}_a)]$,

$$\lim_A \int_\Omega x_a E\left[z \,\middle|\, S\left(\bigcup_A \mathfrak{B}_a\right)\right] d\mu = \int_\Omega x_\infty E\left[z \,\middle|\, S\left(\bigcup_A \mathfrak{B}_a\right)\right] d\mu.$$

Since the functions x_a, $a\in A$, and x_∞ are measurable relative to $S(\bigcup_A \mathfrak{B}_a)$, it follows from well known properties of the conditional expectation that

$$\lim_A \int_\Omega E\left[x_a z \,\middle|\, S\left(\bigcup_A \mathfrak{B}_a\right)\right] d\mu = \int_\Omega E\left[x_\infty z \,\middle|\, S\left(\bigcup_A \mathfrak{B}_a\right)\right] d\mu.$$

Applying the definition of the conditional expectation to the integrals on each side of this equation,

$$\lim_A \int_\Omega x_a z d\mu = \int_\Omega x_\infty z d\mu$$

as was to be shown.

The following two theorems show that P_2 implies P_3 and P_3 implies P_4.

THEOREM 5. *Let $\{x_a, \mathfrak{B}_a, a\in A\}$ be a martingale in $L^p[\mathfrak{B}]$, $1\leqq p<\infty$. If the net $\{x_a, a\in A\}$ converges weakly in $L^p[\mathfrak{B}]$ to x_∞, then the net $\{x_a, a\in A_\infty\}$ is a martingale.*

Proof. Consider any $a'\in A$. It follows from the definition of a martingale that if $S\in\mathfrak{B}_{a'}$, then $\int_S x_a d\mu$ is constant for all $a>a'$. Since $\chi_S\in L^q[\mathfrak{B}]$,

$$\int_S x_{a'} d\mu = \lim_A \int_S x_a d\mu = \lim_A \int_\Omega \chi_S x_a d\mu = \int_\Omega \chi_S x_\infty d\mu = \int_S x_\infty d\mu.$$

This equality is trivial if $a'=\infty$ and $\mathfrak{B}_\infty=\mathfrak{B}$. Therefore, the net $\{x_a, \mathfrak{B}_a, a\in A_\infty\}$ is a martingale.

THEOREM 6. *If $\{x_a, \mathfrak{B}_a, a\in A_\infty\}$ is a martingale in $L^p[\mathfrak{B}]$, $1\leqq p<\infty$, then the net $\{x_a, a\in A\}$ converges strongly in $L^p[\mathfrak{B}]$ to $E[x_\infty | S(\bigcup_A \mathfrak{B}_a)]$.*

Proof. It is clear from the definition of the conditional expectation that x_∞ and $\mathfrak{B}_\infty$ can be replaced by $E[x_\infty | S(\bigcup_A \mathfrak{B}_a)]$ and $S(\bigcup_A \mathfrak{B}_a)$, respectively. It will be assumed that this replacement has been made; that is, x_∞ is measurable relative to $S(\bigcup_A \mathfrak{B}_a)$. Since $L[\bigcup_A \mathfrak{B}_a]$ is dense in $L^p[S(\bigcup_A \mathfrak{B}_a)]$, to every $\epsilon>0$ there corresponds an $x_\epsilon\in L[\bigcup_A \mathfrak{B}_a]$ such that $\|x_\infty - x_\epsilon\|_p<\epsilon/2$. By definition, x_ϵ is of the form

$$x_\epsilon = \sum_{i=1}^n \tau_i \chi_{S_i}$$

where $S_i\in\bigcup_A \mathfrak{B}_a$, $i=1,\cdots,n$, and where the τ_i's are real numbers. By hypothesis, $x_a=E[x_\infty|\mathfrak{B}_a]$ a.e. for every $a\in A$. Hence, for all $a\in A$,

$$\begin{aligned} \|x_\infty - x_a\|_p &= \|x_\infty - E[x_\infty | \mathfrak{B}_a]\|_p \\ &= \|(x_\infty - x_\epsilon) + x_\epsilon - E[(x_\infty - x_\epsilon) + x_\epsilon | \mathfrak{B}_a]\|_p \\ &\leqq \|x_\infty - x_\epsilon\|_p + \|x_\epsilon - E[x_\epsilon | \mathfrak{B}_a]\|_p + \|E[x_\infty - x_\epsilon | \mathfrak{B}_a]\|_p. \end{aligned} \tag{7}$$

Since $|E[x_\infty - x_\epsilon | \mathfrak{B}_a]|^p \leqq E[|x_\infty - x_\epsilon|^p | \mathfrak{B}_a]$ a.e.,

$$\|E[x_\infty - x_\epsilon | \mathfrak{B}_a]\|_p^p \leqq \int_\Omega E[|x_\infty - x_\epsilon|^p | \mathfrak{B}_a] d\mu = \int_\Omega |x_\infty - x_\epsilon|^p d\mu$$
$$< (\epsilon/2)^p.$$

Moreover, since each $S_i \in \bigcup_A \mathfrak{B}_a$, there is an $a_\epsilon \in A$ such that $S_i \in \mathfrak{B}_a$ for all $a > a_\epsilon$, $i = 1, \cdots, n$. For such a, $x_\epsilon = E[x_\epsilon | \mathfrak{B}_a]$ a.e. It follows from (7) that

$$\|x_\infty - x_a\|_p < \epsilon/2 + \|x_\epsilon - x_\epsilon\|_p + \epsilon/2 = \epsilon \qquad \text{for all } a > a_\epsilon.$$

Therefore, $\lim_A \|x_\infty - x_a\|_p = 0$.

Let $\{x_a, \mathfrak{B}_a, a \in A\}$ be a martingale in $L^p[\mathfrak{B}]$ for which there is an $x_\infty \in L^p[\mathfrak{B}]$ such that $\lim_A \|x_a - x_\infty\|_p = 0$. Since strong convergence implies weak convergence, the net $\{x_a, a \in A_\infty\}$ is a martingale according to Theorem 5. That P_4 implies P_1 follows from these remarks and the following theorem.

THEOREM 7. *If $\{x_a, \mathfrak{B}_a, a \in A_\infty\}$ is a martingale in $L^p[\mathfrak{B}]$, $1 \leqq p < \infty$, then the set $\{x_a, a \in A_\infty\}$ is strongly bounded in $L^p[\mathfrak{B}]$ and uniformly integrable.*

Proof. Since $|x_a|^p = |E[x_\infty | \mathfrak{B}_a]|^p \leqq E[|x_\infty|^p | \mathfrak{B}_a]$ a.e., $\|x_a\|_p \leqq \|x_\infty\|_p$ for all $a \in A_\infty$. If $p > 1$, the proof is complete since a strongly bounded subset of $L^p[\mathfrak{B}]$ is necessarily uniformly integrable. Consider the case $p = 1$. Let ξ be any positive number. Since the two functions $|x_a|$ and $|x_\infty|$, $a \in A_\infty$, constitute a semimartingale,

$$\int_{\{|x_a(\omega)| > \xi\}} |x_a| d\mu \leqq \int_{\{|x_a(\omega)| > \xi\}} |x_\infty| d\mu \qquad \text{for all } a \in A_\infty. \tag{8}$$

Moreover, since

$$\xi\mu\{|x_a(\omega)| > \xi\} \leqq \int_{\{|x_a(\omega)| > \xi\}} |x_a| d\mu \leqq \|x_a\|_1 \leqq \|x_\infty\|_1,$$

$\lim_{\xi\to\infty} \mu\{|x_a(\omega)| > \xi\} = 0$ uniformly for $a \in A_\infty$. Combining this result with (8),

$$\lim_{\xi\to\infty} \int_{\{|x_a(\omega)| > \xi\}} |x_a| d\mu \leqq \lim_{\xi\to\infty} \int_{\{|x_a(\omega)| > \xi\}} |x_\infty| d\mu = 0$$

uniformly for $a \in A_\infty$, which completes the proof.

3. **Applications.** Let A be the totality of partitions of the measure space $(\Omega, \mathfrak{B}, \mu)$. If $\pi_1, \pi_2 \in A$, define $\pi_1 > \pi_2$ to mean that π_1 is a refinement of π_2. With this ordering, $(A, >)$ is a directed set. If $\pi \in A$, let $\mathfrak{B}_\pi$ denote the σ-algebra consisting of the empty set and unions of elements in π. Clearly, $\mathfrak{B}_{\pi_1} \subset \mathfrak{B}_{\pi_2}$ whenever $\pi_2 > \pi_1$ and $S(\bigcup_{\pi \in A} \mathfrak{B}_\pi) = \mathfrak{B}$. Consider any $z \in L^p[\mathfrak{B}]$, $1 \leqq p < \infty$. If $\pi = \{S_1, S_2, \cdots\} \in A$, define

$$x_\pi(\omega) = \begin{cases} \dfrac{1}{\mu(S_j)} \displaystyle\int_{S_j} z d\mu & \text{for } \omega \in S_j \text{ if } \mu(S_j) > 0, \\ 0 & \text{for } \omega \in S_j \text{ if } \mu(S_j) = 0. \end{cases}$$

It is easily shown that the net $\{x_\pi, \mathfrak{B}_\pi, \pi \in A\}$ is a martingale. Such a martingale possesses property P_3, where $x_\infty = z$. According to Theorem 6, the net $\{x_\pi, \pi \in A\}$ converges strongly in $L^p[\mathfrak{B}]$ to z since $z = E[z | S(\bigcup_{\pi\in A} \mathfrak{B}_\pi)]$ a.e. This result may be termed a *mean differentiation theorem*. Since strong convergence implies weak convergence,

$$\lim_A \int_\Omega x_a y d\mu = \int_\Omega z y d\mu$$

for all $y \in L^q[\mathfrak{B}]$ (similiar weak differentiation theorem can be found in [9]).

As another application, let μ^* denote the induced outer measure defined on the totality, $\mathfrak{F}$, of subsets of Ω. Consider any $Q \in \mathfrak{F}$ and any partition $\pi = \{S_1, S_2, \cdots\} \in A$. Define

$$x_\pi(\omega) = \begin{cases} \dfrac{\mu^*(Q \cap S_i)}{\mu(S_i)} & \text{if } \omega \in S_i \text{ and } \mu(S_i) > 0, \\ 0 & \text{if } \omega \in S_i \text{ and } \mu(S_i) = 0. \end{cases}$$

Let $\bar{Q}$ be a measure cover for the set Q. Note that $\mu^*(Q \cap S) = \mu(\bar{Q} \cap S)$ for all $S \in \mathfrak{B}$. Therefore, $\mu^*(Q \cap S_i)$ may be replaced by $\int_{S_i} \chi_{\bar{Q}} d\mu$ in the definition of the net $\{x_\pi, \pi \in A\}$. It follows from the above mean differentiation theorem that the net $\{x_\pi, \pi \in A\}$ converges strongly to the characteristic function of the measure cover $\bar{Q}$ (a similiar mean density theorem valid for topological groups can be found in [7, p. 268]). A similar result holds if μ^* and $\bar{Q}$ are replaced by the induced inner measure μ_* and a measure kernel Q'.

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