

LATTICES WITH INVOLUTION⁽¹⁾

BY
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Introduction. By a "lattice with involution," or "*i*-lattice," we shall mean a lattice L together with an involution [1, p. 4] $x \rightarrow x'$ in L . A distributive *i*-lattice in which $x \cap x' \leq y \cup y'$ for all x and y will be called a "normal" *i*-lattice. The underlying lattice of an *l*-group becomes a normal *i*-lattice when x' is defined as the group inverse of x ; also a Boolean algebra becomes a normal *i*-lattice when x' is defined as the complement of x . In this paper *l*-groups and Boolean algebras will always be understood to have the involutions defined above. §1 of the paper contains subdirect decomposition theorems for distributive and normal *i*-lattices, with applications; in §2, as a contribution to the study of nondistributive *i*-lattices, modular and nonmodular *i*-lattices are classified with respect to certain laws each of which, for distributive *i*-lattices, is equivalent to normality; and §§3 and 4 contain some extension and embedding theorems concerning normal *i*-lattices.

1. **Subdirect decomposition of distributive and normal *i*-lattices.** Every *i*-lattice is an algebra [1, p. vii] with operations $\cap, '$ which satisfy the identities $x \cap y = y \cap x$, $x \cap (y \cap z) = (x \cap y) \cap z$, $x' = x' \cap (x \cap y)'$, and $x'' = x$; it may be proved that these identities are independent postulates for *i*-lattices. We shall apply the usual terminology of abstract algebra (cf. [1, pp. viif.]) to *i*-lattices, except that we shall use the terms "*i*-sublattice," "*i*-homomorphism," and "*i*-isomorphism" instead of "subalgebra," "homomorphism," and "isomorphism."

Let L be a distributive *i*-lattice. For elements x, y, p of L we shall set $x \equiv y(C(p))$ if and only if $x \cap p = y \cap p$ and $x' \cap p = y' \cap p$. It is easily verified that this defines a congruence relation $C(p)$ on L , and that

$$(1) \quad C(p) \cap C(q) = C(p \cup q) \text{ for all } p \text{ and } q \text{ in } L.$$

Using [1, p. 28, Lemma 1(ii)] we see that if O is the zero congruence relation on L then

$$(2) \quad C(p) \cap C(p') = O \text{ for all } p \text{ in } L.$$

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Also, since $p \equiv p \cup p'(C(p))$ for all p , $p \geq p'$ if $C(p) = 0$, while, if $p \geq p'$, then $C(p) = 0$ by (1) and (2): thus

$$(3) \quad C(p) = 0 \text{ if and only if } p \geq p'.$$

LEMMA 1. *Let L be a subdirectly irreducible distributive i -lattice. Then*

$$(4) \quad x \text{ is comparable with } x' \text{ for each } x \text{ in } L,$$

and, for elements x, y, z of L ,

$$(5) \quad \text{if } x > x' \text{ and } y > y' \text{ then } x \cap y > (x \cap y)', \text{ and}$$

$$(6) \quad \text{if } y > y' \text{ and } z \leq z' \text{ then } y > z.$$

Proof. Since L is subdirectly irreducible, (4) follows from (2) and (3). If $x > x'$ and $y > y'$ but $x \cap y \not> (x \cap y)'$ then $x \cap y \leq (x \cap y)'$, by (4), and hence $C(x' \cup y') = 0$, by (3); using (1), we deduce that $x' \geq x$ or $y' \geq y$, which contradicts the hypothesis. Suppose now that $y > y'$ and $z \leq z'$; if $(y \cap z')' > y \cap z'$ then, by (5), $y \cap (y' \cup z) > y' \cup (y \cap z') = y \cap (y' \cup z') \geq y \cap (y' \cup z)$, a contradiction; hence $(y \cap z')' \leq y \cap z'$, and it follows that $y \geq y \cap z' \geq y' \cup z \geq z$; but $y \neq z$, hence $y > z$.

For each element x of a given i -lattice we shall set $|x| = x \cup x'$. We shall call an element z of a given i -lattice a “zero” if $z = z'$. We shall denote the i -lattice with four elements and two zeros by \mathfrak{D} . The i -lattice whose underlying lattice is the finite chain with n elements will be denoted by \mathfrak{D}_n . If L and M are i -isomorphic i -lattices we shall write $L \cong M$.

LEMMA 2. *The i -lattices $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$, and \mathfrak{D} are subdirectly irreducible, and are to within i -isomorphism the only subdirectly irreducible distributive i -lattices.*

Proof. Each of the given i -lattices is obviously subdirectly irreducible. Conversely, let L be a subdirectly irreducible distributive i -lattice. For elements x, y of L we shall set $x \sim y$ if and only if one of the following statements is true: (a) $x > x'$ and $y > y'$, (b) $x < x'$ and $y < y'$, (c) $x = x' = y = y'$. Using Lemma 1 and trivial arguments we see that \sim defines a congruence relation on L . Also, for each p in L , we may define congruence relations $D(p)$ and $E(p)$ on L by setting $x \equiv y(D(p))$ if and only if $x \sim y$ and $|x| \cap p = |y| \cap p$, and $x \equiv y(E(p))$ if and only if $x \sim y$ and $|x| \cup p = |y| \cup p$. Using [1, p. 28, Lemma 1(ii)], we see that $D(p) \cap E(p) = 0$ for all p in L and hence, since L is subdirectly irreducible, that

$$(7) \quad \text{either } D(p) = 0 \text{ or } E(p) = 0 \text{ for each } p \text{ in } L.$$

If $x = x'$ for all x in L , then $L \cong \mathfrak{D}_1$; we may therefore assume in the rest of the proof that L has an element c such that $c > c'$. We shall prove that if L has an element x distinct from c and c' then x is a zero of L . First, L cannot

have three distinct elements x such that $x > x'$; for, if L has three distinct such elements, it has a chain $x > y > w$ of such elements, and then $x \equiv y(D(y))$ and $w \equiv y(E(y))$, contradicting (7). It follows that L has at most two such elements, and that they must be comparable if they exist; but if $x > y > y' > x'$, and F is the congruence relation on L with congruence classes $\{x\}$, $\{x'\}$, and $\{w: y' \leq w \leq y\}$ (cf. (6)), then $F \neq 0$, $D(y) \neq 0$, and $F \cap D(y) = 0$, contradicting the subdirect irreducibility of L . Thus, as asserted, every element of L distinct from c and c' is a zero of L ; moreover, by [1, p. 28, Lemma 1(ii)], L has at most two zeros. Thus $L \cong \mathfrak{D}_2$, \mathfrak{D}_3 , or \mathfrak{D} according as L has 0, 1, or 2 zeros. This completes the proof.

It will be convenient to call any i -sublattice of a direct union of i -lattices L_γ a "subdirect union" of the L_γ (cf. [1, p. 91]). With this nomenclature we have

THEOREM 1. *Every distributive i -lattice is i -isomorphic with a subdirect union of i -isomorphic images of \mathfrak{D} .*

Proof. Every algebra A can be represented as a subdirect union of subdirectly irreducible homomorphic images of A (cf. [1, p. 92]). Observing that every i -homomorphic image of a distributive i -lattice is again a distributive i -lattice, and that each \mathfrak{D}_i ($i=1, 2, 3$) is i -isomorphic with an i -sublattice of \mathfrak{D} , we deduce Theorem 1.

THEOREM 2. *Every normal i -lattice is i -isomorphic with a subdirect union of i -isomorphic images of \mathfrak{D}_3 .*

This result is easily deduced from Theorem 1. The well known theorem that every Boolean algebra except \mathfrak{D}_1 is i -isomorphic with a subdirect union of i -isomorphic images of \mathfrak{D}_2 similarly follows from Theorem 2.

If L is an i -lattice, and P is a partly ordered set with involution $t \rightarrow t'$, the cardinal power L^P [1, pp. 8, 25] becomes an i -lattice if we define $f'(t) = (f(t'))'$ for each f in L^P and t in P .

THEOREM 3. *Every normal i -lattice is i -isomorphic with an i -sublattice of a vector lattice.*

Proof. Let L be a normal i -lattice. Then, by Theorem 2, there exists an i -isomorphism ρ of L into a cardinal power $(\mathfrak{D}_3)^\Gamma$, where Γ is an unordered set with involution given by $\gamma' = \gamma$ for each γ in Γ . If we take \mathfrak{D}_3 to be the chain $-1 < 0 < 1$ then ρ becomes an i -isomorphism of L into the vector lattice R^Γ of all real-valued functions on Γ .

An alternative proof of Theorem 3 may be based on [1, p. 140, Corollary] and the existence part of Theorem 7 below (cf. p. 98 of the author's thesis).

Let P be a class of equations for i -lattices; then an i -lattice L will be said to be " P -proper" if none of the equations in P holds in L , and to be " P -

complete" if it is P -proper and no P -proper i -lattice M is such that the set of identities of L is strictly included in the set of identities of M . Let P_0 , P_1 , and P_2 be the one-element classes whose elements are the equations $x=y$, $|x|=|y|$, and $|x|' \cap |y| = |x|'$, respectively; then P_0 -completeness is precisely equational completeness [4].

THEOREM 4. *An i -lattice is P_0 -complete if and only if it is a Boolean algebra with at least two elements⁽²⁾. An i -lattice is P_1 -complete if and only if it is normal but is not a Boolean algebra. An i -lattice is P_2 -complete if and only if it is distributive but is not normal.*

Proof. We observe that if L is a P_2 -proper i -lattice then \mathfrak{D} is an i -homomorphic image of an i -sublattice of L ; indeed if a, b in L are such that $|a|' \cap |b| \neq |a|'$ then $M = \{t: |a|' \leq t \leq |a|\} \cup \{t: |b|' \leq t \leq |b|\} \cup \{t: t \leq |a| \cap |b|\} \cup \{t: t \geq |a|' \cup |b|'\}$ is an i -sublattice of L , and \mathfrak{D} is an i -homomorphic image of M . Using this observation and Theorem 1 the reader may prove by the methods of [4] that an i -lattice is P_2 -complete if and only if it is distributive but not normal. The proofs of the corresponding results for P_0 - and P_1 -completeness are similar but easier.

The results of §1 depend on [1, p. 92, Theorem 10], and hence on the axiom of choice. Although the results of §1 will be used later in the paper, all the results of §§2-4 can be proved without using the axiom of choice.

2. A classification of i -lattices. For elements x, y of any i -lattice set $x \Delta y = (x \cup y) \cap (x' \cup y')$. Then, using Theorems 1 and 2, we may verify that each of the following laws is a necessary and sufficient condition for a given distributive i -lattice to be normal: (A) $x \Delta (y \Delta z) = (x \Delta y) \Delta z$, (B) $x \Delta y = (x \cap y') \cup (x' \cap y)$, (C) $|x \cap y| \cap |x \cup y| = |x| \cap |y|$, (C*) $|x \Delta y| = |x| \cap |y|$, (D) $|x \cap y| \cap |x \cup y| \leq |x| \cap |y|$, (E) $|x \Delta y| \leq |x| \cap |y|$. Similarly the laws (α) if $a \Delta c = b \Delta c$ for some c then $a = b$, and (β) $x \cap (y \Delta z) = (x \cap y) \Delta (x \cap z)$ are each necessary and sufficient for a given distributive i -lattice to be a Boolean algebra; using [2, Theorem 1] we may deduce that *any* i -lattice satisfying (α) is a Boolean algebra. For Boolean algebras, the laws (A), (B), (α), and (β) are well known (cf. [1, pp. 154 f.]), and the laws (C) and (C*) are trivial; for 1-groups, the laws (A), (B), (C), and (C*) are believed to be new. It may be proved that an arbitrary i -lattice satisfies (C*) if and only if it satisfies (C).

Let ω be the set whose elements are the laws (A), (B), (C), (D), (E), (F) $|x|' \leq |y|$, and (M) the modular law. For each $\xi \subseteq \omega$ let ξc be the set of all (Y) in ω such that every i -lattice which satisfies all (X) in ξ necessarily satisfies (Y); then $c: \xi \rightarrow \xi c$ is a closure operation on the subsets of ω . If $\xi \subseteq \omega$ is nonempty and has elements (X), \dots , (Y) we shall write $\xi = \{X \dots Y\}$, and we shall denote the family of all i -lattices which satisfy all (Z) in ξ by

⁽²⁾ Cf. [4, Theorem 3.3; 1, p. 189].

$[X \cdots Y]$. It may be proved that c is the intersection (cf. [5, §1]) of all those closure operations δ on the subsets of ω which are such that $\{A\}\delta \supseteq \{BC\}$, $\{B\}\delta \supseteq \{E\}$, $\{C\}\delta \supseteq \{D\}$, $\{D\}\delta \supseteq \{E\}$, $\{E\}\delta \supseteq \{F\}$, $\{BM\}\delta \supseteq \{A\}$, and $\{EM\}\delta \supseteq \{D\}$; from this result we may deduce

THEOREM 5. *If $\{X \cdots Y\}$ is a nonempty subset of ω then $[X \cdots Y]$ is equal to exactly one of the families $[A]$, $[B]$, $[C]$, $[D]$, $[E]$, $[F]$, $[M]$, $[AM] = [BM]$, $[BC]$, $[BD]$, $[CM]$, $[DM] = [EM]$, and $[FM]$. In the presence of a set of axioms for i -lattices the laws indicated are in each case independent axioms for the corresponding family.*

3. Normal extensions. It is easily proved that an i -lattice satisfying the law (F) of §2 can have at most one zero, and that a modular i -lattice with a zero satisfies (F) if the zero is unique; thus a distributive i -lattice with a zero is normal if and only if it has no other zero. In an i -lattice with a unique zero 0 we shall set $x_+ = x \cup 0$ and $x_- = x' \cup 0$ for each x , and we shall say that x is "positive" if $x \geq 0$; then $x_+ \cap x_- = x \Delta 0$ for all x , and the law $x_+ \cap x_- = 0$ (cf. [1, p. 220, Lemma 4]) is implied by (E) of §2, and implies (F). In this section we shall determine all the normal i -lattices with zero which have a given sublattice of positive elements.

If a is any element of a given distributive lattice P then the ordered pairs (x, y) of elements of P which are such that $x \cap y \leq a \leq x \cup y$ form a normal i -lattice $P(a)$ with operations given by $(x, y) \cap (z, w) = (x \cap z, y \cup w)$ and $(x, y)' = (y, x)$, and the elements (x, y) of $P(a)$ which are such that $a = x \leq y$ or $a = y \leq x$ form an i -sublattice $P\langle a \rangle$ of $P(a)$. If P has a least element O , and L is a normal i -lattice with zero whose sublattice L_+ of positive elements is lattice-isomorphic with P , we shall call L a "normal extension" of P ; then, if λ is a lattice-isomorphism of L_+ onto P , $x \rightarrow (x_+ \lambda, x_- \lambda)$ is an i -isomorphism of L into $P(O)$. Using this observation the reader may prove

THEOREM 6. *Let P be a distributive lattice with least element O . Then every i -sublattice L of $P(O)$ which is such that $P\langle O \rangle \subseteq L$ is a normal extension of P , and every normal extension of P is i -isomorphic with an i -sublattice L of $P(O)$ which is such that $P\langle O \rangle \subseteq L$. All normal extensions of P are i -isomorphic if and only if O is meet-irreducible in P .*

4. Embedding theorems. In §4 if L is a normal i -lattice and $x \in L$ then x^- will denote the principal ideal of all $y \leq x$, \vee will denote the join operation in the complete lattice \bar{L} of all closed ideals [1, p. 59] of L , and Z will denote the set of all $|y|'$ for y in L . Then $Z \in \bar{L}$ and (cf. [3, p. 2]) the mapping $x \rightarrow x^- \vee Z$ is a lattice-homomorphism of L into \bar{L} . Hence $\{x^- \vee Z : x \in L\}$ is a distributive sublattice of \bar{L} ; this lattice will be denoted by L^* .

THEOREM 7. *Let L be a normal i -lattice. Then the set L_Ω of all those ordered pairs $(x^- \vee Z, y^- \vee Z)$ of elements of L^* which are such that $x \leq y'$ is an i -sub-*

lattice of $L^*(Z)$. L_Ω is a normal i -lattice with zero (Z, Z) , and $\pi: x \rightarrow (x^- \vee Z, x'^- \vee Z)$ is an i -isomorphism of L into L_Ω ; π maps L onto L_Ω if and only if L has a zero. If τ is an i -isomorphism of L into a normal i -lattice M with zero 0 , then $\rho: (x^- \vee Z, y^- \vee Z) \rightarrow (x\tau)_+ \cap (y\tau)' (x \leq y')$ is an i -isomorphism of L_Ω onto the i -sublattice of M generated by $L\tau$ and 0 , and $\tau = \pi\rho$. If N is a normal i -lattice with zero having the property: there exists an i -isomorphism π_0 of L into N such that every i -isomorphism τ_0 of L into a normal i -lattice P with zero 0 is of the form $\tau_0 = \pi_0\rho_0$ for some i -isomorphism ρ_0 of N into P , then $N \cong L_\Omega$.

The proof depends on the fact that $x^- \vee Z = y^- \vee Z$ in L^* if and only if $x \cap x' = x \cap y'$ and $y \cap y' = y \cap x'$ in L ; the details will be left to the reader.

If L is any i -lattice we may (and shall) identify the cardinal power $L^{\mathfrak{D}_2}$ (cf. §1) with the set of all ordered pairs (x, y) of elements of L which are such that $x \leq y$.

THEOREM 8. *Let L be a normal i -lattice, and let $M = L^{\mathfrak{D}_2}$. Then $\phi: (x, y) \rightarrow (x^- \vee Z, y'^- \vee Z)$ is an i -homomorphism of M onto L_Ω ; ϕ is an i -isomorphism if and only if L is a Boolean algebra. If ψ is an i -homomorphism of M onto a (distributive) i -lattice N , then N is normal if and only if $\Psi \cong \Phi$, where Φ and Ψ are the congruence relations on M defined by ϕ and ψ respectively.*

The main step in the proof is to show that if N is normal, and $(x, y), (u, v)$ in M are such that $(x, y)\phi = (u, v)\phi$, then $(x \cup u, y \cap v) \in M$ and $(x, y)\psi_\pm = (x \cup u, y \cap v)\psi_\pm = (u, v)\psi_\pm$.

A normal i -lattice L will be said to be "closed" if for every pair x, y of elements of L_Ω such that $x \cap y = 0$ (the zero of L_Ω) there exists z in L_Ω such that $z_+ = x$ and $z_- = y$. A distributive lattice with least element has, to within i -isomorphism, just one closed normal extension. Also, the following embedding theorem may be proved.

THEOREM 9. *Let L be a normal i -lattice. Then the set L^c of all those ordered pairs $(x^- \vee Z, y'^- \vee Z)$ of elements of L^* which are such that $x \Delta y' \in Z$ is an i -sublattice of $L^*(Z)$. L^c is a closed normal i -lattice, and $\pi: x \rightarrow (x^- \vee Z, x'^- \vee Z)$ is an i -isomorphism of L into L^c ; π maps L onto L^c if and only if L is closed. If τ is an i -isomorphism of L into a closed normal i -lattice M then there exists an i -isomorphism σ of L^c into M such that $\tau = \pi\sigma$. If N is a closed normal i -lattice having the property: there exists an i -isomorphism π_0 of L into N such that every i -isomorphism τ_0 of L into a closed normal i -lattice P is of the form $\tau_0 = \pi_0\sigma_0$ for some i -isomorphism σ_0 of N into P , then $N \cong L^c$.*

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