ON SUBGROUPS OF THE ORTHOGONAL GROUP

BY

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0. **Introduction.** We determine in this note the connected Lie subgroups G of the orthogonal group $O(n)$ of degree n such that

$$\dim O(n-1) > \dim G > \dim O(n-3) + \dim O(3). \tag{0.1}$$

D. Montgomery and H. Samelson [5; 6], in this regard, have proved that if

$$\dim O(n) > \dim G \geqq \dim O(n-1)$$

then G is conjugate to the standard subgroup $SO(n-1)$ except $n=4$, 8 and that there does not exist a subgroup G such that

$$\dim O(n-1) > \dim G > \dim O(n-2) + \dim O(2)$$

with a finite number of exceptions for the values of n. But it has not yet been made clear what values of n are exceptional. This note will give an answer to the analogous problem for a wider interval of the dimension of G.

In §1 we shall explain some notations and then, in §2, a relation between irreducibility and absolute irreducibility will be discussed; especially a condition for a representation to be unitary symplectic will be given. In §3 we shall obtain inequalities concerning the dimensions of some irreducible groups. The last §4 concerns the reducibility of G satisfying relation (0.1). The method is due much to H. C. Wang and K. Yano [8] who have determined sufficiently high dimensional subgroups of the projective group. The group G will be determined as the groups which leave invariant a two-dimensional plane.

1. **Preliminaries.** Let G be a group of complex matrices of degree m. If $A \in G$, we may represent it in the form $A = B + (-1)^{1/2}C$, where B and C are real matrices of degree m. We assign to A the real matrix A' of degree $2m$ defined by

$$A' = \begin{pmatrix} B & C \\ -C & B \end{pmatrix}.$$

Then the correspondence $A \to A'$ gives an isomorphism of G with a group G' of real matrices of degree $2m$. Conversely let

$$A' = \begin{pmatrix} B & C \\ -C & B \end{pmatrix}$$

be an element of G'. On putting

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$$P = \frac{1}{2^{1/2}}\begin{pmatrix} E_m & E_m \\ (-1)^{1/2}E_m & -(-1)^{1/2}E_m \end{pmatrix},$$

E_m being the unit matrix of degree m, we have

$$P^{-1}A'P = \begin{pmatrix} B + (-1)^{1/2}C & 0 \\ 0 & B - (-1)^{1/2}C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

As is easily seen, if A is unitary, then A' is real orthogonal and vice versa. On the basis of the above consideration G' will be called the *natural real representation* of G.

Now, throughout this note we denote by k the field of real numbers and by K that of complex numbers and further we adopt the following notations:

$O(n, K)$	the complex orthogonal group of degree n,
$O(n)$	the (real) orthogonal group of degree n,
$SO(n)$	the special orthogonal group of degree n,
$U(n)$	the unitary group of degree n,
$Sp(n)$	the unitary symplectic group of degree $2n$.

2. **Irreducibility and absolute irreducibility.** Let G be a group and ρ_1 its complex representation of degree m. Then the sum $\rho_1+\bar{\rho}_1$ of the representation ρ_1 and its complex conjugate $\bar{\rho}_1$ is equivalent to a representation ρ by real matrices of degree $2m$. If ρ_1 is irreducible and is not equivalent to a real representation, then ρ is irreducible in k but reducible in K. In case ρ_1 is unitary, ρ is equivalent to a real orthogonal representation. Conversely we have the following

LEMMA 1. *If a real representation ρ of degree n of a group is irreducible in k and reducible in K, then n is even, $n=2m$, and is equivalent to the sum $\rho_1+\bar{\rho}_1$ of a complex representation ρ_1 of degree m and its complex conjugate $\bar{\rho}_1$. Furthermore ρ_1 is irreducible. If, moreover, ρ is orthogonal, then ρ_1 is equivalent to a unitary one.*

The proof has been given in detail in [1; 2], so it is omitted.

Next, let G be a group having a unitary representation ρ of degree m $(=2l)$. If ρ is symplectic, i.e. for all a in G we have

$$ {}^t\rho(a)J_l\rho(a) = J_l, \tag{2.1}$$

where

$$J_l = \begin{pmatrix} 0 & E_l \\ -E_l & 0 \end{pmatrix},$$

ρ is equivalent to its complex conjugate $\bar{\rho}$, because (2.1) implies $J_l\rho(a)J_l^{-1} =\bar{\rho}(a)$. The converse problem will be answered affirmatively in Lemma 3 below. To do this we state here lemmas concerning symmetric and skew symmetric matrices with complex coefficients.

LEMMA 2. *Let S be a symmetric matrix of degree m with complex coefficients. If the equality $\bar{S}S=E_m$ holds, there exists a unitary matrix T such that*

$$^tTST = E_m.$$

Proof. We decompose S into the real and imaginary parts, $S=S_1+(-1)^{1/2}S_2$, S_1 and S_2 being real symmetric matrices. The condition $\bar{S}S=E_m$ is expressed by the equalities $S_1^2+S_2^2=E_m$ and $S_1S_2=S_2S_1$. From the latter equality it follows that there exists a common real orthogonal matrix T_0 such that $^tT_0S_1T_0$ and $^tT_0S_2T_0$ are both diagonal matrices:

$$^tT_0S_1T_0 = \begin{pmatrix} s_1' & & 0 \\ & \ddots & \\ 0 & & s_m' \end{pmatrix}, \qquad ^tT_0S_2T_0 = \begin{pmatrix} s_1'' & & 0 \\ & \ddots & \\ 0 & & s_m'' \end{pmatrix},$$

where s_i' and s_i'' $(1\leqq i\leqq m)$ are respectively the characteristic roots of S_1 and S_2. The condition $S_1^2+S_2^2=E_m$ implies $s_i'^2+s_i''^2=1$ $(1\leqq i\leqq m)$. On putting $s_i=s_i'+(-1)^{1/2}s_i''$ we have

$$S_0 = {}^tT_0ST_0 = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_m \end{pmatrix}, \quad |s_i| = 1.$$

If we choose t_i such that $t_i^2=s_i$ $(1\leqq i\leqq m)$, the matrix

$$\bar{T}_1 = \begin{pmatrix} \bar{t}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{t}_m \end{pmatrix}$$

is unitary and $^t\bar{T}_1S_0\bar{T}_1=E_m$. The matrix $T=T_0\bar{T}_1$ is a required matrix.

In an analogous way we can prove the following lemma.

LEMMA 2'. *Let S be a skew-symmetric matrix of degree m with complex coefficients. If the equality $\bar{S}S=-E_m$ holds, then m is even, $m=2l$, and there exists a unitary matrix T such that*

$$^tTST = J_l.$$

Using the above two lemmas we shall prove the following lemma [1].

LEMMA 3. *Let ρ be a unitary representation of degree m of a group. If ρ is irreducible and is equivalent to its complex conjugate $\bar{\rho}$, then ρ is equivalent either to a real orthogonal representation or to a unitary symplectic one. In the latter case m is even.*

Proof. Let G be the group having the representation ρ. On account of the assumed equivalence of ρ and $\bar{\rho}$ there exists a nonsingular matrix S such that

(2.2) $$S\rho(a)S^{-1} = \bar{\rho}(a)$$

for all a in G. Since ρ and $\bar{\rho}$ are unitary, (2.2) implies

(2.3) $${}^t\rho(a)S\rho(a) = S,$$

which shows that ρ leaves invariant such a bilinear form that the matrix of its coefficients is S.

We shall show that S is either symmetric or skew-symmetric. From (2.2) it follows $\bar{S}\bar{\rho}(a)\bar{S}^{-1}=\rho(a)$, which together with (2.2) gives

$$(\bar{S}S)\rho(a) = \rho(a)(\bar{S}S).$$

Since ρ is irreducible, by Schur's lemma, $\bar{S}S$ is a numerical multiple αE_m of the unit matrix:

(2.4) $$\bar{S}S = \alpha E_m,$$

α being a nonzero complex number. (2.4) gives the equality

(2.5) $$S\bar{S} = \bar{\alpha}E_m.$$

(2.4) and (2.5) imply $\alpha=\bar{\alpha}$, i.e. that α is real. Without loss of generality α may be assumed to be ± 1:

(2.6) $$\bar{S}S = S\bar{S} = \pm E_m.$$

Next we shall see that S is unitary. Because of (2.2) the representation ρ leaves invariant the Hermitian matrix H defined by $H={}^t\bar{S}S$, i.e.

$${}^t\bar{\rho}(a)H\rho(a) = H.$$

Since ρ is irreducible and H is always positive-definite, we conclude that H is a numerical multiple βE_m of the unit matrix, β being a positive real number. From (2.6) it follows $|\beta|=1$ and then $\beta=1$, which shows that S is unitary:

(2.7) $${}^t\bar{S}S = E_m.$$

We distinguish two cases according to the sign of α.

(i) $\alpha=+1$. In this case the conditions (2.6) and (2.7) imply that S is symmetric, $S={}^tS$. By Lemma 2 there exists a unitary matrix T such that ${}^tTST=E_m$. It follows by an easy computation that the matrix

$${}^t(T^{-1}\rho(a)T)(T^{-1}\rho(a)T)$$

is the unit matrix. This means that the representation $T^{-1}\rho T$ is unitary and orthogonal, i.e. it is real orthogonal.

(ii) $\alpha=-1$. In this case the conditions (2.6) and (2.7) imply that S is skew-symmetric, $S=-{}^tS$. By Lemma 2′ m is even, $m=2l$, and there exists a unitary matrix T such that ${}^tTST=J_l$. Then a simple calculation shows that the equality

$${}^t(T^{-1}\rho(a)T)J_l(T^{-1}\rho(a)T) = J_l$$

holds good. This means that the representation $T^{-1}\rho T$ is unitary symplectic.

LEMMA 4 [7]. *Let G be a semi-simple Lie group which is not simple. Given an absolutely irreducible real orthogonal representation ρ of G, then ρ is equivalent to the Kronecker product $\rho_1 \otimes \rho_2$ of absolutely irreducible representations ρ_1 and ρ_2. Furthermore ρ_1 and ρ_2 are either both real orthogonal or both unitary symplectic.*

In case ρ is not orthogonal the first half of the lemma holds true if G is simply connected semi-simple or G is a semi-simple Lie algebra.

Proof. Because of the semi-simplicity of G it is written as $G = G_1G_2$ where G_1 and G_2 are semi-simple normal subgroups of G such that every element of G_1 commutes with that of G_2 and $G_1 \cap G_2$ is discrete. Let us denote by $\tilde{\rho}_i$ the restriction of ρ to G_i $(i=1, 2)$. Since $\tilde{\rho}_i$ may be reducible, we decompose $\tilde{\rho}_1$ into irreducible components in K:

$$\tilde{\rho}_1 \approx \rho_1 + \rho_1' + \cdots + \rho_1^{(n_2-1)}. \tag{2.8}$$

From the facts that $\rho_1^{(i)}$ is irreducible in K and that every element of G_2 commutes with that of G_1 it follows that $\rho_1, \rho_1', \cdots, \rho_1^{(n_2-1)}$ are of the same degree n_1 and are equivalent to each other. Thus there exists a nonsingular matrix P such that

$$P^{-1}\tilde{\rho}_1 P = \rho_1 + \cdots + \rho_1 \quad (n_2 \text{ components}). \tag{2.9}$$

Again by irreducibility of ρ_1 and commutativity of G_2 with G_1, for every a_2 in G_2, $P^{-1}\tilde{\rho}_2(a_2)P$ is of the form

$$P^{-1}\tilde{\rho}_2(a_2)P = \begin{pmatrix} b_{11}E_{n_1} & \cdots & b_{1n_2}E_{n_1} \\ \vdots & & \vdots \\ b_{n_21}E_{n_1} & \cdots & b_{n_2n_2}E_{n_1} \end{pmatrix}. \tag{2.10}$$

On putting

$$\rho_2(a_2) = \begin{pmatrix} b_{11} & \cdots & b_{1n_2} \\ \vdots & & \vdots \\ b_{n_21} & \cdots & b_{n_2n_2} \end{pmatrix},$$

we find that the representation ρ_2 is irreducible in K and $P^{-1}\tilde{\rho}_2 P = E_{n_1} \otimes \rho_2$ and then

$$P^{-1}\rho P = \rho_1 \otimes \rho_2. \tag{2.11}$$

It should be noted that the relationship of ρ_1 and ρ_2 is mutual.

To prove the second half of this lemma, we first decompose $\tilde{\rho}_1$ into irreducible components in k. We distinguish two cases.

(i) In case each of the components is absolutely irreducible, by using this decomposition we have

$$P^{-1}\rho P = \rho_1 \otimes \rho_2.$$

Since the decomposition of $\tilde{\rho}_1$ is done in k, ρ_1 and P may be assumed to be real orthogonal, so also is ρ_2 by (2.10). Thus in this case ρ is equivalent to the Kronecker product of two real orthogonal representations.

(ii) In case some one of the components of $\tilde{\rho}_1$ is reducible in K, we shall prove that all of the components are reducible in K and the irreducible components in K are unitary symplectic.

To prove these, let

$$P_1^{-1}\rho_1 P_1 = \sigma_1 + \cdots + \sigma_p$$

be the decomposition of $\tilde{\rho}_1$ into irreducible components in k, where P_1 may be assumed to be real orthogonal. We assume that $\sigma_1, \cdots, \sigma_q$ are reducible in K but $\sigma_{q+1}, \cdots, \sigma_p$ are absolutely irreducible. By Lemma 1 σ_i $(1 \leqq i \leqq q)$ is equivalent to the sum $\sigma_i' + \bar{\sigma}_i'$ of a unitary representation σ_i' and its complex conjugate $\bar{\sigma}_i'$, where σ_i' can not be equivalent to a real one. Since the decompositions σ_i into $\sigma_i' + \bar{\sigma}_i'$ $(1 \leqq i \leqq q)$ are realized by unitary matrices, there exists a unitary matrix P_2 such that

$$P_2^{-1}\tilde{\rho}_2 P_1 = \sigma_1' + \bar{\sigma}_1' + \cdots + \sigma_q' + \bar{\sigma}_q' + \sigma_{q+1} + \cdots + \sigma_p.$$

As we have seen above that all the components of this decomposition are equivalent to each other, σ_1' must be equivalent to the real orthogonal σ_j $(q+1 \leqq j \leqq p)$, which leads to a contradiction. Thus we have

$$\tilde{\rho}_1 \approx \sigma_1' + \bar{\sigma}_1' + \cdots + \sigma_p' + \bar{\sigma}_p' \quad (2p = n_2).$$

Since σ_1' is equivalent to its complex conjugate $\bar{\sigma}_1'$, by Lemma 3 σ_1' is equivalent to a unitary symplectic representation ρ_1. Since the equivalence can be realized by a unitary matrix, from the first part of this lemma there exist a unitary matrix P and a unitary representation ρ_2 of degree n_2 such that

$$P^{-1}\tilde{\rho}_1 P = \rho_1 + \cdots + \rho_1 \quad (n_2 \text{ components}) \quad \text{and} \quad P^{-1}\rho P = \rho_1 \otimes \rho_2.$$

The above consideration shows that ρ_1 is either real orthogonal or unitary symplectic, so also is ρ_2 because the relation of ρ_1 and ρ_2 is mutual. Since we have seen in (i) that if ρ_1 is orthogonal so is ρ_2, we conclude that in case ρ_1 is unitary symplectic so also is ρ_2, which completes the proof.

3. **The dimensions of irreducible subgroups of the orthogonal group.** We obtain in this section, using the results of the preceding section, inequalities concerning the dimensions of irreducible subgroups of the orthogonal group.

LEMMA 5. *Let G be an irreducible group of real orthogonal matrices of degree n. If G is reducible in K, then the inequality*

$$\dim G \leqq n^2/4$$

holds good.

Proof. In view of the assumed irreducibility in k and reducibility in K, by Lemma 1 n is even, $n=2m$, and there exists a nonsingular matrix T such that for all A in G

$$T^{-1}AT = \begin{pmatrix} B & 0 \\ 0 & \overline{B} \end{pmatrix},$$

B being a unitary matrix. This shows that G is the natural real representation of a subgroup of the unitary group $U(m)$. Hence we have

$$\dim G \leqq \dim U(m) = n_2/4.$$

Next, as a condition for a Lie algebra to be semi-simple we have the following lemma due to É. Cartan [3, p. 147].

Lemma 6. *Let g be a Lie algebra of complex matrices of degree n. If g is irreducible, then it is either semi-simple or equivalent to the direct sum of a semi-simple Lie algebra and the Lie algebra consisting of all the matrices of the form cE_n, c being a complex number. If, in addition, g is contained in the Lie algebra of the complex orthogonal group $O(n, K)$, then g is semi-simple.*

The proof is omitted.

Lemma 7. *Let G be an absolutely irreducible Lie group of real orthogonal matrices of degree n. If G is not simple, then the inequality*

$$\dim G \leqq (n^2+2n)/4$$

holds good.

Proof. We denote by r the dimension of G. Since G is semi-simple by Lemma 6 and is not simple by hypothesis, it is written as $G=G_1G_2$ as in the proof of Lemma 4, where we may assume without loss of generality

$$\dim G_1 \geqq \dim G_2 \quad \text{or} \quad \dim G_1 \geqq r/4. \tag{3.1}$$

By Lemma 4 there exist representations ρ_i of degree n_i $(i=1, 2)$ such that the faithful representation $\rho: A \to A$ $(A \in G)$ is equivalent to $\rho_1 \otimes \rho_2$. As is easily seen, ρ_i is a faithful representation of G_i $(i=1, 2)$ provided that ρ is faithful in the proof of Lemma 4. Thus the matric group G is equivalent to $\tilde{G}_1 \otimes \tilde{G}_2$ where $\tilde{G}_i=\rho_i(G_i)$. Since $n_1n_2=n$ and $n_i \geqq 2$, we have

$$n_i \leqq n/4. \tag{3.2}$$

Since by Lemma 4 ρ_i may be assumed to be either a real orthogonal representation or a unitary symplectic one, $\tilde{G}_i$ is contained in $O(n_i)$ or $Sp(n_i/2)$. From the facts that

$$\dim O(n_i) = \frac{1}{2}\, n_i(n_i - 1) \quad \text{and} \quad \dim Sp(n_i/2) = \frac{1}{2}\, n_i(n_i + 1)$$

we have in both cases

$$\dim G_i = \dim \tilde{G}_i \leqq \frac{1}{2} n_i(n_i + 1) \tag{3.3}$$

because $n_i(n_i-1)/2 < n_i(n_i+1)/2$. From (3.1), (3.2) and (3.3) it follows

$$\frac{r}{2} \leqq \frac{n}{4}\left(\frac{n}{2} + 1\right) \quad \text{or} \quad r \leqq \frac{1}{4}(n^2 + 2n).$$

LEMMA 8. *Let g be an absolutely irreducible Lie algebra of real matrices of degree n and g^K the complex form of g. If g is simple but g^K is not simple, then the inequality*

$$\dim g \leqq 2(n - 1)$$

holds good.

Proof. We put $\dim g = r$. Since the semi-simplicity is an implication of the fact that the fundamental quadratic form is non-degenerate, g is semi-simple. g, however, being simple but g^K being not simple, g^K is the direct sum of two complex conjugate ideals h and $\bar{h}$, each of which is simple and of complex dimension $r/2$. Then in the same manner as in Lemma 4 g^K can be written as the Kronecker product of two matric Lie algebras h^* and $\bar{h}^*$ of degree n_1 and n_2 respectively, h^* and $\bar{h}^*$ being isomorphic images of h and $\bar{h}$ respectively. It follows

$$\dim h = \dim h^* = r/2 \leqq n_1^2 - 1 \quad \text{and} \quad \dim \bar{h} \leqq n_1^2 - 1.$$

These imply $r \leqq 2(n-1)$ because $n_1 n_2 = n$.

4. **Reducibility of subgroups of the orthogonal group.** We first prove

LEMMA 9. *Let G be a connected Lie group of real orthogonal matrices of degree n. If $n \geqq 13$ and*

$$\dim O(n) > \dim G \geqq \dim O(n - 3) + \dim O(3) + 1 = \frac{1}{2}(n^2 - 7n + 20),$$

then G is reducible in k.

Proof. Suppose that G is irreducible in k and we shall show that this leads to a contradiction.

If G is irreducible, G is absolutely irreducible. In fact, if G is reducible in K, then by Lemma 5 we have $\dim G \leqq n^2/4$. But this can not occur because the inequality

$$\dim G \geqq \frac{1}{2}(n^2 - 7n + 20) > \frac{n^2}{4}$$

holds for $n \geqq 11$. Thus G must be absolutely irreducible provided that G is irreducible in k. Then the complex form G^K of G is also irreducible. It follows from Lemma 6 that G^K is semi-simple, so also is G.

We distinguish two cases.

(i) G is not simple. In this case by Lemma 7 we must have $\dim G \leqq (n^2+2n)/4$. But by hypothesis we have for $n \geqq 13$

$$\dim G \geqq \frac{1}{2}(n^2 - 7n + 20) > \frac{1}{4}(n^2 + 2n).$$

Thus this case can not occur.

(ii) G is simple. Since $\dim G \geqq (n^2-7n+20)/2 > 2(n-1)$ for $n \geqq 13$, G^K is also simple by Lemma 8. Denoting by g the Lie algebra of G^K, g is one of the simple Lie algebras in the classification of É. Cartan [2, p. 147]. Since g is contained in the Lie algebra of the orthogonal group $O(n, K)$, if we denote by p the rank of the former algebra and by q that of the latter, we have

$$p \leqq q = \left[\frac{n}{2}\right].$$

On account of the assumptions $n \geqq 13$, $n(n-1)/2 > \dim G \geqq (n^2-7n+20)/2$ and $p \leqq q$, the possible types of g are the following.

CASE I. $n = 2q+1$, $g \subset B_q$.

Type of g	Rank of g	Dimension of g	Rank of B_q	n
(B)	$q-1$	$(n-2)(n-3)/2$	q	$2q+1$
(C)	$q-1$	$(n-2)(n-3)/2$	q	$2q+1$
(D)	q	$(n-1)(n-2)/2$	q	$2q+1$
(E)	6	72	6	13
(E)	7	133	8	17
(E)	7	133	9	19
(E)	8	248	11	23
(E)	8	248	12	25
(F)	4	52	6	13

CASE II. $n = 2q$, $g \subset D_q$.

Type of g	Rank of g	Dimension of g	Rank of D_q	n
(B)	$q-1$	$(n-1)(n-2)/2$	q	$2q$
(C)	$q-1$	$(n-1)(n-2)/2$	q	$2q$
(D)	$q-1$	$(n-2)(n-3)/2$	q	$2q$
(E)	6	72	7	14
(E)	7	133	9	18
(E)	8	248	12	24

But the theory of representations of complex simple Lie algebras tells us that in both cases I and II all these Lie algebras do not have irreducible representations of the corresponding degree n, so we may conclude that g can not be simple, which shows that the case (ii) can not occur.

Thus in all cases we have reached contradictions, so that G must be reducible in k.

We can now determine the explicit form of the group G for $n \geqq 14$.

THEOREM. *Let G be a connected Lie subgroup of the orthogonal group $O(n)$ of degree n. We assume that $n \geqq 14$ and*

$$\dim O(n-1) > \dim G > \dim O(n-3) + \dim O(3).$$

Then G leaves invariant one and only one plane, i.e. G consists of all the matrices of the form

$$\textit{either } \begin{pmatrix} E_2 & 0 \\ 0 & A \end{pmatrix}, \quad A \in SO(n-2) \textit{ or } \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad B \in SO(2), \ C \in SO(n-2)$$

up to a conjugation in $O(n)$.

Proof. Since the assumptions of Lemma 9 are satisfied, G is reducible in k. If we denote by V the n-dimensional vector space in k acted on by G, then V is written as the direct sum $V_1 + V_2$ of invariant subspaces V_1 and V_2. On putting $\dim V_1 = m$, we have $\dim V_2 = n - m$ and

$$\dim G \leqq \dim O(m) + \dim O(n-m).$$

We may assume without loss of generality $m \leqq n - m$. On account of the inequality $\dim G > \dim O(3) + \dim O(n-3)$ we have $m = 1$ or $m = 2$.

CASE I. $m = 1$. By connectedness of G every element of V_1 is remained invariant under all the transformations of G. Accordingly every element A of G is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}, \quad A' \in SO(n-1)$$

up to an inner automorphism of $O(n)$. The image G' of G by the isomorphism $A \rightarrow A'$ is a connected Lie subgroup of $O(n-1)$ and satisfies the inequality

$$\dim O(n-1) > \dim G' > \dim O(3) + \dim O(n-3), \qquad n - 1 \geqq 13.$$

By Lemma 9 G' must be also reducible on V_2 in k, so V_2 is the direct sum $U_1 + U_2$ of invariant subspaces U_1 and U_2, where we may assume $\dim U_1 \leqq \dim U_2$. Then $\dim U_1$ must be 1 and again by connectedness of G' every element A' of G' is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A'' \end{pmatrix}, \quad A'' \in SO(n-2)$$

up to a conjugation in $O(n-1)$. The image G'' of G' by the isomorphism $A' \to A''$ is a connected Lie subgroup of $O(n-2)$. Since $\dim G'' > \dim O(n-3)$ G'' must coincide with $SO(n-2)$. Thus G consists of all the matrices of the form

$$\begin{pmatrix} E_2 & 0 \\ 0 & A'' \end{pmatrix}, \quad A'' \in SO(n-2)$$

up to a conjugation in $O(n)$.

CASE II. $m=2$. In this case we may assume that G is irreducible in V_1. In fact, if not, the case reduces to Case I. Furthermore G may be assumed to consist of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A \in O(2),\ B \in O(n-2).$$

Let us denote by G_1 and G_2 the subgroups of G consisting of all the matrices which leave fixed all elements of V_2 and V_1 respectively. Then if $A_1 \in G_1$ and $A_2 \in G_2$ they are written in the forms

$$A_1 = \begin{pmatrix} A' & 0 \\ 0 & E_{n-2} \end{pmatrix}, \quad A' \in O(2), \qquad A_2 = \begin{pmatrix} E_2 & 0 \\ 0 & A'' \end{pmatrix}, \quad A'' \in O(n-2).$$

It is evident that $G_1 \cap G_2$ consists of the unit matrix E_n only and every element of G_1 commutes with that of G_2. We shall see that G is the direct product of G_1 and G_2. To do this it is sufficient to prove $G_1G_2 = G$.

Since $G'' = G/G_1$ is contained in $O(n-2)$ in V_2, from the inequality

$$\dim G'' \geqq \dim G - 1 > \dim O(n-3)$$

it follows that G'' is the special orthogonal group $SO(n-2)$. $G' = G/G_2$ is irreducible in V_1, so it is $SO(2)$. It follows $\dim G_2 = \dim G - 1$. Since $G_2 = G_2/(G_1 \cap G_2)$ is regarded as a subgroup of G'' and

$$\dim G_2 = \dim G - 1 > \dim O(n-3),$$

G_2 coincides with $G'' = SO(n-2)$. Thus we have an isomorphism $G_2 \to G/G_1$ and this is obviously canonical. Therefore we have $G = G_1G_2$. On the other hand it can be proved that G_1 is isomorphic with $SO(2)$. Thus the direct product G of G_1 and G_2 consists of all the matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A \in SO(2),\ B \in SO(n-2).$$

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