ON THE REALIZATION OF HOMOLOGY CLASSES BY SUBMANIFOLDS

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1. **Introduction.** In [1] R. Thom defines when a cohomology class in a manifold is realizable for a closed subgroup of an orthogonal group. He also defines when a homology class is realizable by a submanifold. He then shows the following.

Let m be the dimension of a compact orientable differentiable manifold M. Any cohomology class of dimension $1, 2, m-6, m-5, \cdots, m$ is realizable for the orthogonal group $O(1)=1$, $O(2)$, $O(m-6)$, $O(m-5)$, $\cdots$, $O(m)$ respectively. If a class $z\in H_{m-k}(M; Z)$ can be realized by a submanifold, then the cohomology class $u\in H^k(M; Z)$ which is dual to z satisfies

$$St_p^{2r(p-1)+1}(u) = 0$$

for all integers r and all odd primes p. Here $St_p^{2r(p-1)+1}$ denotes the Steenrod reduced power [4] which operates on the cohomology group with integral coefficients Z. All homology classes with integral coefficients of compact orientable differentiable manifolds of dimension <10 are realizable by submanifolds.

We consider compact differentiable manifolds which are not necessarily orientable, and we ask whether the cup-products and the Steenrod squares [3] of realizable cohomology classes can be realized. It was stated by Thom [1] that cup-products of realizable classes are also realizable. We shall give another proof of this result (see §3 below). As for squares of realizable classes with integers modulo 2 as coefficients, we have the following result. Let u be a cohomology class of dimension n in a compact differentiable manifold of dimension $m+n$. If u is realizable for the group $O(k)\subset O(n)$ $(k\leqq n)$, then the cohomology class $Sq^k(u)$ is also realizable (see §5 below).

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2. **Preliminaries.** In this section, we summarize the theory of Thom [1 and 2]. Let(1) G be a closed subgroup of the orthogonal group $O(n)$. Let S denote an $(n-1)$-sphere fiber space over a finite cell complex K with the structural group G and let A be the mapping cylinder of the projection of S onto K. A becomes a fiber space over K with a closed n-cell $\bar{b}_n$ as fiber. Its projection is induced by that of S. We denote by A' the complement of S in A which is a fiber space over K with fiber, an open n-cell b_n.

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(1) If G is a compact Lie group, then there is an integer n such that G is isomorphic to a subgroup of $O(n)$.

Throughout this paper, $\mathfrak{G}$ will denote the group of integers or integers modulo 2. When dealing with the nonorientable case(2) $\mathfrak{G}$ will denote only the group of integers modulo 2.

By the theory of fiber spaces with open cells as fibers, we have a canonical isomorphism ϕ^* of the r dimensional cohomology group $H^r(K; \mathfrak{G})$ onto the $r+n$ dimensional cohomology group $H^{r+n}(A'; \mathfrak{G}) = H^{r+n}(A, S; \mathfrak{G})$. $H^n(A, S; \mathfrak{G})$ is the first nonvanishing cohomology group.

We denote by $S_{G,n}$ and B_G the universal fiber space and the classifying space respectively, for $(n-1)$-sphere fiber spaces over finite cell complexes of bounded dimension with the structural group G (see [5]). We may assume B_G is a grassmann manifold. Let $A_{G,n}$ be the mapping cylinder of the projection of $S_{G,n}$ onto B_G and let $A'_{G,n}$ be the complement of $S_{G,n}$ in $A_{G,n}$. In particular, we denote $S_{O(n),n}$, $A_{O(n),n}$ and $A'_{O(n),n}$ by $S_{O(n)}$, $A_{O(n)}$ and $A'_{O(n)}$ for the sake of brevity.

Definition 1. Let $M(G, n)$ be the space which we get from $A_{G,n}$ by identifying its boundary $S_{G,n}$ to a point. We call the space $M(G, n)$ the *cell complex corresponding to the subgroup G of $O(n)$*. We denote $M(O(n), n)$ simply by $M(O(n))$.

We have the natural isomorphisms of cohomology groups,

$$H^r(A'_{G,n}; \mathfrak{G}) = H^r(A_{G,n}, S_{G,n}; \mathfrak{G}) = H^r(M(G, n); \mathfrak{G})$$

where $r>0$. It is quite easy to see that $H^0(M(G, n); \mathfrak{G}) = \mathfrak{G}$ $\phi^*_{G,n}$ denotes the canonical isomorphism in the fiber structure $A'_{G,n}$,

$$\phi^*_{G,n}: H^{r-n}(B_G; \mathfrak{G}) \approx H^r(A'_{G,n}; \mathfrak{G}) = H^r(M(G, n); \mathfrak{G})$$

where $r \geqq n$. Therefore the cohomology groups $H^r(M(G, n); \mathfrak{G})$ of dimensions $r \geqq n$ are obtained by raising dimensions of cohomology groups of the classifying space B_G by the integer n. $H^n(M(G, n); \mathfrak{G})$ is the first cohomology group which does not vanish in dimension >0. This group is generated by the class,

$$U_{G,n} = \phi^*_{G,n}(1_G) \in H^n(M(G, n); \mathfrak{G}) \tag{1}$$

where 1_G is the unit class of $H^*(B_G; \mathfrak{G})$. We call the class $U_{G,n}$ the *fundamental class* of $M(G, n)$. We denote $U_{O(n),n}$ simply by $U_{O(n)}$.

Definition 2. Let A be a topological space and let u be a class of $H^n(A; \mathfrak{G})$. u is said to be *realizable for $G \subset O(n)$* or *has G-realization*, if there is a map $f: A \to M(G, n)$ such that $u = f^* U_{G,n}$.

Definition 3. Let W be a submanifold of dimension p in a compact differentiable manifold V of dimension m and of class C^∞, where p is an integer such that $m \geqq p \geqq 0$. The inclusion mapping $i: W \subset V$ induces a homomorphism i_* of the homology group $H_p(W; \mathfrak{G})$ into the homology group $H_p(V; \mathfrak{G})$.

(2) Orientability means that of the fiber space S, therefore, of A. In this case, G is connected.

If a homology class z of $H_p(W; \mathfrak{G})$ is the image of the fundamental class of W, then we say that the class z *is realized by the submanifold* W.

Now we state the following fundamental theorem [1, Chap. II]:

THEOREM 1. *Let V be a compact differentiable manifold of dimension m and of class C^∞, and let n be an integer such that $m \geqq n \geqq 0$. A cohomology class $u \in H^n(V; \mathfrak{G})$ is realizable for the group $G \subset O(n)$ if and only if the dual homology class*(3) *z of u is realized by a submanifold W of dimension $m-n$ and the fiber space consisting of normal vectors on the submanifold W in V has the group G as its structural group.*

Let h be a classifying map of K into B_G for the fiber space S. Then it induces a fiber mapping H of A into $A_{G,n}$. Commutativity holds in the diagram,

$$\begin{array}{ccc} A & \xrightarrow{H} & A_{G,n} \\ p\downarrow & & \downarrow p_{G,n} \\ K & \xrightarrow{h} & B_G \end{array} \tag{2}$$

where p and $p_{G,n}$ are the projections of the fiber structures A and $A_{G,n}$ which are naturally induced by the projections $S \to K$ and $S_{G,n} \to B_G$. The commutativity of (2) induces that of the following diagram of homomorphisms of cohomology groups,

$$\begin{array}{ccc} H^n(A'; \mathfrak{G}) & \xleftarrow{H'^*} & H^n(A'_{G,n}; \mathfrak{G}) \\ \phi^*\uparrow & & \uparrow \phi^*_{G,n} \\ H^0(K; \mathfrak{G}) & \xleftarrow{h^*} & H^0(B_G; \mathfrak{G}) \end{array} \tag{3}$$

where H' is the mapping of A' into $A'_{G,n}$ induced by H.

Obviously, the homomorphism p^* induced by p gives us an isomorphism of $H^r(K; \mathfrak{G})$ onto $H^r(A; \mathfrak{G})$. We put $\phi^*(1) = U_n \in H^n(A'; \mathfrak{G})$ where 1 is the unit class of the cohomology ring $H^*(K; \mathfrak{G})$. Let β^* be the homomorphism of $H^n(A'; \mathfrak{G}) = H^n(A, S; \mathfrak{G})$ into $H^n(A; \mathfrak{G})$ induced by the inclusion map β of $(A, 0)$ into (A, S). The cohomology class $W_n = p^{*-1}\beta^* U_n$ in $H^n(K; \mathfrak{G})$ is called the *fundamental characteristic class* of the $(n-1)$-sphere fiber space S. In below, we shall denote $p^* W_n$ by the same symbol W_n.

The isomorphism ϕ^* has the following important property(4),

$$\phi^*(x \cup y) = p^*x \cup \phi^*y \tag{4}$$

(3) We remark the fact that if V is orientable, then duality means the Poincaré duality, i.e., the isomorphism of the homology group $H_{m-n}(V; Z)$ onto the cohomology group $H^n(V; Z)$, and if V is nonorientable, then we take the coefficient group Z_2 and duality means the isomorphism of Poincaré-Veblen, i.e., the isomorphism of $H_{m-n}(V; Z_2)$ onto $H^n(V; Z_2)$.

(4) The cup-product is defined by the multiplication among integers.

for cohomology classes x, y in $H^*(K; \mathfrak{G})$. If we put $y=1$, then the formula (4) gives us

$$\phi^*(x) = p^*x \cup U_n \tag{5}$$

for every class x in $H^*(K; \mathfrak{G})$. Also ϕ^* commutes with the Steenrod square operations.

The *Stiefel-Whitney class* W_j of dimension j $(0 \leqq j \leqq n)$, is defined as the fundamental characteristic class of the associated $(j-1)$-sphere fiber space over the j-skeleton K_j of K.

THEOREM 2. *The Stiefel-Whitney class W_j of the $(n-1)$-sphere fiber space S satisfies the following formula,*

$$Sq^j U_n = U_n \cup W_j \qquad \text{for } 0 \leqq j \leqq n \tag{6}$$

where Sq^j denotes the Steenrod square operation which raises dimension by j.

The proof of this theorem can be found in [2, Chap. II, Theorem II].

We remark that the relation (6) leads to the topological invariance of the Stiefel-Whitney classes of compact differentiable manifolds.

In the end of this section, we mention a property of the canonical isomorphism for *double fiber structures*. Let A'' be a fiber structure over A' with an open cell as fiber. ϕ'^* denotes the canonical isomorphism for A'' over A'. Obviously, A'' is a fiber structure over K with the open cell which is product cell of the fiber of A' over K and that of A'' over A'. ϕ''^* denotes the isomorphism for A'' over K, then we have the relation

$$\phi''^* = \phi'^*\phi^*. \tag{7}$$

3. Cup-products of realizable cohomology classes.

THEOREM 3. *Suppose that K is a finite cell complex and that $u_1 \in H^{k_1}(K; \mathfrak{G})$ and $u_2 \in H^{k_2}(K; \mathfrak{G})$. If u_1 and u_2 are $O(k_1)$- and $O(k_2)$-realizable, respectively, then the cup-product*(5) *$u_1 \cup u_2$ is $O(k_1+k_2)$-realizable.*

Proof. By Definition 2, there exists two mappings, $f_1: K \rightarrow M(O(k_1))$ and $f_2: K \rightarrow M(O(k_2))$ having the following properties. For the sake of brevity, we use the notations U_1, U_2 instead of $U_{O(k_1)}$, $U_{O(k_2)}$. Then the relations,

$$\overset{*}{f_1} U_1 = u_1, \quad \overset{*}{f_2} U_2 = u_2$$

hold.

Now we shall define a mapping $\overline{H}$ of the product space $M(O(k_1)) \times M(O(k_2))$ into the cell complex $M(O(k_1+k_2))$ as follows: $A_{O(k_1)} \times A_{O(k_2)}$ is a fiber space over the product space $B_{O(k_1)} \times B_{O(k_2)}$ with fiber, the (k_1+k_2)-cell $b_1 \times b_2$ and with structural group, the orthogonal group $O(k_1+k_2)$. Since

(5) The cup-product is defined by the multiplication of the ring $\mathfrak{G}$. Further, Theorem 3 holds generally in the case where the coefficient group is any ring.

$O(k_1+k_2)$ contains the group $O(k_1)\times O(k_2)$, we can take the universal fiber space $A_{O(k_1+k_2)}$ over $B_{O(k_1+k_2)}$ and a classifying map h of $B_{O(k_1)}\times B_{O(k_2)}$ into $B_{O(k_1+k_2)}$ which induces the fiber space $A_{O(k_1)}\times A_{O(k_2)}$ from $A_{O(k_1+k_2)}$. Let H be the mapping of $A_{O(k_1)}\times A_{O(k_2)}$ into $A_{O(k_1+k_2)}$ induced by the classifying map h. Since H maps the boundary of $A_{O(k_1)}\times A_{O(k_2)}$ into that of $A_{O(k_1+k_2)}$, H defines a continuous mapping $\overline{H}$ of the product cell complex $M(O(k_1))\times M(O(k_2))$ into the complex $M(O(k_1+k_2))$.

Let j_1 and j_2 be the projections of $M(O(k_1))\times M(O(k_2))$ onto $M(O(k_1))$ and $M(O(k_2))$ respectively, defined by the formulae,

$$j_1(x, y) = x, \qquad j_2(x, y) = y$$

for each point $(x, y)\in M(O(k_1))\times M(O(k_2))$. Similarly, we define the projections j_1' and j_2' of $A'_{O(k_1)}\times A'_{O(k_2)}$ onto $A'_{O(k_1)}$ and $A'_{O(k_2)}$ respectively. Let U_1' and U_2' be the fundamental classes in the cohomology groups $H^{k_1}(A'_{O(k_1)}; \mathfrak{G})$ and $H^{k_2}(A'_{O(k_2)}; \mathfrak{G})$ which correspond naturally to U_1 and U_2.

We consider a cellular subdivision of the space $A'_{O(k_1)}$ as follows: We can suppose a simplicial subdivision of the base space $B_{O(k_1)}$ which satisfies the condition that each simplex is contained in a coordinate neighborhood. We take product cells of such simplexes with the k_1-cell of the fiber. These cells make a cellular subdivision of $A'_{O(k_1)}$. In the similar way, we can suppose such cellular subdivisions of the space $A'_{O(k_2)}$ and of the space $A'_{O(k_1+k_2)}$. Then we have:

LEMMA 1. *Under the above conditions, the relation*

$$H'^{*}U'_{O(k_1+k_2)} = j_1'^{*}U_1' \cup j_2'^{*}U_2' \tag{8}$$

holds, where H' is the mapping of $A'_{O(k_1)}\times A'_{O(k_2)}$ into $A'_{O(k_1+k_2)}$ induced by H and $U'_{O(k_1+k_2)}$ is the fundamental class in $H^{k_1+k_2}(A'_{O(k_1+k_2)}; \mathfrak{G})$ which correspond naturally to $U_{O(k_1+k_2)}$.

Proof. $A'_{O(k_1)}\times A'_{O(k_2)}$ has a double fiber structure, $A'_{O(k_1)}\times B_{O(k_2)}$ over $B_{O(k_1)}\times B_{O(k_2)}$ and $A'_{O(k_1)}\times A'_{O(k_2)}$ over $A'_{O(k_1)}\times B'_{O(k_2)}$. Let ϕ_1^*, ϕ_2^* denote the canonical isomorphisms for these fiber structures, respectively, and let ϕ^* denote that for $A'_{O(k_1)}\times A'_{O(k_2)}$ over $B_{O(k_1)}\times B_{O(k_2)}$. (7) gives us the relation,

$$\phi^* = \phi_2^*\phi_1^*.$$

By using the commutativity of (3), we have that[6]

$$\begin{aligned} H'^{*}U'_{O(k_1+k_2)} &= H'^{*}\phi^*_{O(k_1+k_2)}(1_{O(k_1+k_2)}) = \phi^*(1) \\ &= \phi_2^*\phi_1^*(1) = \phi_2^*(j_1'^{*}U_1') \\ &= j_1'^{*}U_1' \cup j_2'^{*}U_2'. \end{aligned}$$

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(6) For simplicity, we denote the canonical isomorphism $\phi^*_{O(k_1+k_2),n}$ by $\phi^*_{O(k_1+k_2)}$, if $n = k_1 + k_2$.

Thus Lemma 1 has proved.

We have a diagram of the homomorphisms $\overline{H}^*$, H'^*, $j_1^*\cup j_2^*$, $j_1'^*\cup j_2'^*$ and if the isomorphisms of the cohomology groups induced by identifying the boundaries, $S_{O(k_i)}$ in $A_{O(k_i)}$ $(i=1, 2)$ and $S_{O(k_1+k_2)}$ in $A_{O(k_1+k_2)}$ to a point, respectively,

$$\begin{array}{ccc} H^{k_1}(A'_{O(k_1)}; \mathfrak{G}) \times H^{k_2}(A'_{O(k_2)}; \mathfrak{G}) & = & H^{k_1}(M(O(k_1)); \mathfrak{G}) \times H^{k_2}(M(O(k_2)); \mathfrak{G}) \\ j_1'^* \cup j_2'^* \downarrow & & \downarrow j_1^* \cup j_2^* \\ H^{k_1+k_2}(A'_{O(k_1)} \times A'_{O(k_2)}; \mathfrak{G}) & \longrightarrow & H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathfrak{G}) \\ H'^* \uparrow & & \uparrow \overline{H}^* \\ H^{k_1+k_2}(A'_{O(k_1+k_2)}; \mathfrak{G}) & = & H^{k_1+k_2}(M(O(k_1+k_2)); \mathfrak{G}), \end{array}$$

where the horizontal arrow is a canonical homomorphism of cohomology groups which is induced by the inclusion mapping of pairs of spaces,

$$(A_{O(k_1)} \times A_{O(k_2)}, 0) \rightarrow (A_{O(k_1)} \times A_{O(k_2)}, (A_{O(k_1)} \times A_{O(k_2)})^{\cdot}).$$

Since the above diagram is commutative, the relation (8) leads to the formula,

$$(8') \qquad \overline{H}^* U_{O(k_1+k_2)} = j_1^* U_1 \cup j_2^* U_2,$$

which plays an important roll in the proof of Theorem 3.

Now we define a mapping $f_1 * f_2$ from the cell complex K into the product cell complex $M(O(k_1)) \times M(O(k_2))$ by the formula,

$$f_1 * f_2(a) = f_1(a) \times f_2(a), \qquad a \in K.$$

Then this mapping induces a homomorphism $(f_1 * f_2)^*$ of the cohomology group

$$H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathfrak{G})$$

into the cohomology group $H^{k_1+k_2}(K; \mathfrak{G})$. Using the property that these induced homomorphisms and the cup-products are commutative, we have the relation,

$$(9) \qquad \begin{aligned} (f_1 * f_2)^* (j_1^* U_1 \cup j_2^* U_2) &= f_1^* U_1 \cup f_2^* U_2 \\ &= u_1 \cup u_2, \end{aligned}$$

because it is easily seen that

$$\begin{aligned} f_1 &= j_1(f_1 * f_2), \\ f_2 &= j_2(f_1 * f_2). \end{aligned}$$

We put $f = \overline{H}(f_1 * f_2)$. This is a continuous mapping of K into $M(O(k_1+k_2))$. Combining the formulae (8′) and (9), we obtain the relation,

$$
\begin{aligned}
f^*U_{O(k_1+k_2)} &= (f_1 * f_2)^* H^* U_{O(k_1+k_2)} \\
&= (f_1 * f_2)^* (j_1^* U_1 \cup j_2^* U_2) \qquad (10) \\
&= u_1 \cup u_2.
\end{aligned}
$$

The last formula (10) means that the class $u_1 \cup u_2$ has an $O(k_1+k_2)$-realization.

4. **Applications of Theorem 3.** When the cell complex K stated above is an n dimensional differentiable manifold V of class C^∞, we can use Theorem 1. Then we have the following result.

COROLLARY 1. *Let V be an n dimensional differentiable manifold of class C^∞. Let z_1, z_2 be homology classes of respective dimension $n-k_1$, $n-k_2$ in V, where we suppose that $k_1, k_2 \leqq n$. We take the integers or the integers modulo 2 as coefficients and take only the latter when V is nonorientable. If the classes z_1, z_2 are realized by submanifolds, then their intersection class $z_1 \cdot z_2$ is also realized by a submanifold.*

Proof. The homology class $z_1 \cdot z_2$ is dual to the cohomology class $u_1 \cup u_2$ where the class u_i is the dual of z_i $(i=1, 2)$ (H. Whitney [6]). This fact together with Theorem 1 and Theorem 3 yield Corollary 1.

Theorem 3 shows the existence of a mapping f of K into $M(O(k_1+k_2))$. We state this fact as Corollary 2.

COROLLARY 2. *If there exist continuous mappings f_1, f_2 of the cell complex K into the cell complexes $M(O(k_1))$ and $M(O(k_2))$ which satisfy the condition that $f_1^* U_1 = u_1$, $f_2^* U_2 = u_2$. Then there exists a continuous mapping f of K into $M(O(k_1+k_2))$ such that $f^* U_{O(k_1+k_2)} = u_1 \cup u_2$.*

So far we have proved that in a compact differentiable manifold the cup-product of two realizable classes is realizable. In the next sections, we shall study the Steenrod square operations of realizable cohomology classes.

5. **Squares of classes of n dimension having $O(k)$-realization.** In this section, we shall state theorems on the realizability of Steenrod square of n dimensional cohomology classes which are realizable for $O(k) \subset O(n)$.

Now we denote by V a compact differentiable manifold of dimension $m+n$ and denote by W a submanifold in V of dimension m. Let $z \in H_m(V; Z_2)$ be the homology class defined by W and $u \in H^n(V; Z_2)$ be the cohomology class which is dual to z. Then the following is the main theorem of this note.

THEOREM 4. *If the fiber space $N(W)$ of normal vectors over the submanifold W in V has a field of $(n-k)$-linearly independent vectors where we suppose that $k \leqq n$, then the square $Sq^k(u)$ of the class u can be also realized.*

The condition of Theorem 4 is satisfied if and only if the fiber space $N(W)$ has $O(k)$ as its structural group. So, we can state the above fact simply as follows:

THEOREM 5. *If $u \in H^n(V; Z_2)$ is realizable for $O(k) \subset O(n)$, then $Sq^k(u)$ can be also realized.*

REMARK. If the Stiefel-Whitney class $W_k \in H^k(W; Z_2)$ is realizable, then Theorem 5 follows from Theorem 3 immediately. In the following, however, we shall give the proof without the fact.

In order to prove Theorem 5, we state some preliminary results. By Theorem 1 there exists a mapping f of V into $M(O(k), n)$ such that the homomorphism f^* of $H^n(M(O(k), n); Z_2)$ into $H^n(V; Z_2)$ induced by f satisfies the condition that

$$u = f^* U_{O(k),n}.$$

Next we consider the structure of the fiber space $A_{O(k),n}$ over $B_{O(k)}$.

LEMMA 2. *The fiber structure $A_{O(k),n}$ decomposes into the product of a closed $(n-k)$-cell $\bar{b}_{n-k}$ and $A_{O(k)}$.*

Proof. We can choose a system of coordinate transformations for the fiber space $A_{O(k),n}$, any transformation of which leaves certain $n-k$ coordinates fixed in the fiber, a *closed* n-cell $\bar{b}_n$.

6. **The proof of Theorem 5.** By Lemma 2 we can define a mapping of $A_{O(k),n}$ onto $A_{O(k)}$ which collapses the *closed* $(n-k)$-cell of fiber into a point. Let q be such a map. On the other hand, we denote the identity mapping of $A_{O(k),n}$ onto itself by α. We can define a mapping $\alpha \oplus q$ of $A_{O(k),n}$ into the Whitney sum $A_{O(k),n} \oplus A_{O(k)}$ (see Wu [7]) in such a way that

$$\alpha \oplus q(x) = (\alpha(x), q(x)), \tag{11}$$

for each point x of $A_{O(k),n}$. $A_{O(k),n} \oplus A_{O(k)}$ is a fiber space over $B_{O(k)}$ with $\bar{b}_n \times \bar{b}_k$ as its fiber. Let H denote a fiber mapping of $A_{O(k),n} \oplus A_{O(k)}$ into the universal fiber space over $A_{O(n+k)}$ which is induced by a classifying map. We denote by M the Whitney sum $A_{O(k),n} \oplus A_{O(k)}$ the boundary of which is identified into a point. Let $\bar{H}$ be the mapping of M into $M(O(n+k))$ induced by H. Since $\alpha \oplus q$ maps $S_{O(k),n}$ into the boundary $(A_{O(k),n} \oplus A_{O(k)})^{\cdot}$, it induces a mapping $[\alpha \oplus q]$ of $M(O(k)), n)$ into M. Then we get the following diagram of mappings of spaces,

$$\begin{tikzcd}
 & A_{O(k),n} \arrow[r, "\alpha \oplus q"] \arrow[d, "i"] & A_{O(k),n} \oplus A_{O(k)} \arrow[r, "H"] \arrow[d, "i"] & A_{O(n+k)} \arrow[d, "i"] \\
V \arrow[r, "f"] & M(O(k), n) \arrow[r, "{[\alpha \oplus q]}"] & M \arrow[r, "\bar{H}"] & M(O(n+k))
\end{tikzcd} \tag{12}$$

where the vertical arrow i shows the respective identifying map. In each square, commutativity holds.

The diagram (12) induces the following diagram of homomorphisms of cohomology groups and commutativity holds in each square;

$$
(13)\qquad
\begin{array}{ccccccc}
 & & H^*(A'_{O(k),n}; Z_2) & \xleftarrow{(\alpha\oplus q)^*} & H^*(A'_{O(k),n}\oplus A'_{O(k)}; Z_2) & \xleftarrow{H'^*} & H^*(A'_{O(n+k)}; Z_2) \\
 & & \| & & \| & & \| \\
H^*(V; Z_2) & \xleftarrow{f^*} & H^*(M(O(k), n); Z_2) & \xleftarrow{[\alpha\oplus q]^*} & H^*(M; Z_2) & \xleftarrow{\overline{H}^*} & H^*(M(O(n+k)); Z_2),
\end{array}
$$

where H' is the mapping induced by H.

Let U'_i, $i=k$, n, $n+k$, be the fundamental classes of $A'_{O(k)}$, $A'_{O(k),n}$, $A'_{O(n+k)}$ respectively and let U_i be the fundamental classes of $M(O(k))$, $M(O(k), n)$, $M(O(n+k))$ which correspond canonically to U_i respectively. Let j'_1, j'_2 denote the projections of $A'_{O(k),n}\oplus A'_{O(k)}$ onto $A'_{O(k),n}$ and onto $A'_{O(k)}$ respectively.

By the same argument as Lemma 1, we have the relation,

$$(14)\qquad H'^* U'_{n+k} = j_1'^* U'_n \cup j_2'^* U'_k.$$

From definitions of $\alpha\oplus q$ and of j'_i, we get the formulae,

$$j'_1(\alpha \oplus q) = \alpha,$$
$$j'_2(\alpha \oplus q) = q.$$

They lead to the relations,

$$(\alpha \oplus q)^* j_1'^* = \text{the identity isomorphism},$$
$$(\alpha \oplus q)^* j_2'^* = q^*.$$

Using these formulae together with (14), we obtain the result,

$$
\begin{aligned}
(\alpha \oplus q)^* H'^* U'_{n+k} &= (\alpha \oplus q)^*(j_1'^* U'_n \cup j_2'^* U'_k) \\
&= (\alpha \oplus q)^* j_1'^* U'_n \cup (\alpha \oplus q)^* j_2'^* U'_k \\
&= U'_n \cup q^* U'_k.
\end{aligned}
$$

By definition of the Stiefel-Whitney class of dimension k, we have $W_k = \beta^* q^* U'_k$. Hence we obtain the following,

$$
\begin{aligned}
U'_n \cup q^* U'_k &= U'_n \cup \beta^* q^* U'_k \\
&= U'_n \cup W_k \\
&= Sq^k U'_n.
\end{aligned}
$$

We have, therefore, the relation

$$(15)\qquad (\alpha \oplus q)^* H'^* U'_{n+k} = Sq^k U'_n.$$

(13) and (15) show that

$$(16)\qquad [\alpha \oplus q]^* \overline{H}^* U_{n+k} = Sq^k U_n,$$

where

$$V \xrightarrow{f} M(O(k), n) \xrightarrow{[\alpha + q]} M \xrightarrow{\overline{H}} M(O(n+k)).$$

If we take the composite mapping $\overline{H}[\alpha \oplus q]f$ of V into $M(O(n+k))$, we obtain, by (16), the result that

$$\begin{aligned}(\overline{H}[\alpha \oplus q]f)^* U_{n+k} &= f^*[\alpha \oplus q]^* \overline{H}^* U_{n+k} \\ &= f^* Sq^k U_n \\ &= Sq^k f^* U_n \\ &= Sq^k(u).\end{aligned}$$

Thus the result of Theorem 5 is completely proved.

7. **A general result.** Combining Theorem 3 and Theorem 5, we have a general result on the realizability of cohomology classes generated by cup-products and Steenrod square operations of realizable classes:

THEOREM 6. *Let U_i, $0 \leqq i \leqq r$, be cohomology classes of dimension n_i which are dual to homology classes determined by submanifolds in a compact differentiable manifold V. Suppose each U_i is realizable for $O(k_i) \subset O(n_i)$. Then the cohomology class*

$$Sq^{k_1} U_1 \cup Sq^{k_2} U_2 \cup \cdots \cup Sq^{k_r} U_r$$

can be realized by a submanifold in V.

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