

ON THE REALIZATION OF HOMOLOGY CLASSES BY SUBMANIFOLDS

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1. Introduction. In [1] R. Thom defines when a cohomology class in a manifold is realizable for a closed subgroup of an orthogonal group. He also defines when a homology class is realizable by a submanifold. He then shows the following.

Let m be the dimension of a compact orientable differentiable manifold M . Any cohomology class of dimension 1, 2, $m-6$, $m-5$, \dots , m is realizable for the orthogonal group $O(1) = 1$, $O(2)$, $O(m-6)$, $O(m-5)$, \dots , $O(m)$ respectively. If a class $z \in H_{m-k}(M; Z)$ can be realized by a submanifold, then the cohomology class $u \in H^k(M; Z)$ which is dual to z satisfies

$$St_p^{2r(p-1)+1}(u) = 0$$

for all integers r and all odd primes p . Here $St_p^{2r(p-1)+1}$ denotes the Steenrod reduced power [4] which operates on the cohomology group with integral coefficients Z . All homology classes with integral coefficients of compact orientable differentiable manifolds of dimension < 10 are realizable by submanifolds.

We consider compact differentiable manifolds which are not necessarily orientable, and we ask whether the cup-products and the Steenrod squares [3] of realizable cohomology classes can be realized. It was stated by Thom [1] that cup-products of realizable classes are also realizable. We shall give another proof of this result (see §3 below). As for squares of realizable classes with integers modulo 2 as coefficients, we have the following result. Let u be a cohomology class of dimension n in a compact differentiable manifold of dimension $m+n$. If u is realizable for the group $O(k) \subset O(n)$ ($k \leq n$), then the cohomology class $Sq^k(u)$ is also realizable (see §5 below).

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2. Preliminaries. In this section, we summarize the theory of Thom [1 and 2]. Let ⁽¹⁾ G be a closed subgroup of the orthogonal group $O(n)$. Let S denote an $(n-1)$ -sphere fiber space over a finite cell complex K with the structural group G and let A be the mapping cylinder of the projection of S onto K . A becomes a fiber space over K with a closed n -cell b_n as fiber. Its projection is induced by that of S . We denote by A' the complement of S in A which is a fiber space over K with fiber, an open n -cell b_n .

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(¹) If G is a compact Lie group, then there is an integer n such that G is isomorphic to a subgroup of $O(n)$.

Throughout this paper, \mathbb{G} will denote the group of integers or integers modulo 2. When dealing with the nonorientable case⁽²⁾ \mathbb{G} will denote only the group of integers modulo 2.

By the theory of fiber spaces with open cells as fibers, we have a canonical isomorphism ϕ^* of the r dimensional cohomology group $H^r(K; \mathbb{G})$ onto the $r+n$ dimensional cohomology group $H^{r+n}(A'; \mathbb{G}) = H^{r+n}(A, S; \mathbb{G})$. $H^n(A, S; \mathbb{G})$ is the first nonvanishing cohomology group.

We denote by $S_{G,n}$ and B_G the universal fiber space and the classifying space respectively, for $(n-1)$ -sphere fiber spaces over finite cell complexes of bounded dimension with the structural group G (see [5]). We may assume B_G is a grassmann manifold. Let $A_{G,n}$ be the mapping cylinder of the projection of $S_{G,n}$ onto B_G and let $A'_{G,n}$ be the complement of $S_{G,n}$ in $A_{G,n}$. In particular, we denote $S_{O(n),n}$, $A_{O(n),n}$ and $A'_{O(n),n}$ by $S_{O(n)}$, $A_{O(n)}$ and $A'_{O(n)}$ for the sake of brevity.

DEFINITION 1. Let $M(G, n)$ be the space which we get from $A_{G,n}$ by identifying its boundary $S_{G,n}$ to a point. We call the space $M(G, n)$ the *cell complex corresponding to the subgroup G of O(n)*. We denote $M(O(n), n)$ simply by $M(O(n))$.

We have the natural isomorphisms of cohomology groups,

$$H^r(A'_{G,n}; \mathbb{G}) = H^r(A_{G,n}, S_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})$$

where $r > 0$. It is quite easy to see that $H^0(M(G, n); \mathbb{G}) = \mathbb{G} \phi_{G,n}^*$ denotes the canonical isomorphism in the fiber structure $A'_{G,n}$,

$$\phi_{G,n}^*: H^{r-n}(B_G; \mathbb{G}) \approx H^r(A'_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})$$

where $r \geq n$. Therefore the cohomology groups $H^r(M(G, n); \mathbb{G})$ of dimensions $r \geq n$ are obtained by raising dimensions of cohomology groups of the classifying space B_G by the integer n . $H^n(M(G, n); \mathbb{G})$ is the first cohomology group which does not vanish in dimension > 0 . This group is generated by the class,

$$(1) \quad U_{G,n} = \phi_{G,n}^*(1_G) \in H^n(M(G, n); \mathbb{G})$$

where 1_G is the unit class of $H^*(B_G; \mathbb{G})$. We call the class $U_{G,n}$ the *fundamental class* of $M(G, n)$. We denote $U_{O(n),n}$ simply by $U_{O(n)}$.

DEFINITION 2. Let A be a topological space and let u be a class of $H^n(A; \mathbb{G})$. u is said to be *realizable for G ⊂ O(n)* or *has G-realization*, if there is a map $f: A → M(G, n)$ such that $u = f^* U_{G,n}$.

DEFINITION 3. Let W be a submanifold of dimension p in a compact differentiable manifold V of dimension m and of class C^∞ , where p is an integer such that $m \geq p \geq 0$. The inclusion mapping $i: W ⊂ V$ induces a homomorphism i_* of the homology group $H_p(W; \mathbb{G})$ into the homology group $H_p(V; \mathbb{G})$.

(2) Orientability means that of the fiber space S , therefore, of A . In this case, G is connected.

If a homology class z of $H_p(W; \mathfrak{G})$ is the image of the fundamental class of W , then we say that the class z is *realized by the submanifold W* .

Now we state the following fundamental theorem [1, Chap. II]:

THEOREM 1. *Let V be a compact differentiable manifold of dimension m and of class C^∞ , and let n be an integer such that $m \geq n \geq 0$. A cohomology class $u \in H^n(V; \mathfrak{G})$ is realizable for the group $G \subset O(n)$ if and only if the dual homology class⁽³⁾ z of u is realized by a submanifold W of dimension $m-n$ and the fiber space consisting of normal vectors on the submanifold W in V has the group G as its structural group.*

Let h be a classifying map of K into B_G for the fiber space S . Then it induces a fiber mapping H of A into $A_{G,n}$. Commutativity holds in the diagram,

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{H} & A_{G,n} \\ p \downarrow & & \downarrow p_{G,n} \\ K & \xrightarrow{h} & B_G \end{array}$$

where p and $p_{G,n}$ are the projections of the fiber structures A and $A_{G,n}$ which are naturally induced by the projections $S \rightarrow K$ and $S_{G,n} \rightarrow B_G$. The commutativity of (2) induces that of the following diagram of homomorphisms of cohomology groups,

$$(3) \quad \begin{array}{ccc} H^n(A'; \mathfrak{G}) & \xleftarrow{H'^*} & H^n(A'_{G,n}; \mathfrak{G}) \\ \phi^* \uparrow & & \uparrow \phi_{G,n}^* \\ H^0(K; \mathfrak{G}) & \xleftarrow{h^*} & H^0(B_G; \mathfrak{G}) \end{array}$$

where H' is the mapping of A' into $A'_{G,n}$ induced by H .

Obviously, the homomorphism p^* induced by p gives us an isomorphism of $H^r(K; \mathfrak{G})$ onto $H^r(A; \mathfrak{G})$. We put $\phi^*(1) = U_n \in H^n(A'; \mathfrak{G})$ where 1 is the unit class of the cohomology ring $H^*(K; \mathfrak{G})$. Let β^* be the homomorphism of $H^n(A'; \mathfrak{G}) = H^n(A, S; \mathfrak{G})$ into $H^n(A; \mathfrak{G})$ induced by the inclusion map β of $(A, 0)$ into (A, S) . The cohomology class $W_n = p^{*-1}\beta^*U_n$ in $H^n(K; \mathfrak{G})$ is called the *fundamental characteristic class* of the $(n-1)$ -sphere fiber space S . In below, we shall denote p^*W_n by the same symbol W_n .

The isomorphism ϕ^* has the following important property⁽⁴⁾,

$$(4) \quad \phi^*(x \cup y) = p^*x \cup \phi^*y$$

(3) We remark the fact that if V is orientable, then duality means the Poincaré duality, i.e., the isomorphism of the homology group $H_{m-n}(V; Z)$ onto the cohomology group $H^n(V; Z)$, and if V is nonorientable, then we take the coefficient group Z_2 and duality means the isomorphism of Poincaré-Veblen, i.e., the isomorphism of $H_{m-n}(V; Z_2)$ onto $H^n(V; Z_2)$.

(4) The cup-product is defined by the multiplication among integers.

for cohomology classes x, y in $H^*(K; \mathfrak{G})$. If we put $y=1$, then the formula (4) gives us

$$(5) \quad \phi^*(x) = p^*x \cup U_n$$

for every class x in $H^*(K; \mathfrak{G})$. Also ϕ^* commutes with the Steenrod square operations.

The *Stiefel-Whitney class* W_j of dimension j ($0 \leq j \leq n$), is defined as the fundamental characteristic class of the associated $(j-1)$ -sphere fiber space over the j -skeleton K_j of K .

THEOREM 2. *The Stiefel-Whitney class W_j of the $(n-1)$ -sphere fiber space S satisfies the following formula,*

$$(6) \quad Sq^j U_n = U_n \cup W_j, \quad \text{for } 0 \leq j \leq n$$

where Sq^j denotes the Steenrod square operation which raises dimension by j .

The proof of this theorem can be found in [2, Chap. II, Theorem II].

We remark that the relation (6) leads to the topological invariance of the Stiefel-Whitney classes of compact differentiable manifolds.

In the end of this section, we mention a property of the canonical isomorphism for *double fiber structures*. Let A'' be a fiber structure over A' with an open cell as fiber. ϕ'^* denotes the canonical isomorphism for A'' over A' . Obviously, A'' is a fiber structure over K with the open cell which is product cell of the fiber of A' over K and that of A'' over A' . ϕ''^* denotes the isomorphism for A'' over K , then we have the relation

$$(7) \quad \phi''^* = \phi'^* \phi^*.$$

3. Cup-products of realizable cohomology classes.

THEOREM 3. *Suppose that K is a finite cell complex and that $u_1 \in H^{k_1}(K; \mathfrak{G})$ and $u_2 \in H^{k_2}(K; \mathfrak{G})$. If u_1 and u_2 are $O(k_1)$ - and $O(k_2)$ -realizable, respectively, then the cup-product⁽⁶⁾ $u_1 \cup u_2$ is $O(k_1+k_2)$ -realizable.*

Proof. By Definition 2, there exists two mappings, $f_1: K \rightarrow M(O(k_1))$ and $f_2: K \rightarrow M(O(k_2))$ having the following properties. For the sake of brevity, we use the notations U_1, U_2 instead of $U_{O(k_1)}, U_{O(k_2)}$. Then the relations,

$$f_1^* U_1 = u_1, \quad f_2^* U_2 = u_2$$

hold.

Now we shall define a mapping \bar{H} of the product space $M(O(k_1)) \times M(O(k_2))$ into the cell complex $M(O(k_1+k_2))$ as follows: $A_{O(k_1)} \times A_{O(k_2)}$ is a fiber space over the product space $B_{O(k_1)} \times B_{O(k_2)}$ with fiber, the (k_1+k_2) -cell $b_1 \times b_2$ and with structural group, the orthogonal group $O(k_1+k_2)$. Since

(6) The cup-product is defined by the multiplication of the ring \mathfrak{G} . Further, Theorem 3 holds generally in the case where the coefficient group is any ring.

$O(k_1+k_2)$ contains the group $O(k_1) \times O(k_2)$, we can take the universal fiber space $A_{O(k_1+k_2)}$ over $B_{O(k_1+k_2)}$ and a classifying map h of $B_{O(k_1)} \times B_{O(k_2)}$ into $B_{O(k_1+k_2)}$ which induces the fiber space $A_{O(k_1)} \times A_{O(k_2)}$ from $A_{O(k_1+k_2)}$. Let H be the mapping of $A_{O(k_1)} \times A_{O(k_2)}$ into $A_{O(k_1+k_2)}$ induced by the classifying map h . Since H maps the boundary of $A_{O(k_1)} \times A_{O(k_2)}$ into that of $A_{O(k_1+k_2)}$, H defines a continuous mapping \bar{H} of the product cell complex $M(O(k_1)) \times M(O(k_2))$ into the complex $M(O(k_1+k_2))$.

Let j_1 and j_2 be the projections of $M(O(k_1)) \times M(O(k_2))$ onto $M(O(k_1))$ and $M(O(k_2))$ respectively, defined by the formulae,

$$j_1(x, y) = x, \quad j_2(x, y) = y$$

for each point $(x, y) \in M(O(k_1)) \times M(O(k_2))$. Similarly, we define the projections j'_1 and j'_2 of $A'_{O(k_1)} \times A'_{O(k_2)}$ onto $A'_{O(k_1)}$ and $A'_{O(k_2)}$ respectively. Let U'_1 and U'_2 be the fundamental classes in the cohomology groups $H^{k_1}(A'_{O(k_1)}; \mathbb{G})$ and $H^{k_2}(A'_{O(k_2)}; \mathbb{G})$ which correspond naturally to U_1 and U_2 .

We consider a cellular subdivision of the space $A'_{O(k_1)}$ as follows: We can suppose a simplicial subdivision of the base space $B_{O(k_1)}$ which satisfies the condition that each simplex is contained in a coordinate neighborhood. We take product cells of such simplexes with the k_1 -cell of the fiber. These cells make a cellular subdivision of $A'_{O(k_1)}$. In the similar way, we can suppose such cellular subdivisions of the space $A'_{O(k_2)}$ and of the space $A'_{O(k_1+k_2)}$. Then we have:

LEMMA 1. Under the above conditions, the relation

$$(8) \quad H'^* U'_{O(k_1+k_2)} = j_1'^* U'_1 \cup j_2'^* U'_2$$

holds, where H' is the mapping of $A'_{O(k_1)} \times A'_{O(k_2)}$ into $A'_{O(k_1+k_2)}$ induced by H and $U'_{O(k_1+k_2)}$ is the fundamental class in $H^{k_1+k_2}(A'_{O(k_1+k_2)}; \mathbb{G})$ which correspond naturally to $U_{O(k_1+k_2)}$.

Proof. $A'_{O(k_1)} \times A'_{O(k_2)}$ has a double fiber structure, $A'_{O(k_1)} \times B_{O(k_2)}$ over $B_{O(k_1)} \times B_{O(k_2)}$ and $A'_{O(k_1)} \times A'_{O(k_2)}$ over $A'_{O(k_1)} \times B'_{O(k_2)}$. Let ϕ_1^* , ϕ_2^* denote the canonical isomorphisms for these fiber structures, respectively, and let ϕ^* denote that for $A'_{O(k_1)} \times A'_{O(k_2)}$ over $B_{O(k_1)} \times B_{O(k_2)}$. (7) gives us the relation,

$$\phi^* = \phi_2^* \phi_1^*.$$

By using the commutativity of (3), we have that⁽⁶⁾

$$\begin{aligned} H'^* U'_{O(k_1+k_2)} &= H'^* \phi_{O(k_1+k_2)}^*(1_{O(k_1+k_2)}) = \phi^*(1) \\ &= \phi_2^* \phi_1^*(1) = \phi_2^*(j_1'^* U'_1) \\ &= j_1'^* U'_1 \cup j_2'^* U'_2. \end{aligned}$$

(6) For simplicity, we denote the canonical isomorphism $\phi_{O(k_1+k_2), n}^*$ by $\phi_{O(k_1+k_2)}^*$, if $n = k_1 + k_2$.

Thus Lemma 1 has proved.

We have a diagram of the homomorphisms \bar{H}^* , H'^* , $j_1^* \cup j_2^*$, $j_1'^* \cup j_2'^*$ and if the isomorphisms of the cohomology groups induced by identifying the boundaries, $S_{O(k_i)}$ in $A_{O(k_i)}$ ($i=1, 2$) and $S_{O(k_1+k_2)}$ in $A_{O(k_1+k_2)}$ to a point, respectively,

$$\begin{array}{ccc} H^{k_1}(A'_{O(k_1)}; \mathbb{G}) \times H^{k_2}(A'_{O(k_2)}; \mathbb{G}) & = & H^{k_1}(M(O(k_1)); \mathbb{G}) \times H^{k_2}(M(O(k_2)); \mathbb{G}) \\ j_1'^* \cup j_2'^* \downarrow & & \downarrow j_1^* \cup j_2^* \\ H^{k_1+k_2}(A'_{O(k_1)} \times A'_{O(k_2)}; \mathbb{G}) & \rightarrow & H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathbb{G}) \\ H'^* \uparrow & & \uparrow \bar{H}^* \\ H^{k_1+k_2}(A'_{O(k_1+k_2)}; \mathbb{G}) & = & H^{k_1+k_2}(M(O(k_1+k_2)); \mathbb{G}), \end{array}$$

where the horizontal arrow is a canonical homomorphism of cohomology groups which is induced by the inclusion mapping of pairs of spaces,

$$(A_{O(k_1)} \times A_{O(k_2)}, 0) \rightarrow (A_{O(k_1)} \times A_{O(k_2)}, (A_{O(k_1)} \times A_{O(k_2)})^\circ).$$

Since the above diagram is commutative, the relation (8) leads to the formula,

$$(8') \quad \bar{H}^* U_{O(k_1+k_2)} = j_1^* U_1 \cup j_2^* U_2,$$

which plays an important role in the proof of Theorem 3.

Now we define a mapping $f_1 * f_2$ from the cell complex K into the product cell complex $M(O(k_1)) \times M(O(k_2))$ by the formula,

$$f_1 * f_2(a) = f_1(a) \times f_2(a), \quad a \in K.$$

Then this mapping induces a homomorphism $(f_1 * f_2)^*$ of the cohomology group

$$H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathbb{G})$$

into the cohomology group $H^{k_1+k_2}(K; \mathbb{G})$. Using the property that these induced homomorphisms and the cup-products are commutative, we have the relation,

$$(9) \quad \begin{aligned} (f_1 * f_2)^*(j_1^* U_1 \cup j_2^* U_2) &= f_1^* U_1 \cup f_2^* U_2 \\ &= u_1 \cup u_2, \end{aligned}$$

because it is easily seen that

$$\begin{aligned} f_1 &= j_1(f_1 * f_2), \\ f_2 &= j_2(f_1 * f_2). \end{aligned}$$

We put $f = \bar{H}(f_1 * f_2)$. This is a continuous mapping of K into $M(O(k_1+k_2))$. Combining the formulae (8') and (9), we obtain the relation,

$$\begin{aligned}
 f^*U_{O(k_1+k_2)} &= (f_1 * f_2)^*H^*U_{O(k_1+k_2)} \\
 (10) \quad &= (f_1 * f_2)^*(j_1^*U_1 \cup j_2^*U_2) \\
 &= u_1 \cup u_2.
 \end{aligned}$$

The last formula (10) means that the class $u_1 \cup u_2$ has an $O(k_1+k_2)$ -realization.

4. Applications of Theorem 3. When the cell complex K stated above is an n dimensional differentiable manifold V of class C^∞ , we can use Theorem 1. Then we have the following result.

COROLLARY 1. *Let V be an n dimensional differentiable manifold of class C^∞ . Let z_1, z_2 be homology classes of respective dimension $n - k_1, n - k_2$ in V , where we suppose that $k_1, k_2 \leq n$. We take the integers or the integers modulo 2 as coefficients and take only the latter when V is nonorientable. If the classes z_1, z_2 are realized by submanifolds, then their intersection class $z_1 \cdot z_2$ is also realized by a submanifold.*

Proof. The homology class $z_1 \cdot z_2$ is dual to the cohomology class $u_1 \cup u_2$ where the class u_i is the dual of z_i ($i = 1, 2$) (H. Whitney [6]). This fact together with Theorem 1 and Theorem 3 yield Corollary 1.

Theorem 3 shows the existence of a mapping f of K into $M(O(k_1+k_2))$. We state this fact as Corollary 2.

COROLLARY 2. *If there exist continuous mappings f_1, f_2 of the cell complex K into the cell complexes $M(O(k_1))$ and $M(O(k_2))$ which satisfy the condition that $f_1^*U_1 = u_1, f_2^*U_2 = u_2$. Then there exists a continuous mapping f of K into $M(O(k_1+k_2))$ such that $f^*U_{O(k_1+k_2)} = u_1 \cup u_2$.*

So far we have proved that in a compact differentiable manifold the cup-product of two realizable classes is realizable. In the next sections, we shall study the Steenrod square operations of realizable cohomology classes.

5. Squares of classes of n dimension having $O(k)$ -realization. In this section, we shall state theorems on the realizability of Steenrod square of n dimensional cohomology classes which are realizable for $O(k) \subset O(n)$.

Now we denote by V a compact differentiable manifold of dimension $m+n$ and denote by W a submanifold in V of dimension m . Let $z \in H_m(V; Z_2)$ be the homology class defined by W and $u \in H^n(V; Z_2)$ be the cohomology class which is dual to z . Then the following is the main theorem of this note.

THEOREM 4. *If the fiber space $N(W)$ of normal vectors over the submanifold W in V has a field of $(n-k)$ -linearly independent vectors where we suppose that $k \leq n$, then the square $Sq^k(u)$ of the class u can be also realized.*

The condition of Theorem 4 is satisfied if and only if the fiber space $N(W)$ has $O(k)$ as its structural group. So, we can state the above fact simply as follows:

THEOREM 5. If $u \in H^n(V; Z_2)$ is realizable for $O(k) \subset O(n)$, then $Sq^k(u)$ can be also realized.

REMARK. If the Stiefel-Whitney class $W_k \in H^k(W; Z_2)$ is realizable, then Theorem 5 follows from Theorem 3 immediately. In the following, however, we shall give the proof without the fact.

In order to prove Theorem 5, we state some preliminary results. By Theorem 1 there exists a mapping f of V into $M(O(k), n)$ such that the homomorphism f^* of $H^n(M(O(k), n); Z_2)$ into $H^n(V; Z_2)$ induced by f satisfies the condition that

$$u = f^* U_{O(k), n}.$$

Next we consider the structure of the fiber space $A_{O(k), n}$ over $B_{O(k)}$.

LEMMA 2. The fiber structure $A_{O(k), n}$ decomposes into the product of a closed $(n-k)$ -cell b_{n-k} and $A_{O(k)}$.

Proof. We can choose a system of coordinate transformations for the fiber space $A_{O(k), n}$, any transformation of which leaves certain $n-k$ coordinates fixed in the fiber, a closed n -cell b_n .

6. The proof of Theorem 5. By Lemma 2 we can define a mapping of $A_{O(k), n}$ onto $A_{O(k)}$ which collapses the closed $(n-k)$ -cell of fiber into a point. Let q be such a map. On the other hand, we denote the identity mapping of $A_{O(k), n}$ onto itself by α . We can define a mapping $\alpha \oplus q$ of $A_{O(k), n}$ into the Whitney sum $A_{O(k), n} \oplus A_{O(k)}$ (see Wu [7]) in such a way that

$$(11) \quad \alpha \oplus q(x) = (\alpha(x), q(x)),$$

for each point x of $A_{O(k), n}$. $A_{O(k), n} \oplus A_{O(k)}$ is a fiber space over $B_{O(k)}$ with $b_n \times b_k$ as its fiber. Let H denote a fiber mapping of $A_{O(k), n} \oplus A_{O(k)}$ into the universal fiber space over $A_{O(n+k)}$ which is induced by a classifying map. We denote by M the Whitney sum $A_{O(k), n} \oplus A_{O(k)}$ the boundary of which is identified into a point. Let \bar{H} be the mapping of M into $M(O(n+k))$ induced by H . Since $\alpha \oplus q$ maps $S_{O(k), n}$ into the boundary $(A_{O(k), n} \oplus A_{O(k)})^\circ$, it induces a mapping $[\alpha \oplus q]$ of $M(O(k), n)$ into M . Then we get the following diagram of mappings of spaces,

$$(12) \quad \begin{array}{ccccc} A_{O(k), n} & \xrightarrow{\alpha \oplus q} & A_{O(k), n} \oplus A_{O(k)} & \xrightarrow{H} & A_{O(n+k)} \\ f \downarrow & & i \downarrow & & \bar{H} \downarrow \\ V \rightarrow M(O(k), n) & \xrightarrow{[\alpha \oplus q]} & M & \xrightarrow{} & M(O(n+k)) \end{array}$$

where the vertical arrow i shows the respective identifying map. In each square, commutativity holds.

The diagram (12) induces the following diagram of homomorphisms of cohomology groups and commutativity holds in each square;

$$(13) \quad \begin{array}{c} H^*(A'_{O(k),n}; Z_2) \xleftarrow{(\alpha \oplus q)^*} H^*(A'_{O(k),n} \oplus A'_{O(k)}; Z_2) \xleftarrow{H'^*} H^*(A'_{O(n+k)}; Z_2) \\ \| \qquad \qquad \qquad \| \qquad \qquad \qquad \| \\ H^*(V; Z_2) \xleftarrow{f^*} H^*(M(O(k), n); Z_2) \xleftarrow{[\alpha \oplus q]^*} H^*(M; Z_2) \xleftarrow{\bar{H}^*} H^*(M(O(n+k)); Z_2), \end{array}$$

where H' is the mapping induced by H .

Let U'_i , $i = k, n, n+k$, be the fundamental classes of $A'_{O(k)}$, $A'_{O(n),n}$, $A'_{O(n+k)}$ respectively and let U_i be the fundamental classes of $M(O(k))$, $M(O(k), n)$, $M(O(n+k))$ which correspond canonically to U'_i respectively. Let j'_1, j'_2 denote the projections of $A'_{O(k),n} \oplus A'_{O(k)}$ onto $A'_{O(k),n}$ and onto $A'_{O(k)}$ respectively.

By the same argument as Lemma 1, we have the relation,

$$(14) \quad H'^* U'_{n+k} = j'^*_1 U'_n \cup j'^*_2 U'_k.$$

From definitions of $\alpha \oplus q$ and of j'_i , we get the formulae,

$$\begin{aligned} j'_1(\alpha \oplus q) &= \alpha, \\ j'_2(\alpha \oplus q) &= q. \end{aligned}$$

They lead to the relations,

$$\begin{aligned} (\alpha \oplus q)^* j'^*_1 &= \text{the identity isomorphism,} \\ (\alpha \oplus q)^* j'^*_2 &= q^*. \end{aligned}$$

Using these formulae together with (14), we obtain the result,

$$\begin{aligned} (\alpha \oplus q)^* H'^* U'_{n+k} &= (\alpha \oplus q)^*(j'^*_1 U'_n \cup j'^*_2 U'_k) \\ &= (\alpha \oplus q)^* j'^*_1 U'_n \cup (\alpha \oplus q)^* j'^*_2 U'_k \\ &= U'_n \cup q^* U'_k. \end{aligned}$$

By definition of the Stiefel-Whitney class of dimension k , we have $W_k = \beta^* q^* U'_k$. Hence we obtain the following,

$$\begin{aligned} U'_n \cup q^* U'_k &= U'_n \cup \beta^* q^* U'_k \\ &= U'_n \cup W_k \\ &= Sq^k U'_n. \end{aligned}$$

We have, therefore, the relation

$$(15) \quad (\alpha \oplus q)^* H'^* U'_{n+k} = Sq^k U'_n.$$

(13) and (15) show that

$$(16) \quad [\alpha \oplus q]^* \bar{H}^* U'_{n+k} = Sq^k U_n,$$

where

$$V \xrightarrow{f} M(O(k), n) \xrightarrow{[\alpha + q]} M \xrightarrow{\bar{H}} M(O(n+k)).$$

If we take the composite mapping $\bar{H}[\alpha \oplus q]f$ of V into $M(O(n+k))$, we obtain, by (16), the result that

$$\begin{aligned} (\bar{H}[\alpha \oplus q]f)^* U_{n+k} &= f^* [\alpha \oplus q]^* \bar{H}^* U_{n+k} \\ &= f^* Sq^k U_n \\ &= Sq^k f^* U_n \\ &= Sq^k(u). \end{aligned}$$

Thus the result of Theorem 5 is completely proved.

7. A general result. Combining Theorem 3 and Theorem 5, we have a general result on the realizability of cohomology classes generated by cup-products and Steenrod square operations of realizable classes:

THEOREM 6. *Let U_i , $0 \leq i \leq r$, be cohomology classes of dimension n_i which are dual to homology classes determined by submanifolds in a compact differentiable manifold V . Suppose each U_i is realizable for $O(k_i) \subset O(n_i)$. Then the cohomology class*

$$Sq^{k_1} U_1 \cup Sq^{k_2} U_2 \cup \dots \cup Sq^{k_r} U_r$$

can be realized by a submanifold in V .

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