ON THE REALIZATION OF HOMOLOGY CLASSES BY SUBMANIFOLDS

BY

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1. Introduction. In [1] R. Thom defines when a cohomology class in a manifold is realizable for a closed subgroup of an orthogonal group. He also defines when a homology class is realizable by a submanifold. He then shows the following.

Let \( m \) be the dimension of a compact orientable differentiable manifold \( M \). Any cohomology class of dimension \( 1, 2, m-6, m-5, \cdots, m \) is realizable for the orthogonal group \( O(1) = 1, O(2), O(m-6), O(m-5), \cdots, O(m) \) respectively. If a class \( z \in H_{m-k}(M; \mathbb{Z}) \) can be realized by a submanifold, then the cohomology class \( u \in H^k(M; \mathbb{Z}) \) which is dual to \( z \) satisfies

\[
S_l^{2r(p-1)+1}(u) = 0
\]

for all integers \( r \) and all odd primes \( p \). Here \( S_l^{2r(p-1)+1} \) denotes the Steenrod reduced power [4] which operates on the cohomology group with integral coefficients \( \mathbb{Z} \). All homology classes with integral coefficients of compact orientable differentiable manifolds of dimension \(<10\) are realizable by submanifolds.

We consider compact differentiable manifolds which are not necessarily orientable, and we ask whether the cup-products and the Steenrod squares [3] of realizable cohomology classes can be realized. It was stated by Thom [1] that cup-products of realizable classes are also realizable. We shall give another proof of this result (see §3 below). As for squares of realizable classes with integers modulo 2 as coefficients, we have the following result. Let \( u \) be a cohomology class of dimension \( n \) in a compact differentiable manifold of dimension \( m+n \). If \( u \) is realizable for the group \( O(k) \subset O(n) \) (\( k \leq n \)), then the cohomology class \( S_l^k(u) \) is also realizable (see §5 below).

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2. Preliminaries. In this section, we summarize the theory of Thom [1 and 2]. Let \( G \) be a closed subgroup of the orthogonal group \( O(n) \). Let \( S \) denote an \( (n-1) \)-sphere fiber space over a finite cell complex \( K \) with the structural group \( G \) and let \( A \) be the mapping cylinder of the projection of \( S \) onto \( K \). \( A \) becomes a fiber space over \( K \) with a closed \( n \)-cell \( b_n \) as fiber. Its projection is induced by that of \( S \). We denote by \( A' \) the complement of \( S \) in \( A \) which is a fiber space over \( K \) with fiber, an open \( n \)-cell \( b_n \).

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(1) If \( G \) is a compact Lie group, then there is an integer \( n \) such that \( G \) is isomorphic to a subgroup of \( O(n) \).
Throughout this paper, \( \mathbb{G} \) will denote the group of integers or integers modulo 2. When dealing with the nonorientable case \(^\text{(2)}\) \( \mathbb{G} \) will denote only the group of integers modulo 2.

By the theory of fiber spaces with open cells as fibers, we have a canonical isomorphism \( \phi^* \) of the \( r \) dimensional cohomology group \( H^r(K; \mathbb{G}) \) onto the \( r+n \) dimensional cohomology group \( H^{r+n}(A'_n; \mathbb{G}) = H^{r+n}(A, S; \mathbb{G}) \). \( H^n(A, S; \mathbb{G}) \) is the first nonvanishing cohomology group.

We denote by \( S_{G,n} \) and \( B_G \) the universal fiber space and the classifying space respectively, for \((n-1)\)-sphere fiber spaces over finite cell complexes of bounded dimension with the structural group \( G \) (see [5]). We may assume \( B_G \) is a grassmann manifold. Let \( A_{G,n} \) be the mapping cylinder of the projection of \( S_{G,n} \) onto \( B_G \) and let \( A'_{G,n} \) be the complement of \( S_{G,n} \) in \( A_{G,n} \). In particular, we denote \( S_{O(n),n} \) and \( A_{O(n),n} \) by \( S_{O(n)} \) and \( A_{O(n)} \) for the sake of brevity.

**DEFINITION 1.** Let \( M(G, n) \) be the space which we get from \( A_{G,n} \) by identifying its boundary \( S_{G,n} \) to a point. We call the space \( M(G, n) \) the cell complex corresponding to the subgroup \( G \) of \( O(n) \). We denote \( M(O(n), n) \) simply by \( M(O(n)) \).

We have the natural isomorphisms of cohomology groups,

\[
H^r(A'_{G,n}; \mathbb{G}) = H^r(A_{G,n}, S_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})
\]

where \( r > 0 \). It is quite easy to see that \( H^0(M(G, n); \mathbb{G}) = \mathbb{G} \phi^*_{G,n} \) denotes the canonical isomorphism in the fiber structure \( A'_{G,n} \),

\[
\phi^*_{G,n}: H^{r-n}(B_G; \mathbb{G}) \cong H^r(A'_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})
\]

where \( r \geq n \). Therefore the cohomology groups \( H^r(M(G, n); \mathbb{G}) \) of dimensions \( r \geq n \) are obtained by raising dimensions of cohomology groups of the classifying space \( B_G \) by the integer \( n \). \( H^n(M(G, n); \mathbb{G}) \) is the first cohomology group which does not vanish in dimension \( >0 \). This group is generated by the class,

\[
U_{G,n} = \phi^*_{G,n}(1_G) \in H^n(M(G, n); \mathbb{G})
\]

where \( 1_G \) is the unit class of \( H^*(B_G; \mathbb{G}) \). We call the class \( U_{G,n} \) the fundamental class of \( M(G, n) \). We denote \( U_{O(n),n} \) simply by \( U_{O(n)} \).

**DEFINITION 2.** Let \( A \) be a topological space and let \( u \) be a class of \( H^n(A; \mathbb{G}) \). \( u \) is said to be realizable for \( G \subset O(n) \) or has \( G \)-realization, if there is a map \( f: A \to M(G, n) \) such that \( u = f^* U_{G,n} \).

**DEFINITION 3.** Let \( W \) be a submanifold of dimension \( p \) in a compact differentiable manifold \( V \) of dimension \( m \) and of class \( C^\infty \), where \( p \) is an integer such that \( m \geq p \geq 0 \). The inclusion mapping \( i: W \subset V \) induces a homomorphism \( i_\* \) of the homology group \( H_p(W; \mathbb{G}) \) into the homology group \( H_p(V; \mathbb{G}) \).

\(^{(2)}\) Orientability means that of the fiber space \( S \), therefore, of \( A \). In this case, \( G \) is connected.
If a homology class \( z \) of \( H_p(W; \mathcal{O}) \) is the image of the fundamental class of \( W \), then we say that the class \( z \) is realized by the submanifold \( W \).

Now we state the following fundamental theorem [1, Chap. II]:

**Theorem 1.** Let \( V \) be a compact differentiable manifold of dimension \( m \) and of class \( C^\infty \), and let \( n \) be an integer such that \( m \geq n \geq 0 \). A cohomology class \( u \in H^n(V; \mathcal{O}) \) is realizable for the group \( G \subset O(n) \) if and only if the dual homology class\(^{(6)} \) \( \tilde{u} \) of \( u \) is realized by a submanifold \( W \) of dimension \( m-n \) and the fiber space consisting of normal vectors on the submanifold \( W \) in \( V \) has the group \( G \) as its structural group.

Let \( h \) be a classifying map of \( K \) into \( B_G \) for the fiber space \( S \). Then it induces a fiber mapping \( H \) of \( A \) into \( A_{G,n} \). Commutativity holds in the diagram,

\[
\begin{array}{ccc}
A & \to & A_{G,n} \\
\downarrow p & & \downarrow p_{G,n} \\
K & \to & B_G \\
\end{array}
\]

(2)

where \( p \) and \( p_{G,n} \) are the projections of the fiber structures \( A \) and \( A_{G,n} \) which are naturally induced by the projections \( S \to K \) and \( S_{G,n} \to B_G \). The commutativity of (2) induces that of the following diagram of homomorphisms of cohomology groups,

\[
\begin{array}{ccc}
H^n(A'; \mathcal{O}) & \to & H^n(A_{G,n}; \mathcal{O}) \\
\phi^* \uparrow & & \uparrow \phi_{G,n}^* \\
H^0(K; \mathcal{O}) & \to & H^0(B_G; \mathcal{O}) \\
\end{array}
\]

(3)

where \( H' \) is the mapping of \( A' \) into \( A_{G,n} \) induced by \( H \).

Obviously, the homomorphism \( p^* \) induced by \( p \) gives us an isomorphism of \( H^r(K; \mathcal{O}) \) onto \( H^r(A; \mathcal{O}) \). We put \( \phi^*(1) = U_n \in H^n(A'; \mathcal{O}) \) where \( 1 \) is the unit class of the cohomology ring \( H^*(K; \mathcal{O}) \). Let \( \beta^* \) be the homomorphism of \( H^n(A'; \mathcal{O}) = H^n(A, S; \mathcal{O}) \) into \( H^n(A; \mathcal{O}) \) induced by the inclusion map \( \beta \) of \( (A, 0) \) into \( (A, S) \). The cohomology class \( W_n = p^* - \beta^* U_n \) in \( H^n(K; \mathcal{O}) \) is called the fundamental characteristic class of the \((n-1)\)-sphere fiber space \( S \). In below, we shall denote \( p^* W_n \) by the same symbol \( W_n \).

The isomorphism \( \phi^* \) has the following important property\(^{(4)}\),

\[
\phi^*(x \cup y) = p^* x \cup \phi^* y
\]

\(^{(6)}\) We remark the fact that if \( V \) is orientable, then duality means the Poincaré duality, i.e., the isomorphism of the homology group \( H_{m-n}(V; \mathbb{Z}) \) onto the cohomology group \( H^n(V; \mathbb{Z}) \), and if \( V \) is nonorientable, then we take the coefficient group \( \mathbb{Z}_2 \) and duality means the isomorphism of Poincaré-Veblen, i.e., the isomorphism of \( H_{m-n}(V; \mathbb{Z}_2) \) onto \( H^n(V; \mathbb{Z}_2) \).

\(^{(4)}\) The cup-product is defined by the multiplication among integers.
for cohomology classes $x$, $y$ in $H^*(K; \mathcal{O})$. If we put $y = 1$, then the formula (4) gives us
\[ \phi^*(x) = p^*x \cup U_n \]
for every class $x$ in $H^*(K; \mathcal{O})$. Also $\phi^*$ commutes with the Steenrod square operations.

The Stiefel-Whitney class $W_j$ of dimension $j$ ($0 \leq j \leq n$), is defined as the fundamental characteristic class of the associated $(j-1)$-sphere fiber space over the $j$-skeleton $K_j$ of $K$.

**Theorem 2.** The Stiefel-Whitney class $W_j$ of the $(n-1)$-sphere fiber space $S$ satisfies the following formula,
\[ \text{Sq}^j U_n = U_n \cup W_j \]
for $0 \leq j \leq n$ where $\text{Sq}^j$ denotes the Steenrod square operation which raises dimension by $j$.

The proof of this theorem can be found in [2, Chap. II, Theorem II].

We remark that the relation (6) leads to the topological invariance of the Stiefel-Whitney classes of compact differentiable manifolds.

In the end of this section, we mention a property of the canonical isomorphism for *double fiber structures*. Let $A''$ be a fiber structure over $A'$ with an open cell as fiber. $\phi^*$ denotes the canonical isomorphism for $A''$ over $A'$. Obviously, $A''$ is a fiber structure over $K$ with the open cell which is product cell of the fiber of $A'$ over $K$ and that of $A''$ over $A'$. $\phi''^*$ denotes the isomorphism for $A''$ over $K$, then we have the relation
\[ \phi''^* = \phi'^* \phi^*. \]

**3. Cup-products of realizable cohomology classes.**

**Theorem 3.** Suppose that $K$ is a finite cell complex and that $u_1 \in H^k(K; \mathcal{O})$ and $u_2 \in H^k(K; \mathcal{O})$. If $u_1$ and $u_2$ are $O(k_1)$- and $O(k_2)$-realizable, respectively, then the cup-product $(\text{k})$ $u_1 \cup u_2$ is $O(k_1 + k_2)$-realizable.

**Proof.** By Definition 2, there exists two mappings, $f_1: K \to M(O(k_1))$ and $f_2: K \to M(O(k_2))$ having the following properties. For the sake of brevity, we use the notations $U_1$, $U_2$ instead of $U_{O(k_1)}$, $U_{O(k_2)}$. Then the relations,
\[ f_1^* U_1 = u_1, \quad f_2^* U_2 = u_2 \]
hold.

Now we shall define a mapping $\overline{H}$ of the product space $M(O(k_1)) \times M(O(k_2))$ into the cell complex $M(O(k_1 + k_2))$ as follows: $A_{O(k_1)} \times A_{O(k_2)}$ is a fiber space over the product space $B_{O(k_1)} \times B_{O(k_2)}$ with fiber, the $(k_1 + k_2)$-cell $b_1 \times b_2$ and with structural group, the orthogonal group $O(k_1 + k_2)$. Since

$(\text{k})$ The cup-product is defined by the multiplication of the ring $\mathcal{O}$. Further, Theorem 3 holds generally in the case where the coefficient group is any ring.
O(k_1+k_2) contains the group \( O(k_1) \times O(k_2) \), we can take the universal fiber space \( A_{0(k_1+k_2)} \) over \( B_{O(k_1+k_2)} \) and a classifying map \( h \) of \( B_{O(k_1)} \times B_{O(k_2)} \) into \( B_{O(k_1+k_2)} \) which induces the fiber space \( A_{0(k_1)} \times A_{0(k_2)} \) from \( A_{0(k_1+k_2)} \). Let \( H \) be the mapping of \( A_{0(k_1)} \times A_{0(k_2)} \) into \( A_{0(k_1+k_2)} \) induced by the classifying map \( h \). Since \( H \) maps the boundary of \( A_{0(k_1)} \times A_{0(k_2)} \) into that of \( A_{0(k_1+k_2)} \), \( H \) defines a continuous mapping \( H \) of the product cell complex \( M(O(k_1)) \times M(O(k_2)) \) into the complex \( M(O(k_1+k_2)) \).

Let \( j_1 \) and \( j_2 \) be the projections of \( M(O(k_1)) \times M(O(k_2)) \) onto \( M(O(k_1)) \) and \( M(O(k_2)) \) respectively, defined by the formulae,

\[
j_1(x, y) = x, \quad j_2(x, y) = y
\]

for each point \((x, y) \in M(O(k_1)) \times M(O(k_2))\). Similarly, we define the projections \( j'_1 \) and \( j'_2 \) of \( A'_0(k_1) \times A'_0(k_2) \) onto \( A'_0(k_1) \) and \( A'_0(k_2) \) respectively. Let \( U'_1 \) and \( U'_2 \) be the fundamental classes in the cohomology groups \( H^{k_1}(A'_0(k_1); \mathbb{Z}) \) and \( H^{k_2}(A'_0(k_2); \mathbb{Z}) \) which correspond naturally to \( U_1 \) and \( U_2 \).

We consider a cellular subdivision of the space \( A'_0(k_1) \) as follows: We can suppose a simplicial subdivision of the base space \( B_{O(k_1)} \) which satisfies the condition that each simplex is contained in a coordinate neighborhood. We take product cells of such simplexes with the \( k_1 \)-cell of the fiber. These cells make a cellular subdivision of \( A'_0(k_1) \). In the similar way, we can suppose such cellular subdivisions of the space \( A'_0(k_2) \) and of the space \( A'_0(k_1+k_2) \). Then we have:

**Lemma 1.** Under the above conditions, the relation

\[
H^* U'_{0(k_1+k_2)} = j'_1^* U'_1 \cup j'_2^* U'_2
\]

holds, where \( H' \) is the mapping of \( A'_0(k_1) \times A'_0(k_2) \) into \( A'_0(k_1+k_2) \) induced by \( H \) and \( U'_{0(k_1+k_2)} \) is the fundamental class in \( H^{k_1+k_2}(A'_0(k_1+k_2); \mathbb{Z}) \) which correspond naturally to \( U_{0(k_1+k_2)} \).

**Proof.** \( A'_0(k_1) \times A'_0(k_2) \) has a double fiber structure, \( A'_0(k_1) \times B_{O(k_2)} \) over \( B_{O(k_1)} \times B_{O(k_2)} \) and \( A'_0(k_1) \times A'_0(k_2) \) over \( A'_0(k_1) \times B'_{O(k_2)} \). Let \( \phi_1^*, \phi_2^* \) denote the canonical isomorphisms for these fiber structures, respectively, and let \( \phi^* \) denote that for \( A'_0(k_1) \times A'_0(k_2) \) over \( B_{O(k_1)} \times B_{O(k_2)} \). (7) gives us the relation,

\[
\phi^* = \phi_2^* \phi_1^*.
\]

By using the commutativity of (3), we have that\(^{(6)}\)

\[
H^* U'_{0(k_1+k_2)} = H^* \phi_0^*_{(k_1+k_2)}(1_{0(k_1+k_2)}) = \phi^*(1) = \phi_2^* \phi_1^*(1) = \phi_2^*(j_1^* U'_1) = j'_1^* U'_1 \cup j'_2^* U'_2.
\]

\(^{(6)}\) For simplicity, we denote the canonical isomorphism \( \phi_0^*_{(k_1+k_2)n} \) by \( \phi_0^*_{(k_1+k_2)} \) if \( n = k_1 + k_2 \).
Thus Lemma 1 has proved.

We have a diagram of the homomorphisms $\mathbf{H}^*, H^* (* = 1, 2)$ and if the isomorphisms of the cohomology groups induced by identifying the boundaries, $\mathcal{S}_{O(k_i)}$ in $A_{O(k_i)} (i = 1, 2)$ and $\mathcal{S}_{O(k_1+k_2)}$ in $A_{O(k_1+k_2)}$ to a point, respectively,

\[
\begin{align*}
H^{k_1}(A'_o(k_1); \mathcal{O}) \times H^{k_2}(A'_o(k_2); \mathcal{O}) & = H^{k_1}(M(O(k_1)); \mathcal{O}) \times H^{k_2}(M(O(k_2)); \mathcal{O}) \\
\downarrow_{j_1 \cup j_2} & \quad \downarrow_{j_1 \cup j_2} \\
H^{k_1+k_2}(A'_o(k_1) \times A'_o(k_2); \mathcal{O}) & \rightarrow H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathcal{O}) \\
\mathbf{H}^* & \uparrow \\
H^{k_1+k_2}(A'_o(k_1+k_2); \mathcal{O}) & = H^{k_1+k_2}(M(O(k_1+k_2)); \mathcal{O}),
\end{align*}
\]

where the horizontal arrow is a canonical homomorphism of cohomology groups which is induced by the inclusion mapping of pairs of spaces,

\[
(A_{O(k_1)}, A_{O(k_2)}, \mathcal{O}) \rightarrow (A_{O(k_1)} \times A_{O(k_2)}, (A_{O(k_1)} \times A_{O(k_2)})'.
\]

Since the above diagram is commutative, the relation (8) leads to the formula,

(8') \[
\mathbf{H}^* U_{O(k_1+k_2)} = j_1^* U_1 \cup j_2^* U_2,
\]

which plays an important roll in the proof of Theorem 3.

Now we define a mapping $f_1 \ast f_2$ from the cell complex $K$ into the product cell complex $M(O(k_1)) \times M(O(k_2))$ by the formula,

\[
f_1 \ast f_2(a) = f_1(a) \times f_2(a), \quad a \in K.
\]

Then this mapping induces a homomorphism $(f_1 \ast f_2)^*$ of the cohomology group

\[
H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathcal{O})
\]

into the cohomology group $H^{k_1+k_2}(K; \mathcal{O})$. Using the property that these induced homomorphisms and the cup-products are commutative, we have the relation,

(9) \[
(f_1 \ast f_2)^* (j_1^* U_1 \cup j_2^* U_2) = f_1^* U_1 \cup f_2^* U_2 = u_1 \cup u_2,
\]

because it is easily seen that

\[
f_1 = j_1(f_1 \ast f_2), \quad f_2 = j_2(f_1 \ast f_2).
\]

We put $f = \mathbf{H}(f_1 \ast f_2)$. This is a continuous mapping of $K$ into $M(O(k_1+k_2))$. Combining the formulae (8') and (9), we obtain the relation,
\[ f^*U_{O(k_1+k_2)} = (f_1 * f_2)^*H^*U_{O(k_1+k_2)} \]
\[ = (f_1 * f_2)^*(j_1^*U_1 \cup j_2^*U_2) \]
\[ = U_1 \cup U_2. \]

The last formula (10) means that the class \( U_1 \cup U_2 \) has an \( O(k_1+k_2) \) realization.

4. Applications of Theorem 3. When the cell complex \( K \) stated above is an \( n \) dimensional differentiable manifold \( V \) of class \( C^\infty \), we can use Theorem 1. Then we have the following result.

**Corollary 1.** Let \( V \) be an \( n \) dimensional differentiable manifold of class \( C^\infty \). Let \( z_1, z_2 \) be homology classes of respective dimension \( n - k_1, n - k_2 \) in \( V \), where we suppose that \( k_1, k_2 \leq n \). We take the integers or the integers modulo 2 as coefficients and take only the latter when \( V \) is nonorientable. If the classes \( z_1, z_2 \) are realized by submanifolds, then their intersection class \( z_1 \cdot z_2 \) is also realized by a submanifold.

**Proof.** The homology class \( z_1 \cdot z_2 \) is dual to the cohomology class \( U_1 \cdot U_2 \) where the class \( U_i \) is the dual of \( z_i \) (i = 1, 2) (H. Whitney [6]). This fact together with Theorem 1 and Theorem 3 yield Corollary 1.

Theorem 3 shows the existence of a mapping \( f \) of \( K \) into \( M(O(k_1+k_2)) \). We state this fact as Corollary 2.

**Corollary 2.** If there exist continuous mappings \( f_1, f_2 \) of the cell complex \( K \) into the cell complexes \( M(O(k_1)) \) and \( M(O(k_2)) \) which satisfy the condition that \( f_1^*U_1 = U_1, f_2^*U_2 = U_2 \). Then there exists a continuous mapping \( f \) of \( K \) into \( M(O(k_1+k_2)) \) such that \( f^*U_{O(k_1+k_2)} = U_1 \cup U_2 \).

So far we have proved that in a compact differentiable manifold the cup-product of two realizable classes is realizable. In the next sections, we shall study the Steenrod square operations of realizable cohomology classes.

5. Squares of classes of \( n \) dimension having \( O(k) \)-realization. In this section, we shall state theorems on the realizability of Steenrod square of \( n \) dimensional cohomology classes which are realizable for \( O(k) \subset O(n) \).

Now we denote by \( V \) a compact differentiable manifold of dimension \( m+n \) and denote by \( W \) a submanifold in \( V \) of dimension \( m \). Let \( z \in H_m(V; Z_2) \) be the homology class defined by \( W \) and \( u \in H^n(V; Z_2) \) be the cohomology class which is dual to \( z \). Then the following is the main theorem of this note.

**Theorem 4.** If the fiber space \( N(W) \) of normal vectors over the submanifold \( W \) in \( V \) has a field of \( (n-k) \)-linearly independent vectors where we suppose that \( k \leq n \), then the square \( Sq^k(u) \) of the class \( u \) can be also realized.

The condition of Theorem 4 is satisfied if and only if the fiber space \( N(W) \) has \( O(k) \) as its structural group. So, we can state the above fact simply as follows:
Theorem 5. If \( u \in H^n(V; \mathbb{Z}_2) \) is realizable for \( O(k) \subseteq O(n) \), then \( Sq^k(u) \) can be also realized.

Remark. If the Stiefel-Whitney class \( W_k \in H^k(W; \mathbb{Z}_2) \) is realizable, then Theorem 5 follows from Theorem 3 immediately. In the following, however, we shall give the proof without the fact.

In order to prove Theorem 5, we state some preliminary results. By Theorem 1 there exists a mapping \( f \) of \( V \) into \( M(O(k), n) \) such that the homomorphism \( f^* \) of \( H^n(M(O(k), n); \mathbb{Z}_2) \) into \( H^n(V; \mathbb{Z}_2) \) induced by \( f \) satisfies the condition that

\[
\alpha = f^*U_0(*)
\]

Next we consider the structure of the fiber space \( A_{O(k),n} \) over \( B_{O(k)} \).

**Lemma 2.** The fiber structure \( A_{O(k),n} \) decomposes into the product of a closed \((n-k)\)-cell \( b_{n-k} \) and \( A_{O(k)} \).

**Proof.** We can choose a system of coordinate transformations for the fiber space \( A_{O(k),n} \), any transformation of which leaves certain \( n-k \) coordinates fixed in the fiber, a closed \( n \)-cell \( b_n \).

6. The proof of Theorem 5. By Lemma 2 we can define a mapping of \( A_{O(k),n} \) onto \( A_{O(k)} \) which collapses the closed \((n-k)\)-cell of fiber into a point. Let \( q \) be such a map. On the other hand, we denote the identity mapping of \( A_{O(k),n} \) onto itself by \( \alpha \). We can define a mapping \( \alpha \oplus q \) of \( A_{O(k),n} \) into the Whitney sum \( A_{O(k),n} \oplus A_{O(k)} \) (see Wu [7]) in such a way that

\[
(11) \quad \alpha \oplus q(x) = (\alpha(x), q(x)),
\]

for each point \( x \) of \( A_{O(k),n} \). \( A_{O(k),n} \oplus A_{O(k)} \) is a fiber space over \( B_{O(k)} \) with \( b_n \times b_k \) as its fiber. Let \( H \) denote a fiber mapping of \( A_{O(k),n} \oplus A_{O(k)} \) into the universal fiber space over \( A_{O(k)} \) which is induced by a classifying map. We denote by \( M \) the Whitney sum \( A_{O(k),n} \oplus A_{O(k)} \) the boundary of which is identified into a point. Let \( \overline{H} \) be the mapping of \( M \) into \( M(O(n+k)) \) induced by \( H \). Since \( \alpha \oplus q \) maps \( S_{O(k),n} \) into the boundary \( (A_{O(k),n} \oplus A_{O(k)})' \), it induces a mapping \([\alpha \oplus q]\) of \( M(O(k), n) \) into \( M \). Then we get the following diagram of mappings of spaces,

\[
\begin{array}{ccc}
A_{O(k),n} & \xrightarrow{\alpha \oplus q} & A_{O(k),n} \oplus A_{O(k)} \xrightarrow{H} A_{O(n+k)} \\
V \xrightarrow{f} M(O(k), n) & \xrightarrow{[\alpha \oplus q]} & M \xrightarrow{\overline{H}} M(O(n+k))
\end{array}
\]

where the vertical arrow \( i \) shows the respective identifying map. In each square, commutativity holds.

The diagram (12) induces the following diagram of homomorphisms of cohomology groups and commutativity holds in each square;
Let \( U', i = k, n, n+k \), be the fundamental classes of \( \mathcal{A}'(k), \mathcal{A}'(k), \mathcal{A}'(k) \) respectively and let \( U_i \) be the fundamental classes of \( M(O(k)), M(O(k), n), M(O(n+k)) \) which correspond canonically to \( U_i \), respectively. Let \( j^i_1, j^i_2 \) denote the projections of \( \mathcal{A}'(k) \oplus \mathcal{A}'(k) \) onto \( \mathcal{A}'(k) \) and onto \( \mathcal{A}'(k) \) respectively.

By the same argument as Lemma 1, we have the relation,

\[
H^* U'_n U'_k = j^i_1 U'_n \cup j^i_2 U'_k.
\]

From definitions of \( \alpha \oplus q \) and of \( j^i_1 \), we get the formulae,

\[
j^i_1 (\alpha \oplus q) = \alpha,
\]

\[
j^i_2 (\alpha \oplus q) = q.
\]

They lead to the relations,

\[
(\alpha \oplus q)^* j^i_1 = \text{the identity isomorphism},
\]

\[
(\alpha \oplus q)^* j^i_2 = q^*.
\]

Using these formulae together with (14), we obtain the result,

\[
(\alpha \oplus q)^* H^* U'_n U'_k = (\alpha \oplus q)^* (j^i_1 U'_n \cup j^i_2 U'_k)
\]

\[
= (\alpha \oplus q)^* j^i_1 U'_n \cup (\alpha \oplus q)^* j^i_2 U'_k
\]

\[
= U'_n \cup q^* U'_k.
\]

By definition of the Stiefel-Whitney class of dimension \( k \), we have \( W_k = \beta^* q^* U'_k \). Hence we obtain the following,

\[
U'_n \cup q^* U'_k = U'_n \cup \beta^* q^* U'_k
\]

\[
= U'_n \cup W_k
\]

\[
= Sq^* U'_n.
\]

We have, therefore, the relation

\[
(\alpha \oplus q)^* H^* U'_n U'_k = Sq^* U'_n.
\]

(13) and (15) show that

\[
[\alpha \oplus q]^* H^* U'_n U'_k = Sq^* U'_n,
\]
where
\[ V \xrightarrow{f} M(O(k), n) \xrightarrow{[\alpha + q]} M \xrightarrow{H} M(O(n + k)). \]

If we take the composite mapping \( H[\alpha \oplus q]f \) of \( V \) into \( M(O(n + k)) \), we obtain, by (16), the result that
\[
(H[\alpha \oplus q]f)^*u_{n+k} = f^*[\alpha \oplus q]^*H^*u_{n+k} \\
= f^*Sq^k u_n \\
= Sq^k f^* u_n \\
= Sq^k(u).
\]

Thus the result of Theorem 5 is completely proved.

7. **A general result.** Combining Theorem 3 and Theorem 5, we have a general result on the realizability of cohomology classes generated by cup-products and Steenrod square operations of realizable classes:

**Theorem 6.** Let \( U_i, 0 \leq i \leq r \), be cohomology classes of dimension \( n_i \), which are dual to homology classes determined by submanifolds in a compact differentiable manifold \( V \). Suppose each \( U_i \) is realizable for \( O(k_i) \subset O(n_i) \). Then the cohomology class
\[
Sq^{k_1} U_1 \cup Sq^{k_2} U_2 \cup \cdots \cup Sq^{k_r} U_r
\]
can be realized by a submanifold in \( V \).

**References**


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