ON THE REALIZATION OF HOMOLOGY CLASSES
BY SUBMANIFOLDS

BY
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1. Introduction. In [1] R. Thom defines when a cohomology class in a
manifold is realizable for a closed subgroup of an orthogonal group. He also
defines when a homology class is realizable by a submanifold. He then shows
the following.

Let \( m \) be the dimension of a compact orientable differentiable manifold
\( M \). Any cohomology class of dimension 1, 2, \( m-6 \), \( m-5 \), \( \cdots \), \( m \) is realizable
for the orthogonal group \( O(1)=1, O(2), O(m-6), O(m-5), \cdots, O(m) \)
respectively. If a class \( z \in H_{m-k}(M; \mathbb{Z}) \) can be realized by a submanifold, then
the cohomology class \( u \in H^k(M; \mathbb{Z}) \) which is dual to \( z \) satisfies

\[
S_t^{2r(p-1)+1}(u) = 0
\]

for all integers \( r \) and all odd primes \( p \). Here \( S_t^{2r(p-1)+1} \) denotes the Steenrod
reduced power [4] which operates on the cohomology group with integral
coefficients \( \mathbb{Z} \). All homology classes with integral coefficients of compact
orientable differentiable manifolds of dimension < 10 are realizable by sub-
manifolds.

We consider compact differentiable manifolds which are not necessarily
orientable, and we ask whether the cup-products and the Steenrod squares
[3] of realizable cohomology classes can be realized. It was stated by Thom
[1] that cup-products of realizable classes are also realizable. We shall give
another proof of this result (see §3 below). As for squares of realizable classes
with integers modulo 2 as coefficients, we have the following result. Let \( u \) be
a cohomology class of dimension \( n \) in a compact differentiable manifold of
dimension \( m+n \). If \( u \) is realizable for the group \( O(k) \subset O(n) \) (\( k \leq n \)), then the
cohomology class \( Sq^k(u) \) is also realizable (see §5 below).

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suggestions.

2. Preliminaries. In this section, we summarize the theory of Thom
[1 and 2]. Let \(^{(')} \) \( G \) be a closed subgroup of the orthogonal group \( O(n) \). Let \( S \)
denote an \( (n-1) \)-sphere fiber space over a finite cell complex \( K \) with the
structural group \( G \) and let \( A \) be the mapping cylinder of the projection of \( S \)
onto \( K \). \( A \) becomes a fiber space over \( K \) with a closed \( n \)-cell \( b_n \) as fiber. Its
projection is induced by that of \( S \). We denote by \( A' \) the complement of \( S \)
in \( A \) which is a fiber space over \( K \) with fiber, an open \( n \)-cell \( b_n \).

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\(^{(1)} \) If \( G \) is a compact Lie group, then there is an integer \( n \) such that \( G \) is isomorphic to a
subgroup of \( O(n) \).
Throughout this paper, $\mathbb{G}$ will denote the group of integers or integers modulo 2. When dealing with the nonorientable case $^2$ $\mathbb{G}$ will denote only the group of integers modulo 2.

By the theory of fiber spaces with open cells as fibers, we have a canonical isomorphism $\phi^*$ of the $r$ dimensional cohomology group $H^r(K; \mathbb{G})$ onto the $r+n$ dimensional cohomology group $H^{r+n}(A'; \mathbb{G}) = H^{r+n}(A, S; \mathbb{G})$. $H^n(A, S; \mathbb{G})$ is the first nonvanishing cohomology group.

We denote by $S_{G,n}$ and $B_G$ the universal fiber space and the classifying space respectively, for $(n-1)$-sphere fiber spaces over finite cell complexes of bounded dimension with the structural group $G$ (see [5]). We may assume $B_G$ is a grassmann manifold. Let $A_{G,n}$ be the mapping cylinder of the projection of $S_{G,n}$ onto $B_G$ and let $A'_{G,n}$ be the complement of $S_{G,n}$ in $A_{G,n}$. In particular, we denote $S_{O(n),n}, A_{O(n),n}$ and $A'_{O(n),n}$ by $S_{O(n)}, A_{O(n)}$ and $A'_{O(n)}$ for the sake of brevity.

**Definition 1.** Let $M(G, n)$ be the space which we get from $A_{G,n}$ by identifying its boundary $S_{G,n}$ to a point. We call the space $M(G, n)$ the cell complex corresponding to the subgroup $G$ of $O(n)$. We denote $M(O(n), n)$ simply by $M(O(n))$.

We have the natural isomorphisms of cohomology groups,

$$H^r(A'_{G,n}; \mathbb{G}) = H^r(A_{G,n}, S_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})$$

where $r > 0$. It is quite easy to see that $H^0(M(G, n); \mathbb{G}) = \mathbb{G}$ $\phi_{G,n}^*$ denotes the canonical isomorphism in the fiber structure $A'_{G,n}$,

$$\phi_{G,n}^* : H^{r-n}(B_G; \mathbb{G}) \approx H^r(A'_{G,n}; \mathbb{G}) = H^r(M(G, n); \mathbb{G})$$

where $r \geq n$. Therefore the cohomology groups $H^r(M(G, n); \mathbb{G})$ of dimensions $r \geq n$ are obtained by raising dimensions of cohomology groups of the classifying space $B_G$ by the integer $n$. $H^n(M(G, n); \mathbb{G})$ is the first cohomology group which does not vanish in dimension $>0$. This group is generated by the class,

$$U_{G,n} = \phi_{G,n}^*(1_G) \in H^n(M(G, n); \mathbb{G})$$

where $1_G$ is the unit class of $H^*(B_G; \mathbb{G})$. We call the class $U_{G,n}$ the fundamental class of $M(G, n)$. We denote $U_{O(n),n}$ simply by $U_{O(n)}$.

**Definition 2.** Let $A$ be a topological space and let $u$ be a class of $H^n(A; \mathbb{G})$. $u$ is said to be realizable for $G \subset O(n)$ or has $G$-realization, if there is a map $f : A \to M(G, n)$ such that $u = f^*U_{G,n}$.

**Definition 3.** Let $W$ be a submanifold of dimension $p$ in a compact differentiable manifold $V$ of dimension $m$ and of class $C^\infty$, where $p$ is an integer such that $m \geq p \geq 0$. The inclusion mapping $i : W \subset V$ induces a homomorphism $i_*^*$ of the homology group $H_p(W; \mathbb{G})$ into the homology group $H_p(V; \mathbb{G})$.

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$^2$ Orientability means that of the fiber space $S$, therefore, of $A$. In this case, $G$ is connected.
If a homology class \( z \) of \( H_p(W; \mathbb{G}) \) is the image of the fundamental class of \( W \), then we say that the class \( z \) is realized by the submanifold \( W \).

Now we state the following fundamental theorem [1, Chap. II]:

**Theorem 1.** Let \( V \) be a compact differentiable manifold of dimension \( m \) and of class \( C^\infty \), and let \( n \) be an integer such that \( m \geq n \geq 0 \). A cohomology class \( u \in H^n(V; \mathbb{G}) \) is realizable for the group \( G \subset O(n) \) if and only if the dual homology class \( z \) of \( u \) is realized by a submanifold \( W \) of dimension \( m - n \) and the fiber space consisting of normal vectors on the submanifold \( W \) in \( V \) has the group \( G \) as its structural group.

Let \( h \) be a classifying map of \( K \) into \( B_G \) for the fiber space \( S \). Then it induces a fiber mapping \( H \) of \( A \) into \( A_G, n \). Commutativity holds in the diagram,

\[
\begin{array}{c}
A \xrightarrow{H} A_G, n \\
p \downarrow \quad \downarrow p_G, n \\
K \xrightarrow{h} B_G
\end{array}
\]

where \( p \) and \( p_G, n \) are the projections of the fiber structures \( A \) and \( A_G, n \) which are naturally induced by the projections \( S \to K \) and \( S_G, n \to B_G \). The commutativity of (2) induces that of the following diagram of homomorphisms of cohomology groups,

\[
\begin{array}{c}
H^r(A'; \mathbb{G}) \xleftarrow{\phi^*} H^r(A_G, n; \mathbb{G}) \\
\phi^* \uparrow \quad \uparrow p_G, n^*
\end{array}
\]

\[
\begin{array}{c}
H^0(K; \mathbb{G}) \xleftarrow{h^*} H^0(B_G; \mathbb{G}) \\
\phi^* \uparrow \quad \uparrow \phi_G, n^*
\end{array}
\]

where \( H' \) is the mapping of \( A' \) into \( A_G, n \) induced by \( H \).

Obviously, the homomorphism \( p^* \) induced by \( p \) gives us an isomorphism of \( H^r(K; \mathbb{G}) \) onto \( H^r(A; \mathbb{G}) \). We put \( \phi^*(1) = U_n \in H^n(A'; \mathbb{G}) \) where 1 is the unit class of the cohomology ring \( H^*(K; \mathbb{G}) \). Let \( \beta^* \) be the homomorphism of \( H^n(A'; \mathbb{G}) = H^n(A, S; \mathbb{G}) \) into \( H^n(A; \mathbb{G}) \) induced by the inclusion map \( \beta \) of \((A, 0)\) into \((A, S)\). The cohomology class \( W_n = p^* - \beta^* U_n \) in \( H^0(K; \mathbb{G}) \) is called the fundamental characteristic class of the \((n - 1)\)-sphere fiber space \( S \). In below, we shall denote \( p^* W_n \) by the same symbol \( W_n \).

The isomorphism \( \phi^* \) has the following important property(4),

\[
\phi^*(x \cup y) = p^* x \cup \phi^* y
\]

(4) We remark the fact that if \( V \) is orientable, then duality means the Poincaré duality, i.e., the isomorphism of the homology group \( H_{n-1}(V; \mathbb{Z}) \) onto the cohomology group \( H^n(V; \mathbb{Z}) \), and if \( V \) is nonorientable, then we take the coefficient group \( \mathbb{Z}_2 \) and duality means the isomorphism of Poincaré-Veblen, i.e., the isomorphism of \( H_{n-1}(V; \mathbb{Z}_2) \) onto \( H^n(V; \mathbb{Z}_2) \).

(4) The cup-product is defined by the multiplication among integers.
for cohomology classes $x, y$ in $H^*(K; \mathbb{G})$. If we put $y=1$, then the formula (4) gives us

$$\phi^*(x) = p^*x \cup U_n$$

for every class $x$ in $H^*(K; \mathbb{G})$. Also $\phi^*$ commutes with the Steenrod square operations.

The Stiefel-Whitney class $W_j$ of dimension $j$ $(0 \leq j \leq n)$, is defined as the fundamental characteristic class of the associated $(j-1)$-sphere fiber space over the $j$-skeleton $K_j$ of $K$.

**Theorem 2.** The Stiefel-Whitney class $W_j$ of the $(n-1)$-sphere fiber space $S$ satisfies the following formula,

$$Sq^j U_n = U_n \cup W_j$$

for $0 \leq j \leq n$ where $Sq^j$ denotes the Steenrod square operation which raises dimension by $j$.

The proof of this theorem can be found in [2, Chap. II, Theorem II].

We remark that the relation (6) leads to the topological invariance of the Stiefel-Whitney classes of compact differentiable manifolds.

In the end of this section, we mention a property of the canonical isomorphism for double fiber structures. Let $A''$ be a fiber structure over $A'$ with an open cell as fiber. $\phi''^*$ denotes the canonical isomorphism for $A''$ over $A'$. Obviously, $A''$ is a fiber structure over $K$ with the open cell which is product cell of the fiber of $A'$ over $K$ and that of $A''$ over $A'$. $\phi^*$ denotes the isomorphism for $A''$ over $K$, then we have the relation

$$\phi''^* = \phi^* \phi^*.$$

3. Cup-products of realizable cohomology classes.

**Theorem 3.** Suppose that $K$ is a finite cell complex and that $u_1 \in H^{k_1}(K; \mathbb{G})$ and $u_2 \in H^{k_2}(K; \mathbb{G})$. If $u_1$ and $u_2$ are $O(k_1)$- and $O(k_2)$-realizable, respectively, then the cup-product\(^{(4)}\) $u_1 \cup u_2$ is $O(k_1 + k_2)$-realizable.

**Proof.** By Definition 2, there exists two mappings, $f_1: K \to M(O(k_1))$ and $f_2: K \to M(O(k_2))$ having the following properties. For the sake of brevity, we use the notations $U_1, U_2$ instead of $U_{O(k_1)}, U_{O(k_2)}$. Then the relations,

$$f_1^* U_1 = u_1, \quad f_2^* U_2 = u_2$$

hold.

Now we shall define a mapping $\overline{H}$ of the product space $M(O(k_1)) \times M(O(k_2))$ into the cell complex $M(O(k_1 + k_2))$ as follows: $A_{O(k_1)} \times A_{O(k_2)}$ is a fiber space over the product space $B_{O(k_1)} \times B_{O(k_2)}$ with fiber, the $(k_1 + k_2)$-cell $b_1 \times b_2$ and with structural group, the orthogonal group $O(k_1 + k_2)$. Since

\(^{(4)}\) The cup-product is defined by the multiplication of the ring $\mathbb{G}$. Further, Theorem 3 holds generally in the case where the coefficient group is any ring.
Let $j_1$ and $j_2$ be the projections of $M(O(k_1)) \times M(O(k_2))$ onto $M(O(k_1))$ and $M(O(k_2))$ respectively, defined by the formulae,

$$j_1(x, y) = x, \quad j_2(x, y) = y$$

for each point $(x, y) \in M(O(k_1)) \times M(O(k_2))$. Similarly, we define the projections $j'_1$ and $j'_2$ of $A'_0(k_1) \times A'_0(k_2)$ onto $A'_0(k_1)$ and $A'_0(k_2)$ respectively. Let $U'_1$ and $U'_2$ be the fundamental classes in the cohomology groups $H^{k_1}(A'_0(k_1); \mathbb{R})$ and $H^{k_2}(A'_0(k_2); \mathbb{R})$ which correspond naturally to $U_1$ and $U_2$.

We consider a cellular subdivision of the space $A'_0(k_1)$ as follows: We can suppose a simplicial subdivision of the base space $B'_0(k_1)$ which satisfies the condition that each simplex is contained in a coordinate neighborhood. We take product cells of such simplexes with the $k_1$-cell of the fiber. These cells make a cellular subdivision of $A'_0(k_1)$. In the similar way, we can suppose such cellular subdivisions of the space $A'_0(k_2)$ and of the space $A'_0(k_1+k_2)$. Then we have:

**Lemma 1.** Under the above conditions, the relation

$$H^* U'_0(k_1+k_2) = j'_1*U'_1 \cup j'_2*U'_2$$

holds, where $H'$ is the mapping of $A'_0(k_1+k_2)$ into $A'_0(k_1+k_2)$ induced by $H$ and $U'_0(k_1+k_2)$ is the fundamental class in $H^{k_1+k_2}(A'_0(k_1+k_2); \mathbb{R})$ which correspond naturally to $U_0(k_1+k_2)$.

**Proof.** $A'_0(k_1) \times A'_0(k_2)$ has a double fiber structure, $A'_0(k_1) \times B'_0(k_2)$ over $B'_0(k_1) \times B'_0(k_2)$ and $A'_0(k_1) \times A'_0(k_2)$ over $A'_0(k_1) \times B'_0(k_2)$. Let $\phi_1$, $\phi_2$ denote the canonical isomorphisms for these fiber structures, respectively, and let $\phi^*$ denote that for $A'_0(k_1) \times A'_0(k_2)$ over $B'_0(k_1) \times B'_0(k_2)$. (7) gives us the relation,

$$\phi = \phi_2\phi_1.$$ 

By using the commutativity of (3), we have that

$$H^* U'_0(k_1+k_2) = H^* \phi_0 U'_0(k_1+k_2) = \phi_1* U'_1 \cup \phi_2* U'_2.$$ 

(6) For simplicity, we denote the canonical isomorphism $\phi_0^* (k_1+k_2)$ by $\phi_0^* (k_1+k_2)$, if $n = k_1 + k_2.$
Thus Lemma 1 has proved.

We have a diagram of the homomorphisms $\Pi^*$, $H^*$, $j_1^* \cup j_2^*$, $j_1'^* \cup j_2'^*$ and if the isomorphisms of the cohomology groups induced by identifying the boundaries, $S_{0(\pm)}$ in $A_{0(\pm)}$ ($i=1, 2$) and $S_{0(k_1+k_2)}$ in $A_{0(k_1+k_2)}$ to a point, respectively,

$$H^{k_1}(A_{0(k_1)}; \mathfrak{G}) \times H^{k_2}(A_{0(k_2)}; \mathfrak{G}) = H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathfrak{G}),$$

where the horizontal arrow is a canonical homomorphism of cohomology groups which is induced by the inclusion mapping of pairs of spaces,

$$(A_{0(k_1)} \times A_{0(k_2)}, \mathfrak{G}) \rightarrow (A_{0(k_1)} \times A_{0(k_2)}, (A_{0(k_1)} \times A_{0(k_2)}).$$

Since the above diagram is commutative, the relation (8) leads to the formula,

$$(8') \Pi^* U_{0(k_1+k_2)} = j_1^* U_1 \cup j_2^* U_2,$$

which plays an important roll in the proof of Theorem 3.

Now we define a mapping $f_1 \ast f_2$ from the cell complex $K$ into the product cell complex $M(O(k_1)) \times M(O(k_2))$ by the formula,

$$f_1 \ast f_2(a) = f_1(a) \times f_2(a), \quad a \in K.$$  

Then this mapping induces a homomorphism $(f_1 \ast f_2)^*$ of the cohomology group

$$H^{k_1+k_2}(M(O(k_1)) \times M(O(k_2)); \mathfrak{G}),$$

into the cohomology group $H^{k_1+k_2}(K; \mathfrak{G})$. Using the property that these induced homomorphisms and the cup-products are commutative, we have the relation,

$$(f_1 \ast f_2)^* (j_1^* U_1 \cup j_2^* U_2) = f_1^* U_1 \cup f_2^* U_2$$

because it is easily seen that

$$f_1 = j_1(f_1 \ast f_2),$$

$$f_2 = j_2(f_1 \ast f_2).$$

We put $f = \Pi(f_1 \ast f_2)$. This is a continuous mapping of $K$ into $M(O(k_1+k_2))$. Combining the formulae $(8')$ and $(9)$, we obtain the relation,
\[ f^*U_{0(k_1+k_2)} = (f_1 \ast f_2)^*H^*U_{0(k_1+k_2)} \]
\[ = (f_1 \ast f_2) \ast (j_1^*U_1 \cup j_2^*U_2) \]
\[ = u_1 \cup u_2. \]

The last formula (10) means that the class \( u_1 \cup u_2 \) has an \( O(k_1+k_2) \) realization.

4. Applications of Theorem 3. When the cell complex \( K \) stated above is an \( n \) dimensional differentiable manifold \( V \) of class \( C^\infty \), we can use Theorem 1. Then we have the following result.

**Corollary 1.** Let \( V \) be an \( n \) dimensional differentiable manifold of class \( C^\infty \). Let \( z_1, z_2 \) be homology classes of respective dimension \( n-k_1, n-k_2 \) in \( V \), where we suppose that \( k_1, k_2 \leq n \). We take the integers or the integers modulo 2 as coefficients and take only the latter when \( V \) is nonorientable. If the classes \( z_1, z_2 \) are realized by submanifolds, then their intersection class \( z_1 \cdot z_2 \) is also realized by a submanifold.

**Proof.** The homology class \( z_1 \cdot z_2 \) is dual to the cohomology class \( u_1 \cup u_2 \) where the class \( u_1 \) is the dual of \( z_1 \) (\( i=1, 2 \)) (H. Whitney [6]). This fact together with Theorem 1 and Theorem 3 yield Corollary 1.

Theorem 3 shows the existence of a mapping \( f \) of \( K \) into \( M(O(k_1+k_2)) \). We state this fact as Corollary 2.

**Corollary 2.** If there exist continuous mappings \( f_1, f_2 \) of the cell complex \( K \) into the cell complexes \( M(O(k_1)) \) and \( M(O(k_2)) \) which satisfy the condition that \( f_1^*U_1 = u_1, f_2^*U_2 = u_2 \). Then there exists a continuous mapping \( f \) of \( K \) into \( M(O(k_1+k_2)) \) such that \( f^*U_{0(k_1+k_2)} = u_1 \cup u_2 \).

So far we have proved that in a compact differentiable manifold the cup-product of two realizable classes is realizable. In the next sections, we shall study the Steenrod square operations of realizable cohomology classes.

5. Squares of classes of \( n \) dimension having \( O(k) \)-realization. In this section, we shall state theorems on the realizability of Steenrod square of \( n \) dimensional cohomology classes which are realizable for \( O(k) \subset O(n) \).

Now we denote by \( V \) a compact differentiable manifold of dimension \( m+n \) and denote by \( W \) a submanifold in \( V \) of dimension \( m \). Let \( z \in H_m(V; Z_2) \) be the homology class defined by \( W \) and \( u \in H^n(V; Z_2) \) be the cohomology class which is dual to \( z \). Then the following is the main theorem of this note.

**Theorem 4.** If the fiber space \( N(W) \) of normal vectors over the submanifold \( W \) in \( V \) has a field of \( (n-k) \)-linearly independent vectors where we suppose that \( k \leq n \), then the square \( Sq^k(u) \) of the class \( u \) can be also realized.

The condition of Theorem 4 is satisfied if and only if the fiber space \( N(W) \) has \( O(k) \) as its structural group. So, we can state the above fact simply as follows:
Theorem 5. If \( u \in H^n(V; \mathbb{Z}_2) \) is realizable for \( O(k) \subset O(n) \), then \( Sq^k(u) \) can be also realized.

Remark. If the Stiefel-Whitney class \( W_k \in H^k(W; \mathbb{Z}_2) \) is realizable, then Theorem 5 follows from Theorem 3 immediately. In the following, however, we shall give the proof without the fact.

In order to prove Theorem 5, we state some preliminary results. By Theorem 1 there exists a mapping \( f \) of \( V \) into \( M(O(k), n) \) such that the homomorphism \( f^* \) of \( H^n(M(O(k), n); \mathbb{Z}_2) \) into \( H^n(V; \mathbb{Z}_2) \) induced by \( f \) satisfies the condition that

\[
u = f^* U_0(\ast), \ast.
\]

Next we consider the structure of the fiber space \( A_{O(k), n} \) over \( B_{O(k)} \).

Lemma 2. The fiber structure \( A_{O(k), n} \) decomposes into the product of a closed \( (n-k) \)-cell \( b_{n-k} \) and \( A_{O(k)} \).

Proof. We can choose a system of coordinate transformations for the fiber space \( A_{O(k), n} \), any transformation of which leaves certain \( n-k \) coordinates fixed in the fiber, a closed \( n \)-cell \( b_n \).

6. The proof of Theorem 5. By Lemma 2 we can define a mapping of \( A_{O(k), n} \) onto \( A_{O(k)} \) which collapses the closed \( (n-k) \)-cell of fiber into a point. Let \( q \) be such a map. On the other hand, we denote the identity mapping of \( A_{O(k), n} \) onto itself by \( \alpha \). We can define a mapping \( \alpha \oplus q \) of \( A_{O(k), n} \) into the Whitney sum \( A_{O(k), n} \oplus A_{O(k)} \) (see Wu [7]) in such a way that

\[
\alpha \oplus q(x) = (\alpha(x), q(x)),
\]

for each point \( x \) of \( A_{O(k), n} \). \( A_{O(k), n} \oplus A_{O(k)} \) is a fiber space over \( B_{O(k)} \) with \( b_n \times b_k \) as its fiber. Let \( H \) denote a fiber mapping of \( A_{O(k), n} \oplus A_{O(k)} \) into the universal fiber space over \( A_{O(n+k)} \) which is induced by a classifying map. We denote by \( M \) the Whitney sum \( A_{O(k), n} \oplus A_{O(k)} \) the boundary of which is identified into a point. Let \( \overline{H} \) be the mapping of \( M \) into \( M(O(n+k)) \) induced by \( H \). Since \( \alpha \oplus q \) maps \( S_{O(k), n} \) into the boundary \( (A_{O(k), n} \oplus A_{O(k)})^c \), it induces a mapping \( [\alpha \oplus q] \) of \( M(O(k), n) \) into \( M \). Then we get the following diagram of mappings of spaces,

\[
\begin{array}{ccc}
A_{O(k), n} & \xrightarrow{\alpha \oplus q} & A_{O(k), n} \oplus A_{O(k)} \\
& \downarrow{i} & \downarrow{i} \\
V \rightarrow M(O(k), n) & \xrightarrow{[\alpha \oplus q]} & M \rightarrow M(O(n+k))
\end{array}
\]

where the vertical arrow \( i \) shows the respective identifying map. In each square, commutativity holds.

The diagram (12) induces the following diagram of homomorphisms of cohomology groups and commutativity holds in each square;
\[
H^*(A_0(k), n; Z_2) \xleftarrow{(\alpha \oplus q)^*} H^*(A_0(k), n \oplus A_0(k); Z_2) \xrightarrow{H^*} H^*(A_0(n+k); Z_2)
\]
(13)
\[
H^*(V; Z_2) \xleftarrow{[\alpha \oplus q]^*} H^*(M; Z_2) \xrightarrow{H^*} H^*(M(O(n+k)); Z_2),
\]
where \( H^* \) is the mapping induced by \( H \).

Let \( U'_i, i = k, n, n+k, \) be the fundamental classes of \( A_0(k), A_0(k), n, A_0(n+k) \) respectively and let \( U_i \) be the fundamental classes of \( M(O(k)), M(O(k), n), M(O(n+k)) \) which correspond canonically to \( U_i \) respectively. Let \( j'_1, j'_2 \) denote the projections of \( A_0(k), n \oplus A_0(k) \) onto \( A'_0(k), n \) and onto \( A'_0(k) \) respectively.

By the same argument as Lemma 1, we have the relation,

\[
H'^* U'_{n+k} = j'_1 U'_n \cup j'_2 U'_k.
\]
(14)

From definitions of \( \alpha \oplus q \) and of \( j'_i \), we get the formulae,

\[
\begin{align*}
  j'_1(\alpha \oplus q) &= \alpha, \\
  j'_2(\alpha \oplus q) &= q.
\end{align*}
\]

They lead to the relations,

\[
\begin{align*}
  (\alpha \oplus q)^* j'_1 &= \text{the identity isomorphism}, \\
  (\alpha \oplus q)^* j'_2 &= q^*.
\end{align*}
\]

Using these formulae together with (14), we obtain the result,

\[
(\alpha \oplus q)^* H'^* U'_{n+k} = (\alpha \oplus q)^* (j'^*_1 U'_n \cup j'^*_2 U'_k)
\]
\[
= (\alpha \oplus q)^* j'^*_1 U'_n \cup (\alpha \oplus q)^* j'^*_2 U'_k
\]
\[
= U'_n \cup q^* U'_k.
\]

By definition of the Stiefel-Whitney class of dimension \( k \), we have \( W_k = \beta q^* U'_k \). Hence we obtain the following,

\[
U'_n \cup q^* U'_k = U'_n \cup \beta q^* U'_k
\]
\[
= U'_n \cup W_k
\]
\[
= Sq^k U'_n.
\]

We have, therefore, the relation

(15) \[
(\alpha \oplus q)^* H'^* U'_{n+k} = Sq^k U'_n.
\]

(13) and (15) show that

(16) \[
[\alpha \oplus q]^* H^* U_{n+k} = Sq^k U_n,
\]
where
\[ V \xrightarrow{f} M(O(k), n) \xrightarrow{[\alpha + q]} H \xrightarrow{M} M(O(n + k)) \]

If we take the composite mapping \( H[\alpha \oplus q]f \) of \( V \) into \( M(O(n + k)) \), we obtain, by (16), the result that
\[
(H[\alpha \oplus q]f)^*U_{n+k} = f^*[\alpha \oplus q]^*H^*U_{n+k}
= f*Sq^kU_n
= Sq^k(u).
\]

Thus the result of Theorem 5 is completely proved.

7. A general result. Combining Theorem 3 and Theorem 5, we have a general result on the realizability of cohomology classes generated by cup-products and Steenrod square operations of realizable classes:

**Theorem 6.** Let \( U_i, 0 \leq i \leq r \), be cohomology classes of dimension \( n_i \) which are dual to homology classes determined by submanifolds in a compact differentiable manifold \( V \). Suppose each \( U_i \) is realizable for \( O(k_i) \subset O(n_i) \). Then the cohomology class
\[
Sq^{k_1}U_1 \cup Sq^{k_2}U_2 \cup \cdots \cup Sq^{k_r}U_r
\]
can be realized by a submanifold in \( V \).

**References**


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