BOUNDARY BEHAVIOR OF GENERALIZED ANALYTIC FUNCTIONS

BY
KENNETH HOFFMAN

1. **Introduction.** Recently, Arens and Singer [3] have studied a generalization of part of the theory of analytic functions in the unit disc, arising from a study of summable functions on partially ordered, locally compact groups. It is our purpose here to study the natural abstractions to this context of classical theorems such as Fatou's general Dirichlet principle, Riesz's theorem on functions of Hardy's class H_p, and Priwaloff's theorem on the measure of the set of boundary zeros of an analytic function. We shall also consider other questions, such as the existence of Cauchy measures. We begin by summarizing some of the Arens-Singer results.

Let G be a locally-compact abelian group, and let G_+ be a closed semi-group in G, such that the interior of G_+ is dense on G_+ and generates the group G. Let Δ be the set of all *characters* of G_+, i.e., continuous homomorphisms of G_+ into the unit disc of the complex plane. We make Δ into a topological space, using the topology of uniform convergence on compact subsets of G_+. If Γ is the (topological) character group of G, each element of Γ determines a homomorphism of G_+ into the disc, and the so determined (one-one) embedding of Γ in Δ is a homeomorphism of Γ with a closed subset of Δ.

In the classical case, when G is the discrete group of integers and G_+ is the semi-group of non-negative integers, the space Δ is the unit disc in the plane and Γ is the unit circle.

There exists in Δ a polar decomposition, generalizing that in the unit disc.

THEOREM 1.1. *Each element ζ in Δ is expressible in the form*

$$\zeta = \rho\alpha \tag{1.11}$$

where ρ is a non-negative element of Δ and α is in Γ.

Of course, ρ is uniquely defined by $\rho(x) = |\zeta(x)|$; however, α is not generally unique.

Consider the Banach algebra $L_1(G)$, the multiplication being the convolution

$$(f * g)(x) = \int_G f(x - y)g(y)dy.$$

Let A_1 be the subalgebra of $L_1(G)$ consisting of those functions f for which

Presented to the Society December 28, 1956; received by the editors October 3, 1956.

$f(x)=0$ when x is not in G_+. Let $H(A_1)$ be the canonical space of complex homomorphisms (regular maximal ideals) of the Banach algebra A_1. Each element h of $H(A_1)$ is uniquely representable in the form

$$h(f) = \int_{G_+} f(x)\zeta(x)dx \tag{1.21}$$

for some ζ in Δ. Conversely, given any ζ in Δ, (1.21) determines an element h in $H(A_1)$.

THEOREM 1.3. *The one-one correspondence determined by* (1.21) *is a homeomorphism of* Δ *and* $H(A_1)$.

We thus identify Δ and $H(A_1)$ and will refer to an element of Δ as a character of G_+, a homomorphism of A_1, or a regular maximal ideal of A_1. With this identification, each function f in A_1 determines a continuous function $\hat{f}$ on Δ by

$$\hat{f}(\zeta) = \int_{G_+} f(x)\zeta(x)dx. \tag{1.31}$$

In the classical case, the representing functions $\hat{f}$ are those functions continuous on the disc Δ, analytic in the interior, such that the restriction of $\hat{f}$ to Γ has an absolutely convergent Fourier series.

The well-known maximum modulus principle of function theory takes the following form, in general.

THEOREM 1.4. *For the algebra* A_1, Γ *is the Silov boundary of the space of maximal ideals* Δ. *That is, for each element* f *of* A_1, *the maximum modulus of the representing function* $\hat{f}$ *is taken on the set* Γ; *and* Γ *is the unique minimal closed set in* Δ *having this property.*

Thus, by analogy with the classical case, we shall refer to Δ as the disc and to Γ as the boundary of Δ.

We next state the generalization of the familiar Poisson boundary integral representation.

THEOREM 1.5. *For each point* ζ *of* Δ, *there is a regular Baire measure* m_ζ *on* Γ *such that for every* f *in* A_1

$$\hat{f}(\zeta) = \int_{\Gamma} \hat{f}(\alpha)m_\zeta(d\alpha). \tag{1.51}$$

We should perhaps remark that if ρ is a fixed non-negative element of Δ and $\zeta=\rho\alpha$ (1.1), then for any continuous function ϕ

$$\int_{\Gamma} \phi(\beta)m_\zeta(d\beta) = \int_{\Gamma} \phi(\alpha\beta)m_\rho(d\beta). \tag{1.52}$$

The measures m_ζ are not generally unique; however, if $G_+ \cup G_+^{-1} = G$, which is the case, for instance, when G is linearly ordered and G_+ is the set of elements not less than the identity, each m_ζ is unique.

Before specializing the group G, we list one further observation of the Arens-Singer paper. If Π is the closed half-plane $\operatorname{Re}(w) \geqq 0$, each ρ in Δ determines a continuous mapping of Π into Δ. We map the complex number w into the element ρ^w defined by

$$\rho^w(x) = \begin{cases} \exp w \log \rho(x), & \rho(x) \neq 0, \\ 0, & \rho(x) = 0. \end{cases} \tag{1.61}$$

THEOREM 1.7. *For each f in A_1, the function $\hat{f}(\rho^{u+iv})$ is holomorphic for u positive and continuous and bounded on all of Π.*

In those cases in which G is discrete, the above analytic functions are almost periodic; however, no use will be made of that fact in this paper.

2. **The influence of archimedean order.** We now confine our attention to those situations in which G is archimedean-linearly ordered (subgroup of the additive group of real numbers) and G_+ is the set of elements not less than zero. The topology of G is to be discrete. We shall treat the elements of G as real numbers, in particular using $>$ for the order relation in G.

When G is discrete, Δ possesses a point ζ_0, which we shall call the origin of Δ, defined by

$$\zeta_0(x) = \begin{cases} 1, & x = 0, \\ 0, & x > 0. \end{cases} \tag{2.01}$$

The measure on Γ associated with ζ_0 (1.5) is Haar measure. At times, we shall write simply $\zeta = 0$ to mean $\zeta = \zeta_0$. The origin plays a much more distinguished role in general than it does in the classical situation. We shall not elaborate much here on this distinguished nature of ζ_0. We might say that it is roughly due to the fact that the "Taylor series" of an analytic function need not have a first nonzero coefficient.

It will be convenient for our later purposes if we obtain at this point a more precise picture of Δ. Note that the archimedean order of G guarantees that no element of Δ different from ζ_0 ever assumes the value zero. This makes the polar decomposition (1.1) unique. Also, if an element of Δ assumes the value 1 at any point of G_+ other than the zero, it is identically 1 and is the unit element of Δ. Thus, we shall write $0 < \rho < 1$ to mean $0 < \rho(x) < 1$ for all nonzero x in G_+.

THEOREM 2.1. *If $0 < \rho \leqq 1$, there is a positive number r, $0 < r \leqq 1$, such that*

$$\rho(x) = r^x. \tag{2.11}$$

Conversely, if $0 < r \leqq 1$, (2.11) defines an element ρ of Δ. The correspondence *determined by* (2.11) *between the real segment* $0<\rho<1$ *of* Δ *and the unit interval* (0, 1) *is a homeomorphism.*

Proof. Since $\rho(y)=\rho(x)\rho(y-x)$ for x less than y, we see that ρ is monotone decreasing. This monotonicity keeps ρ bounded near zero, from which it follows that ρ is continuous in the linear topology of G, and consequently that ρ is of the form (2.11). The second statement of the theorem is obvious. A typical neighborhood of ρ_0 in Δ has the form

$$U = \{\rho;\ |\rho(x_i) - \rho_0(x_i)| < \epsilon,\ i = 1, \cdots, n\}. \tag{2.12}$$

Thus, it is clear that ρ is near ρ_0 if and only if r is near r_0, which proves the third statement.

In view of this theorem, we shall use the symbol ρ interchangeably for the element of Δ and the number r.

We recall the continuous mapping of the real line into Γ defined in (1.61) sending v into ρ^{iv}. Now (2.1) makes it clear that the image of the line under this mapping is the same subgroup of Γ for each $0<\rho<1$. We shall call this subgroup Λ. As Arens and Singer showed [3, Theorem 7.2], Λ is a dense subgroup of Γ; also, Λ has Haar measure zero, unless G is the group of integers. It is perhaps well to normalize our notation in Λ, defining Λ as the one-parameter subgroup of Γ consisting of the characters α_v, defined for each real v by

$$\alpha_v(x) = e^{-ivx}. \tag{2.13}$$

The harmonic measures m_ρ, $\rho>0$, determined by (1.5), are supported by the subgroup Λ.

THEOREM 2.2. *If* $0<\rho<1$, *then for each bounded Baire function* ϕ *on* Γ

$$\int_\Gamma \phi(\alpha)m_\rho(d\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\rho^{iv})(1+v^2)^{-1}dv. \tag{2.21}$$

Proof. The representation is proved in [3, Theorem 5.5]; the uniqueness follows from the linear order of G, as remarked after (1.5).

Once again using (2.1), direct computation shows that

$$\int_\Gamma \phi(\alpha)m_\rho(d\alpha) = \int_{-\infty}^{\infty}\phi(\alpha_v)c_u(v)dv, \tag{2.22}$$

where $u=-\log\rho$ and c_u is the Cauchy density

$$c_u(v) = u[\pi(u^2+v^2)]^{-1}. \tag{2.23}$$

One useful conclusion from (2.23) is that the measures m_ρ, $0<\rho<1$, are mutually absolutely continuous.

We conclude this section by exhibiting a local direct product decomposition of the character group Γ. The consequences (2.5 and 2.6) of this decomposition will be extremely important to us later. Fix an element x_0 in G_+, $x_0 \neq 0$, and let Z be the compact subgroup of Γ of those characters which assume the value 1 at x_0.

THEOREM 2.3. *The character group* Γ *is locally isomorphic to the direct product of the unit circle,* $|z|=1$, *and the subgroup* Z.

Proof. By isomorphism, we of course mean topological isomorphism. Let α_0 be in Γ, and select the unique number t between 0 and $2\pi/x_0$, such that $\alpha_0(x_0)=\alpha_t(x_0)$ (see 2.13). Let β be the element $\alpha_0\alpha_t^{-1}$ in Z. Consider the neighborhood W of α_0 of the form

$$W = \{\alpha;\, \alpha(x_0) \neq -\alpha_0(x_0)\}. \tag{2.31}$$

Then there is a one-one correspondence between W and the product $U\times Z$, where U is the open subset of the circle obtained by deleting that t_0 such that $\alpha_{t_0}(x_0)=-\alpha_0(x_0)$. The correspondence is, of course, $\alpha\leftrightarrow(t,\beta)$ as outlined above. That this is an algebraic isomorphism is obvious. We need only verify that the topologies are the same. This fact is easy to see after one notes the elementary fact (observed in the proof of (2.1)) that two values of t are close to one another if and only if the corresponding α_t's are close to one another at some fixed x in G (or finitely many x's). The details are somewhat cumbersome to write down, consequently, we omit them, realizing that the situation is clear after a few moments thought.

Note that, according to the proof of (2.3), two direct product neighborhoods will cover Γ. It now follows that, except for a normalization, the Haar measure of Γ is locally the product of the Lebesgue measure of the unit circle and the Haar measure of Z [2, Theorem 2.3]. In the notation of (2.3), we may thus state that for any bounded Baire function ϕ on Γ

$$\int_W \phi(\alpha)d\alpha = K\int_Z\int_U \phi(\alpha_t\beta)dtd\beta, \tag{2.41}$$

where K is a constant of (measure) normalization.

THEOREM 2.5. *Let S be a Borel set in* Γ *and* $0<\rho<1$. *If for each* α *in* Γ, $m_{\rho\alpha}(S)=0$, *then S has Haar measure zero.*

Proof. Let k_S be the characteristic function of the set S. Again with the notation of (2.3), we may apply the Fubini theorem to (2.41) to conclude that

$$\int_W k_S(\alpha)d\alpha = K\int_Z d\beta\int_U k_S(\alpha_t\beta)dt. \tag{2.51}$$

Hence,

$$\int_W k_S(\alpha)d\alpha \leqq K \max_\beta \int_U k_S(\alpha_t\beta)dt = K \max_\beta M(S_\beta), \tag{2.52}$$

where $S_\beta = U \times \{\beta\}$ and M is Lebesgue measure on the circle. According to (1.52),

$$m_\rho(S_\beta) = m_{\rho\beta}(S) = 0. \tag{2.53}$$

By (2.22),

$$m_\rho(S_\beta) = \int_\Gamma k_S(\alpha\beta)m_\rho(d\alpha) = \int_{-\infty}^{\infty} k_S(d_v\beta)c_u(v)dv. \tag{2.54}$$

As $m_\rho(S_\beta) = 0$ and c_u is a positive kernel, it is clear that

$$M(S_\beta) = \int_U k_S(\alpha_v\beta)dv = 0. \tag{2.55}$$

It follows from (2.52) that

$$\int_W k_S(\alpha)d\alpha = 0, \tag{2.56}$$

and since two neighborhoods W will cover Γ, the Haar measure of S is zero.

This theorem is of course trivial in the classical case, in which each m_ζ is absolutely continuous with respect to Haar measure; however, in general we know that the measures m_ζ are mutually singular with Haar measure, being supported by translates of the one-parameter subgroup Λ, which has Haar measure zero. Thus, (2.5) represents a useful cementing relation between the measures m_ζ, $\zeta \neq 0$, and Haar measure.

THEOREM 2.6. *Let ρ be a non-negative element of Δ, and let $p > 0$. Suppose that the Baire function ϕ on Γ has the property that for some fixed M and every β in Γ*

$$\int_\Gamma |\phi(\alpha\beta)|^p m_\rho(d\alpha) \leqq M. \tag{2.61}$$

In other words, suppose that for the elements $\zeta = \rho\beta$, ϕ is uniformly bounded in the norms $L_p(m_\zeta)$. Then ϕ is in $L_p(\Gamma)$.

Proof. In view of the mutual absolute continuity of the measures m_ρ, as implied by (2.21), we may assume $\rho = e^{-1}$. Then by (2.23),

$$\int_\Gamma |\phi(\beta\alpha)|^p m_\rho(d\alpha) = \int_{-\infty}^{\infty} |\phi(\beta\alpha_v)|^p(1 + v^2)^{-1}dv. \tag{2.62}$$

Using the local decomposition of (2.3), $W = U \times Z$, we see that, since $(1+v^2)^{-1}$ is bounded away from zero on the interval U, the integrals

$$\int_U |\phi(\alpha_v\beta)|^p dv \tag{2.63}$$

must be uniformly bounded, say by N. As in (2.41),

$$\int_W |\phi(\alpha)|^p d\alpha = K\int_Z d\beta \int_U |\phi(\beta\alpha_v)|^p dv. \tag{2.64}$$

Since the group Z is compact, we have

$$\int_W |\phi(\alpha)|^p d\alpha \leqq KN. \tag{2.65}$$

3. **Analytic functions in the half-plane.** We have seen (1.61) that there is a natural continuous mapping of the half-plane $\Pi = \{\mathrm{Re}\,(w) \geqq 0\}$ into the disc Δ, defined by sending w into e^{-w}. If g is an "analytic" function in the interior of Δ, i.e., if g can be uniformly approximated on compact subsets of the interior of Δ by functions $\hat{f}$ with f in A_1, then the function $\tilde{g}$ defined by

$$\tilde{g}(w) = g(e^{-w}) \tag{3.01}$$

is holomorphic (and almost periodic) in the interior of Π (see 1.7). Furthermore, the Poisson-type boundary integrals defined by the harmonic measures on Γ are transformed via this mapping into boundary integrals on Π involving the Cauchy density functions (2.22). Thus, the study of analytic functions and/or boundary value problems in Π will be very useful for a similar study in the disc Δ. We shall therefore present some of the theory of analytic functions in the half-plane, basing the discussion on the more familiar theory in the classical unit disc. Most of these theorems in the half-plane are well-known; however, I believe the Riesz theorem (3.9) is more general than any to be found in the literature.

In this section, we denote the classical unit disc of the complex plane by Δ_0 and its boundary by Γ_0. Let ϕ be the conformal map of Δ_0 onto Π (plus the point at infinity) defined by

$$\phi(z) = (i + z)/(i - z). \tag{3.02}$$

Then, of course, ϕ maps Γ_0 onto the imaginary axis of Π (plus the point at infinity). Every Baire function f on the boundary of Π gives rise to a Baire function F on Γ_0 by means of the composition

$$F(\theta) = f(\phi(e^{i\theta})). \tag{3.03}$$

Direct computation shows that the familiar Poisson kernels

$$P_r(\theta) = (1 - r^2)[2\pi(1 - 2r\cos\theta + r^2)]^{-1} \tag{3.04}$$

are transformed by ϕ into the Cauchy densities

(3.05) $$c_u(v) = u[\pi(u^2 + v^2)]^{-1}$$

in the following sense (see, for instance, [7]). The function f in (3.03) is summable with respect to the Cauchy density c_1 (3.05) exactly when F (3.03) is summable on the circle Γ_0; and, when F is summable, the functions

(3.06) $$g(u + iv) = \int_{-\infty}^{\infty} f(v - t)c_u(t)dt$$

and

(3.07) $$G(re^{i\theta}) = \int_0^{2\pi} F(\theta - t)P_r(t)dt$$

are related by

(308) $$G(z) = g(\phi(z)).$$

In (3.06), $f(v)$ is (of course) actually $f(iv)$.

The Dirichlet problem is one of the famous problems in the function theory of the unit disc and has the following general solution [4, p. 152; 12, p. 54].

THEOREM 3.1. *Let F be a real-valued summable function on Γ_0 and define G inside Δ_0, by* (3.07). *Then G is a harmonic function in Δ_0 with the property that, for almost every θ, $G(z)$ tends to $F(\theta)$ as z approaches $e^{i\theta}$ along any path nontangential with Γ_0.*

Since ϕ (3.02) is conformal and maps sets of measure zero into sets of (linear) measure zero, our remarks above combine with (3.1) to yield directly the analogous result for the half-plane.

THEOREM 3.2. *Let f be a real-valued* (*Baire*) *function on* $(-\infty, \infty)$ *which is summable with respect to the Cauchy density c_1. If g is the function defined on Π by* (3.06), *then g is a harmonic function in the interior of Π such that, for almost every v, $g(w)$ tends to $f(v)$ as w approaches iv along any path nontangential with the imaginary axis of Π.*

We consider next Fatou's theorem on bounded analytic functions, which in Δ_0 has the following form [4, p. 147].

THEOREM 3.3. *Let G be a bounded function, analytic in the interior of Δ_0. Then, for almost every θ, the limit*

(3.31) $$F(\theta) = \operatorname*{limit}_{r=1} G(re^{i\theta})$$

exists, and (3.07) *holds.*

There is a similar result in II. If g is a bounded function, holomorphic in the interior of Π, and G is defined by (3.08), then G is a bounded analytic function in the interior of Δ_0 to which we may apply Fatou's result. With G represented as in (3.07), we define a function f via (3.03), and as before, (3.06) will hold. We state our observations formally.

THEOREM 3.4. *Let g be a bounded analytic function in the interior of the half-plane* Π. *Then, for almost every v, the limit*

$$f(v) = \operatorname*{limit}_{u=0} g(u + iv) \tag{3.41}$$

exists, and (3.06) *holds.*

We turn to a theorem which is somewhat more difficult to obtain in the half-plane. Let H_p $(p>0)$ denote the class of functions G, analytic inside Δ_0, with the property that the functions

$$G_r(\theta) = G(re^{i\theta}) \tag{3.51}$$

are uniformly bounded in L_p-norm. In 1923, F. Riesz [11] proved the following [see also 12, p. 162].

THEOREM 3.6. *If G is in $H_p(\Delta_0)$, then for almost every θ, the limit* (3.31) *exists and defines a function F in $L_p(\Gamma_0)$, such that the functions G_r converge to F in the L_p-norm. If $p \geqq 1$, then* (3.07) *holds.*

This result includes the Fatou theorem (3.3) as a special case. We have stated Fatou's theorem separately since it is of independent interest, and also since Riesz's proof made use of (3.3) and a Blaschke product decomposition of functions in H_p. In [8], Hille and Tamarkin proved an analogous result for the class of analytic functions in the half-plane such that the Lebesgue L_p-norms of the function along lines parallel to the boundary are uniformly bounded. This class is somewhat restricted, since any such function is continuous at infinity. We shall prove a Riesz-type theorem for a larger class of functions than Hille and Tamarkin considered; nevertheless, certain ideas in our proof are contained in their work.

Let $\tilde{H}_p$ $(p>0)$ be the class of functions g, holomorphic for Re $(w) \geqq 0$ and bounded on Re $(w) \geqq u_0 > 0$, such that the integrals

$$\int_{-\infty}^{\infty} | g(u + iv) |^p (1 + v^2)^{-1} dv \tag{3.71}$$

are uniformly bounded for $u>0$. What we need to know is the following.

LEMMA 3.8. *If g is in $\tilde{H}_p$, and the function G is defined in Δ_0 by* (3.08), *then G is in $H_p = H_p(\Delta_0)$.*

Proof. The proof will be similar to that of [8, Lemma 2.5]. Let C be a circle, $|z| = r$, in Δ_0. We wish to show that there exists a uniform bound for the integrals

$$\int_{-\pi}^{\pi} | G_r(\theta) |^p d\theta = (2\pi r)^{-1} \int_C | G(z) |^p | dz | . \tag{3.81}$$

Each line Re $(w)=b$ in Π maps into Δ_0 as a circle Γ_b, tangent to Γ_0 at $z=i$. Choose b small enough that C lies in the interior of Γ_b. Now let Γ_n be the circle in Δ_0 concentric with Γ_b, but of radius $1/n$ less. A theorem of Gabriel [5] states that, as $|G|^p$ is subharmonic inside Γ_n and continuous on Γ_n, we must have

$$\int_C | G(z) |^p | dz | \leqq \int_{\Gamma_n} | G(z) |^p | dz | \tag{3.82}$$

whenever n is large enough that C lies inside Γ_n. We would like the same inequality with Γ_b in place of Γ_n; however, we must somehow avoid the difficulty presented by the fact that G is not continuous (necessarily) on Γ_b. The obvious thing to attempt is to show that

$$\operatorname*{limit}_{n\to\infty} \int_{\Gamma_n} | G(z) |^p | dz | = \int_{\Gamma_b} | G(z) |^p | dz | . \tag{3.83}$$

Now Γ_n corresponds (via 3.02) to a circle C_n in Π which is centered on the positive real axis. As n increases, one real intercept of C_n approaches zero, and the other tends to infinity. We are interested in the integrals

$$\int_{\Gamma_n} | G(z) |^p dz = 2 \int_{C_n} | g(w) |^p | 1 + w^2 |^{-1} | dw | . \tag{3.84}$$

Since

$$\int_{\Gamma_b} | G(z) |^p | dz | = 2 \int_{-\infty}^{\infty} | g(b + iv) |^p [(1 + b)^2 + v^2]^{-1} dv, \tag{3.85}$$

we shall prove (3.83) if we show that

$$\begin{aligned} &\operatorname*{limit}_{n\to\infty} \int_{C_n} | g(w) |^p | 1 + w^2 |^{-1} | dw | \\ &\qquad = \int_{-\infty}^{\infty} | g(b + iv) |^p [(1 + b)^2 + v^2]^{-1} dv. \end{aligned} \tag{3.86}$$

Let $t>0$, and choose T such that

$$\int_{-T}^{T} | g(b + iv) |^p [(1 + b)^2 + v^2]^{-1} dv \tag{3.87}$$

differs in magnitude from the right-hand member of (3.86) by less than $t/2$. When n is sufficiently large, the circle C_n will possess two sub-arcs which lie in the strip $-T\leqq \text{Im } (w) \leqq T$. Let γ_n be the one of these two arcs which is nearer the imaginary axis. As $|g|$ is, by hypothesis, bounded for Re $(w) \geqq b$, the presence of the function $|1+w^2|^{-1}$ in the right side of (3.84) guarantees that when T is sufficiently large

$$\int_{C_n-\gamma_n} |g(w)|^p |1+w^2|^{-1} |dw| < t/4. \tag{3.88}$$

Assume that T has been chosen large enough that (for all sufficiently large n) condition (3.88) is satisfied. Now all we need show is that for large n the integral

$$\int_{\gamma_n} |g(w)|^p |1+w^2|^{-1} |dw| \tag{3.89}$$

differs from (3.87) by less than $t/4$. But, this follows immediately from the fact that the integrand of (3.89) is uniformly continuous for $w=b+iv$, $-T\leqq v\leqq T$, and the fact that the points $w=u+iv$ on γ_n are tending uniformly to the points $(b+iv)$.

To review, we have established (3.83). In view of (3.82), and the fact that g is of class $\tilde{H}_p$, the integrals (3.81) are uniformly bounded, which tells us that the function G is of class H_p in Δ_0.

The lemma (3.8) leads us, by our familiar pattern, to the desired Riesz-type theorem for the half-plane.

THEOREM 3.9. *Let g be in $\tilde{H}_p(\Pi)$. Then, for almost every v, the limit* (3.41) *exists, and the functions $g_u(v)=g(u+iv)$ converge to the function f in the L_p-norm of the measure* $(1+v^2)^{-1}dv$. *If $p\geqq 1$, then* (3.06) *holds.*

As the final result of this section, we state the half-plane analogue of a theorem of Priwaloff [9] concerning analytic functions in the disc Δ_0. Priwaloff showed that if an analytic function in the open unit disc has nontangential boundary values vanishing on a set of positive measure, the analytic function is identically zero. We obtain directly for the half-plane the following.

THEOREM 3.91. *Let g be an analytic function in the half-plane* Re $(w)\geqq 0$ *such that for each v in the set S*

$$\operatorname*{limit}_{w=iv} g(w) = 0 \tag{3.92}$$

as w approaches iv along any path nontangential with the imaginary axis. If S has positive measure, then g is identically zero.

We shall be primarily interested in this result when g is represented in the form (3.06). In this case, the fulfillment of the first hypothesis of (3.91) is guaranteed by (3.2).

4. Boundary value problems in Δ. We direct our attention now to boundary value problems in the disc Δ. Every bounded Baire function F on the boundary Γ gives rise to a function G inside Δ by means of

$$G(\rho\alpha) = \int_{\Gamma} F(\alpha\beta)m_{\rho}(d\beta). \tag{4.01}$$

In the classical case, (4.01) becomes the familiar Poisson formula (3.07). In this case, there is no point in restricting oneself to bounded Baire functions, as F may be any summable function on the circle; however, in the general setting, one must consider only functions F which are summable with respect to each m_{ζ}. Accordingly, we introduce the following notation. Let $\mathfrak{L}\mathfrak{H}_1$ be the class of Baire functions on Γ, summable with respect to all the harmonic measures m_{ζ} with $0<|\zeta|<1$. Those functions in $\mathfrak{L}\mathfrak{H}_1$ which are in addition Haar summable will be said to be of class $\mathfrak{L}_1$, i.e., $\mathfrak{L}_1=\mathfrak{L}\mathfrak{H}_1\cap L_1$. The classes $\mathfrak{L}\mathfrak{H}_p$ and $\mathfrak{L}_p$ are defined similarly.

In the classical theory, it is also true that the function G is continuous for $|\zeta|<1$, since each m_{ζ} is absolutely continuous with respect to the Haar measure of Γ and has a continuous Radon-Nikodym derivative. This is a unique property of the classical situation, as in all other cases there exist F's which give rise to totally discontinuous functions G. As an example of this, we offer for F the characteristic function of the one-parameter subgroup Λ (§2). The associated G has the value zero at the origin, and for $\rho\neq 0$,

$$G(\rho\alpha) = \begin{cases} 1, & \alpha\in\Lambda, \\ 0, & \alpha\notin\Lambda. \end{cases} \tag{4.02}$$

In view of the inequality $\|G_{\rho}\|_1\leqq\|F\|_1$, one can see that for any F such that $\|F\|_1=0$, either G is discontinuous or identically zero. One can see from these remarks that boundary-value problems in Δ will present some new and interesting questions.

In order to see that things are not completely bizarre, we prove the following two theorems.

THEOREM 4.1. *If the function F in* (4.01) *is continuous, then G, equipped with F as boundary values, is continuous on all of* Δ.

Proof. The function F can be uniformly approximated by "trigonometric polynomials"

$$P(\alpha) = \sum_{k=1}^{n} \lambda_k(x_k, \alpha). \tag{4.11}$$

This follows from the Stone-Weierstrass theorem (see also [10, p. 160]). Hence it will suffice to prove the theorem for functions P (4.11). Each such P is of the form $P=\hat{f}+\bar{\hat{h}}$, where f and h are in A_1 (§1), where f corresponds to the x_k greater than or equal to zero and h to the negatives of the remaining x_k. The result follows immediately from (1.5).

THEOREM 4.2. *Let F be in $\mathcal{LH}_1$ and for* $|\zeta| > 0$ *define G by* (4.01). *Then for* $0 < \rho < 1$, *G is continuous as a function of ρ.*

Proof. The proof is immediate from the representation (2.22) and the fact that the segment $0 < \rho < 1$ is homeomorphic to the unit interval (2.1).

We now wish to show that the harmonic measures m_ρ behave like approximate identities in $\mathcal{L}_1$. The following lemma will be needed.

LEMMA 4.3. *Let U be a neighborhood of* 1 *in* Γ. *Then* $\text{limit}_{\rho=1} m_\rho(U') = 0$.

Proof. Let F be a non-negative continuous function such that $F(1) = 0$ and $F(\alpha) = 1$ on U'. Then, using (4.1),

$$\lim_{\rho=1} m_\rho(U') \leqq \text{limit } G_\rho(1) = F(1) = 0.$$

THEOREM 4.4. *Let F be in the class $\mathcal{L}_1$. If G is defined by* (4.01), *then* $\text{limit}_{\rho=1} \|F - G_\rho\|_1 = 0$.

Proof. The L_1-norm involved is that of Haar measure. From (4.01), and the fact that $m_\rho(\Gamma) = 1$, we see that

$$\|F - G_\rho\|_1 \leqq \int_\Gamma \|F - F_\beta\|_1 m_\rho(d\beta). \tag{4.41}$$

Let t be positive. As F is in $L_1(\Gamma)$, there is a neighborhood U of 1 such that β in U implies $\|F - F_\beta\|_1 < t/2$. By Lemma 4.3, we have $m_\rho(U') < t/4\|F\|_1$, for sufficiently large ρ. For such ρ,

$$\|F - G_\rho\|_1 \leqq \int_U \|F - F_\beta\|_1 m_\rho(d\beta) + \int_{U'} \|F - F_\beta\|_1 m_\rho(d\beta), \tag{4.42}$$

and each of these integrals is less than $t/2$; hence, we are done.

As our last item of business in this section we shall present an extension to the disc Δ of the Dirichlet principle (3.1). Now obviously we cannot hope for a result which is completely analogous to Fatou's general solution of the Dirichlet problem, for we have no definition of harmonicity; what should harmonic function mean in Δ? Any definition of harmonic function would certainly require the function to be continuous, and in view of the fact that (4.01) does not always define a continuous function in Δ, we shall have difficulty in obtaining a useful definition. For the present, we disregard the word "harmonic" and state the following Dirichlet-type principle.

THEOREM 4.5. *Let F be in $\mathcal{LH}_1$. Then* (4.01) *defines a function G in* $0 < |\zeta| < 1$ *with this property: the set S, of characters α such that* $\text{limit}_{\rho=1} G(\rho\alpha) \neq F(\alpha)$, *has m_ζ-measure zero for* $0 \leqq |\zeta| < 1$.

Proof. Fix ζ in Δ, $\zeta \neq 0$. We wish to show that $m_\zeta(S) = 0$. As the measures m_ρ are mutually absolutely continuous (§2), we may assume $\zeta = e^{-1}\beta$. Consider the mapping ϕ of the half-plane Re $(w) \geqq 0$ into Δ defined by

$$\phi(w) = \beta e^{-w}. \tag{4.51}$$

Define the function f on the imaginary axis of the half-plane by

$$f(iv) = F(\beta e^{-iv}) = (F \circ \phi)(iv), \tag{4.52}$$

and (see 3.05) let

$$g(u + iv) = \int_{-\infty}^{\infty} f(it) c_u(t - v) dt. \tag{4.53}$$

Now (4.53) is meaningful, as one can see by (2.2) that $g = G_\beta \circ \phi$. By the representation of elements ρ in Δ (2.1),

$$\operatorname*{limit}_{\rho=1} G(\rho\beta e^{-iv}) = F(\beta e^{-iv}) \tag{4.54}$$

if and only if

$$\operatorname*{limit}_{u=0} g(u + iv) = f(iv). \tag{4.55}$$

Notice that, since each G_ρ is a Baire function, continuous as a function of ρ (4.2), S is a Borel set. If k_S is the characteristic function of S,

$$\int_\Gamma k_S(\alpha) m_\zeta(d\alpha) = \int_{-\infty}^{\infty} k_S(\beta e^{-iv}) c_1(v) dv. \tag{4.56}$$

By (3.2), the Dirichlet theorem for the half-plane, (4.55) holds for almost every v. Thus, the equivalence of (4.54) and (4.55) tells us that the right-hand member of (4.56) is zero, i.e., $m_\zeta(S) = 0$. All that remains to be shown is that the Haar measure of S is zero. This is immediate from (2.5).

We remark that, if F is in $L_p(\Gamma)$ for some $p \geqq 1$, the Lebesgue dominating convergence principle guarantees that

$$\operatorname*{limit}_{\rho=1} \|F - G_\rho\|_p = 0.$$

5. **The generalized Riesz theorem.** In this section, we shall establish in Δ a generalization of the classical theorem of F. Riesz (3.6), and consider some of its consequences.

If $p > 0$, let $H_p = H_p(\Delta)$ denote the class of functions G, analytic (§3) in the interior of Δ, with the property that for some fixed ρ_0, $0 < \rho_0 < 1$, the integrals

$$\int_\Gamma | G(\rho\alpha\beta) |^p m_{\rho_0}(d\alpha) \tag{5.01}$$

are bounded, uniformly in ρ and β. This class is independent of ρ_0.

THEOREM 5.1. *Let G be in the class H_p. Then*

$$F(\alpha) = \operatorname*{limit}_{\rho=1} G(\rho\alpha) \tag{5.11}$$

exists, except on a set which has m_ζ-measure zero for $0 \leqq |\zeta| < 1$. The function F is in $\mathfrak{L}_p$ (§4), *and*

$$\operatorname*{limit}_{\rho=1} \|F - G\|_p^\zeta = 0. \tag{5.12}$$

If $p \geqq 1$, G is representable as the harmonic integral of F (4.01).

Proof. The proof proceeds in a manner entirely analogous to that of (4.5), upon applying the Riesz-type theorem in the half-plane (which we established in (3.9)) to the functions $g(w) = G_\beta(e^{-w})$. Two points in the procedure are worthy of illumination. First, the functions g in the half-plane are bounded on Re $(w) \geqq b > 0$, since this region corresponds to the closed subdisc of Δ defined by $|\zeta| \leqq e^{-b}$. Second, the Haar summability of $|F|^p$ and the convergence of G_ρ to F in the Haar L_p-norm are guaranteed by Theorem (2.6).

As a special case of this theorem, we obtain the generalization of Fatou's Theorem (3.3).

COROLLARY 5.2.[1] *Let G be a bounded analytic function in the interior of Δ. Then the limit* (5.31) *exists, except on a set which has m_ζ-measure zero for $0 \leqq |\zeta| < 1$, and G is representable in the form* (4.01).

In the classical situation, the case $p = 1$ of the Riesz theorem is of special interest. Theorem (3.6) with $p = 1$ is rather easily shown to be equivalent to the following.

THEOREM 5.3. *Let μ be a bounded* (*Radon*) *Borel measure on the unit circle, $|z| = 1$. If the Fourier-Stieltjes coefficients*

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-in\theta}\mu(d\theta) \tag{5.31}$$

are zero for $n < 0$, then μ is absolutely continuous with respect to Lebesgue measure.

A modern proof of (5.3) has been given by Helson [6]. A comparable result in our present context would, of course, be desirable. No similar result has yet been obtained. Perhaps the direct generalization of (5.3) would be the following. If μ is a bounded Borel measure on Γ such that the Fourier-Stieltjes coefficients

$$\hat{\mu}(x) = \int_\Gamma (x, \alpha)^- \mu(d\alpha) \tag{5.32}$$

---

[1] A proof of Fatou's theorem in this context has been found independently by J. Wermer. His proof is virtually identical with that presented here.

are zero when x is not in G_+, then μ is absolutely continuous with respect to the Haar measure of Γ. However, this statement is false, as shown by the following example, which was provided by R. Arens. For each continuous function ϕ, define

$$\int_\Gamma \phi(\alpha)\mu(d\alpha) = \int_{-\infty}^{\infty} \phi(e^{-it})(1 - it)^{-2}dt. \tag{5.33}$$

Then μ is an analytic measure, i.e., $\hat{\mu}(x)=0$ for x not in G_+, whereas (except in the classical case) μ is mutually singular with Haar measure, being supported by the one-parameter subgroup Λ (§2). Perhaps if one assumes that μ is (as a functional) a weak limit of absolutely continuous measures, or that each convolution $\mu * m_\rho$ is absolutely continuous, a conclusion like that of (5.3) will be valid. The answer remains unknown.

6. **Further results on boundary values of analytic functions.** An object worthy of much study is the Banach algebra A, of functions continuous on all of the disc Δ and analytic in its interior. We are not prepared to discuss the structure (or ideal theory) of the algebra A, but we should observe the following.

THEOREM 6.1. *If F is a continuous function on Γ and G is defined by (4.01), then a necessary and sufficient condition that G (with boundary values F) be in A is that for each x not in G_+*

$$\hat{F}(x) = \int_\Gamma (x, \alpha)^- F(\alpha)d\alpha = 0. \tag{6.11}$$

Proof. The necessity is clear. For the sufficiency, all that we must demonstrate is that F can be approximated uniformly on Γ by functions $\hat{h}$ with h in $A_1 = L_1(G_+)$. To show this, let $t>0$, and find a continuous function ϕ on Γ such that

$$\|\phi * F - F\|_\infty < t. \tag{6.12}$$

By the Plancherel theorem, $h = \hat{\phi}\hat{F}$ is in $L_1(G_+)$ and its Fourier transform (1.31) is $\phi * F$; hence, we are done.

It was shown by Arens and Singer [3, 8.2] that if G belongs to A, and if G vanishes on an open subset of Γ, then G is identically zero. In another paper [1], Arens established the more general result that for G in A

$$\int_\Gamma \log |G(\alpha)| m_\zeta(d\alpha) > -\infty \tag{6.13}$$

provided G is not identically zero. Consequently, the set of points of the boundary Γ at which $G(\alpha)=0$ must have m_ζ-measure zero. We prove a similar result, where G is not assumed to have continuous boundary values.

THEOREM 6.2. *Let G be an analytic function in the interior of Δ which is* *representable in the form* (4.01) (*that is, G is in H_1*). *If for some ζ, $0 \leqq |\zeta| < 1$, the boundary value function F vanishes on a set of positive m_ζ-measure, then G is identically zero.*

Proof. Suppose F vanishes on a set of positive $m_{\rho\beta}$-measure, $\rho > 0$. We may assume $\rho = e^{-1}$. Define the analytic function g in the half-plane Π by $g(w) = G(\beta e^{-w})$, and define f on the imaginary axis of Π by $f(it) = F(\beta e^{-it})$. Then g has the form (3.06). Furthermore, the representation (2.21) of m_ρ shows that f vanishes on a set of positive linear measure. It follows from the classical Priwaloff theorem (3.91) that g is identically zero. Hence, for each w, $G(\beta e^{-w}) = 0$. The elements βe^{-w} form a dense subset of Δ (§2), and as G is continuous, G is identically zero.

If F vanishes on a set of positive Haar measure, we see from (2.5) that F vanishes on a set of positive m_ζ-measure for some $\zeta = e^{-1}\beta$. This completes the proof.

We conclude our discussion by considering the Cauchy measures, so important in classical analysis. On the unit circle, there not only exist harmonic (Poisson) measures m_ρ such that

$$\hat{f}(\rho\alpha) = \int_\Gamma \hat{f}(\alpha\beta) m_\rho(d\beta) \tag{6.31}$$

for functions f in $A_1 = L_1(G_+)$, but there also exist Cauchy measures

$$c_\rho(d\theta) = e^{i\theta}[2\pi(e^{i\theta} - \rho)]^{-1} d\theta \tag{6.32}$$

such that

$$\hat{f}(\rho\alpha) = \int_\Gamma \hat{f}(\alpha\beta) c_\rho(d\beta) \tag{6.33}$$

for f in A_1, while if $\hat{f}(0) = 0$,

$$\int_\Gamma [\hat{f}(\alpha\beta)]^- c_\rho(d\beta) = 0. \tag{6.34}$$

This amounts to saying that the measures c_ρ have Fourier-Stieltjes transforms

$$\hat{c}_\rho(x) = \begin{cases} \hat{m}_\rho(x), & x \leqq 0, \\ 0, & x > 0. \end{cases} \tag{6.35}$$

It is convenient to describe (6.35) by saying that c_ρ is an *analytic contraction* of m_ρ. We inquire whether "Cauchy" measures exist in general setting, and to this end we prove the following.

THEOREM 6.4. *Let μ be a real, bounded, Borel measure on Γ which is symmetric, i.e., $\mu(E^{-1}) = \mu(E)$. If μ is mutually singular with Haar measure, and if*

$$\hat{\mu}(0) = \int_\Gamma \mu(d\alpha) \neq 0, \tag{6.41}$$

then μ has no analytic contraction.

Proof. Suppose ν is an analytic contraction of μ. Let $\nu=\nu_r+i\nu_i$, where ν_r and ν_i are real measures. Since μ is real and symmetric, $\hat{\mu}$ is real-valued and symmetric, i.e., $\hat{\mu}(-x)=\hat{\mu}(x)$. Since $\hat{\mu}(x)=\hat{\nu}(x)$ for $x\leqq 0$,

$$2\hat{\nu}_r(x) = \begin{cases} \hat{\mu}(x), & x \neq 0, \\ 2\hat{\mu}(0), & x = 0. \end{cases} \tag{6.42}$$

As ν_r is uniquely determined by its Fourier-Stieltjes transform, and since the Haar measure m_0 has the transform

$$\hat{m}_0(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{6.43}$$

we must have

$$\nu_r = 1/2[\mu + \hat{\mu}(0)m_0]. \tag{6.44}$$

Thus, μ is absolutely continuous with respect to ν_r. Let $|\mu|$ be the total variation of μ, etc. By the mutual singularity of m_0 and μ

$$|\nu_r| = 1/2[|\mu| + |\hat{\mu}(0)|\cdot m_0]. \tag{6.45}$$

Now, let $\nu_i=g\cdot\nu_r+\nu'$ be the Lebesgue decomposition of ν_i relative to ν_r. Let A and B be the Borel sets exhibiting the mutual singularity of ν_r and ν', and let k_A be the characteristic function of A. Consider the function

$$h = fk_A(1 + ig)^{-1}, \tag{6.46}$$

where f is the bounded Baire function such that $\mu=f\cdot\nu_r$. A routine computation shows that $\mu=h\nu$ (note that h is bounded). Let C be a Borel set of Haar measure 1 such that $|\mu|(C)=0$. Now select a sequence of real-valued "trigonometric polynomials" P_n (see proof of 4.1), such that

$$\operatorname*{limit}_{n\to\infty}\int_\Gamma |k_C(\alpha) - P_n(\alpha)|\,|\nu|(d\alpha) = 0. \tag{6.47}$$

In view of (6.45),

$$\left|\int_\Gamma |k_C(\alpha) - P_n(\alpha)|\,\mu(d\alpha)\right| \leqq \int_\Gamma |k_C(\alpha) - P_n(\alpha)|\,|\nu_r|(d\alpha). \tag{6.480}$$

Consequently,

$$\operatorname*{limit}_{n\to\infty}\int_\Gamma |k_C(\alpha) - P_n(\alpha)|\,\mu(d\alpha) = 0. \tag{6.481}$$

The polynomials P_n, being real-valued, can be expressed in the form $P_n=a_n+p_n+\bar{p}_n$, where a_n is a real constant and p_n is of the form

$$p_n(\alpha) = \sum_{j=1}^{k_n} \lambda_j(x_j, \alpha) \qquad (x_j > 0).$$

Since $\hat{\nu}(x) = 0$ for $x > 0$, and ν is an analytic contraction of μ,

$$\int_\Gamma P_n(\alpha)\nu(d\alpha) = \int_\Gamma [a_n + p_n(\alpha)]\nu(d\alpha) = \int_\Gamma [a_n + p_n(\alpha)]\mu(d\alpha). \tag{6.482}$$

Our assumptions about the set C tell us that

$$\int_\Gamma k_C(\alpha)\mu(d\alpha) = 0, \tag{6.483}$$

so that (6.481)

$$\lim_{n\to\infty} \int_\Gamma P_n(\alpha)\mu(d\alpha) = 0. \tag{6.484}$$

Since μ is a real measure

$$\lim_{n\to\infty} \mathrm{Re}\left[\int_\Gamma [a_n + p_n(\alpha)]\mu(d\alpha)\right] = 0. \tag{6.485}$$

Thus, by (6.482),

$$\lim_{n\to\infty} \mathrm{Re}\left[\int_\Gamma P_n(\alpha)\nu(d\alpha)\right] = \lim_{n\to\infty} \int_\Gamma P_n(\alpha)\nu_r(d\alpha) = 0. \tag{6.486}$$

By (6.44) and (6.47)

$$\lim_{n\to\infty} \int_\Gamma P_n(\alpha)\nu_r(d\alpha) = \int_\Gamma k_C(\alpha)\nu_r(d\alpha) = \hat{\mu}(0)/2. \tag{6.847}$$

Thus, $\hat{\mu}(0) = 0$, a contradiction.

Theorem (6.4) applied to $\mu = m_\rho$ shows us that as long as the measures m_ρ are mutually singular with Haar measure, i.e., except in the classical case, no Cauchy measures exist.

Bibliography

1. Richard Arens, *The boundary integral of* log $|\phi|$ *for generalized analytic functions*, Trans. Amer. Math. Soc. vol. 86 (1957) pp. 57–69.

2. ———, *Homeomorphisms preserving measure in a group*, Ann. of Math. vol. 60, no. 3 (1954) pp. 454–457.

3. Richard Arens and I. M. Singer, *Generalized analytic functions*, Trans. Amer. Math. Soc. vol. 81, no. 2 (1956) pp. 379–393.

4. Ludwig Bieberbach, *Lehrbuch der Functionentheorie*, vol. II, New York, Chelsea, 1945.

5. R. M. Gabriel, *An inequality concerning the integrals of positive subharmonic functions along certain curves*, J. London Math. Soc. vol. 5 (1930) pp. 129–131.

6. Henry Helson, *On a theorem of F. and M. Riesz*, Colloquium Mathematicum vol. 3 (1955).

7. E. Hille and J. D. Tamarkin, *On a theorem of Paley and Wiener*, Ann. of Math. (2) vol. 34 (1933) pp. 606–614.

8. ———, *On the absolute integrability of Fourier transforms*, Fund. Math. vol. 25 (1935) pp. 329–352.

9. N. N. Lusin and I. Priwaloff, *Sur l'unicité et multiplicité des fonctions analytiques*, Ann. École Norm. vol. 42 (1925) pp. 143–191.

10. Lynn Loomis, *Abstract harmonic analysis*, Van Nostrand, 1953.

11. F. Riesz, *Uber die Randwerte einer Analytischen Function*, Math. Zeit. vol. 18 (1923).

12. Antoni Zygmund, *Trigonometrical series*, Dover, 1955.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF.