

# TRANSFORMATION GROUPS ON A HOMOLOGICAL MANIFOLD

BY

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**1. Introduction.** In a recent paper of Montgomery-Samelson-Zippin [1], the following theorem is proved. If a compact Lie group acts as a topological transformation group on an  $n$ -dimensional manifold such that the highest dimension of any orbit is  $r$ , then the union of all the orbits of dimension  $\leq k$ ,  $0 \leq k < r$ , is a closed set of dimension  $\leq n - r + k - 1$ . Hence the singular set, i.e., the union of all the orbits not of the highest dimension, is a closed set of dimension  $\leq n - 2$ . The purpose of the present paper is to generalize this theorem to a homological manifold. This generalization together with some other results of this paper will be used in a further study [2] of transformation groups on a homological manifold.

Let  $X$  be an  $n$ -dimensional manifold and let  $G$  be a compact Lie group acting topologically on  $X$ . For each  $x \in X$ ,  $G_x$  denotes the isotropic subgroup of  $G$  at  $x$ . It is known [3] that every point  $p$  of  $X$  has an open neighborhood  $U$  such that there is a subset  $Y$  of  $U$  containing  $p$  and an open cell  $Q$  in  $G$  containing the identity such that (i) whenever  $g \in G$  and  $x \in Y$ ,  $g(x) \in Y$  if and only if  $g \in G_p$  and (ii)  $(g, x) \rightarrow g(x)$  defines a homeomorphism of  $Q \times Y$  onto  $U$ .  $Y$  is called a slice at  $p$  and may be assumed to be connected. As one can easily see,  $G_p$  may be regarded as a transformation group on  $Y$  and its orbit space coincides with that of  $G(Y)$  acted on by  $G$ . Moreover, an orbit in  $G(Y)$  is of the same type as  $G(p)$  if and only if it intersects  $Y$  at a single point. Therefore certain properties of the set of the orbits in  $G(Y)$  which are of the same type as  $G(p)$ , can be obtained by studying the fixed point set of  $G_p$  in  $Y$ .

If  $G$  acts differentiably on  $X$ , we may have open cells as slices [4] and then the proof of the cited theorem can be greatly simplified by using the remark of the preceding paragraph. For the nondifferentiable case it is not known whether  $Y$  is a manifold, as it is still an open question whether a topological space is a  $k$ -dimensional manifold if the product of the space and the real line is known to be an  $(n+1)$ -dimensional manifold. However,  $Y$  can be shown to be a homological manifold. Therefore it seems more natural to begin with a homological manifold  $X$  and then to proceed to the proof just as in the differential case. Moreover, it is pointed out in [1] that the original proof depends only on the local homology property of a manifold, and this gives us another justification to deal with homological manifolds rather than ordinary manifolds.

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Received by the editors July 2, 1956.

By a homological  $n$ -manifold we mean a connected, finite-dimensional, locally compact Hausdorff space which is of type  $P_n$  in the sense of Smith [5], where the dimension is defined to be the highest Lebesgue covering dimension of any compact subset of the space. But we can not use the group  $\mathbb{C}_q$  of integers modulo a fixed prime number  $q$  as the coefficient group as Smith did in [5], because we need the property  $P_n$  with respect to various  $\mathbb{C}_q$  at the same time. It will be seen later that we can meet this requirement by using the group  $\mathfrak{P}$  of reals modulo 1 as coefficient group. Another reason for using  $\mathfrak{P}$  as coefficient group is that it follows from [6] that the dimension of such a homological  $n$ -manifold is exactly equal to  $n$ . Note that the homological manifolds in our sense and the generalized manifolds defined in Wilder [7] are alike to a great extent.

To conclude the introduction the author wishes to express his gratitude to Professor Montgomery for his valuable suggestions and encouragement when this paper was prepared.

**2. A homology theory on  $\mathfrak{A}_{LC}$ .** We shall use a compact abelian group  $\mathbb{C}$  as coefficient group which of course can be the additive group  $\mathfrak{P}$  of reals modulo 1 or the additive group  $\mathbb{C}_q$  of integers modulo a prime number  $q$ . Let  $\mathfrak{A}_{LC}$  be the admissible category defined to consist of all the pairs  $(X, A)$  such that  $X$  is a locally compact Hausdorff space and  $A$  is a closed subset of  $X$ , and to consist of all the maps  $f$  of such pairs such that the inverse image of any compact set under  $f$  is compact. Let  $\mathfrak{A}_c$  be the subcategory of  $\mathfrak{A}_{LC}$  which consists of all the compact pairs and all the maps of such pairs. By the one-point-compactification process, we can use the Čech homology theory on  $\mathfrak{A}_c$  to define a homology theory on  $\mathfrak{A}_{LC}$  [8]. This homology theory on  $\mathfrak{A}_{LC}$  satisfies the Eilenberg-Steenrod axioms and the continuity axiom [8] and will be used throughout this paper.

Whenever  $(X, A) \in \mathfrak{A}_{LC}$ ,  $H_k(X, A; \mathbb{C})$  denotes the  $k$ th homology group of  $(X, A)$ , i.e., the  $k$ th Čech homology group of  $(X \cup \omega, A \cup \omega)$  with coefficients in  $\mathbb{C}$ , where  $X \cup \omega$  is the one-point-compactification of  $X$ .  $\partial$  denotes the boundary operator of a homology sequence and  $\Delta$  denotes the boundary operator of a Mayer-Vietoris sequence.

If  $f: (X, A) \rightarrow (Y, B)$  is a map in  $\mathfrak{A}_{LC}$ , the notation

$$H_k(X, A; \mathbb{C}) \xrightarrow{f_*} H_k(Y, B; \mathbb{C})$$

means the homomorphism of  $H_k(X, A; \mathbb{C})$  into  $H_k(Y, B; \mathbb{C})$  induced by  $f$ . If in particular  $f$  is the inclusion map, we write only

$$H_k(X, A; \mathbb{C}) \rightarrow H_k(Y, B; \mathbb{C})$$

and call it the *natural* homomorphism of  $H_k(X, A; \mathbb{C})$  into  $H_k(Y, B; \mathbb{C})$ . Whenever  $S$  is a subset or an element of  $H_k(X, A; \mathbb{C})$ ,  $S|(Y, B)$  will denote the natural homomorphic image of  $S$  in  $H_k(Y, B; \mathbb{C})$ . Any arrow to which a symbol  $\approx$  is attached will indicate an isomorphism onto.

LEMMA 1. Let  $(X, A) \in \mathfrak{A}_{LC}$  and let  $e \in H_k(X, A; \mathbb{C})$ . Let  $Y$  be a closed subset of  $X$ . Then  $e|_{(X, Y \cup A)} = 0$  if and only if  $e \in H_k(Y \cup A, A; \mathbb{C})|_{(X, A)}$ . Moreover, there exists a minimal  $Y$  satisfying these conditions, and such a set  $Y$  is contained in  $(X - A)^-$ .

LEMMA 2. Let  $(X, A) \in \mathfrak{A}_{LC}$  and let  $Y$  be a closed  $t$ -cell with boundary  $B$ . Then there is a natural isomorphism  $i$  of  $H_k(X, A; \mathbb{C})$  onto  $H_{k+t}(X \times Y, (X \times B) \cup (A \times Y); \mathbb{C}) \cup (A \times Y); \mathbb{C}$ . If  $f: (X, A) \rightarrow (X', A')$  is a map in  $\mathfrak{A}_{LC}$  and  $j: (Y, B) \rightarrow (Y, B)$  is the identity map, then the commutativity relation  $(f \times j)_* i = i_*$  holds in the diagram

$$\begin{array}{ccc} H_k(X, A; \mathbb{C}) & \xrightarrow{i} & H_{k+t}(X \times Y, (X \times B) \cup (A \times Y); \mathbb{C}) \\ \uparrow f_* & & \uparrow (f \times j)_* \\ H_k(X', A'; \mathbb{C}) & \xrightarrow{i} & H_{k+t}(X' \times Y, (X' \times B) \cup (A' \times Y); \mathbb{C}). \end{array}$$

3. **The property  $P_n$ .** A subset of a locally compact Hausdorff space is called *bounded* if its closure is compact. If  $X$  is a locally compact Hausdorff space and  $x$  is a point of  $X$ , a *neighborhood* of  $x$  (in  $X$ ) will always mean a *bounded open* subset of  $X$  containing  $x$ .

DEFINITION. Let  $X$  be a locally compact Hausdorff space and let  $n$  be a non-negative integer. A non-null bounded open subset  $U$  of  $X$  and a subgroup  $S$  of  $H_n(X, X - U; \mathbb{C})$  are said to form a *fundamental  $(\mathbb{C}, n)$ -pair  $(U, S)$*  if the following conditions are satisfied.

- (i)  $S$  is isomorphic to  $\mathbb{C}$ .
- (ii) Whenever  $W$  is a non-null open subset of  $U$ , the natural homomorphism of  $H_n(X, X - U; \mathbb{C})$  into  $H_n(X, X - W; \mathbb{C})$  maps  $S$  isomorphically onto  $S|_{(X, X - W)}$ .
- (iii) Whenever  $y$  is a point of  $U$  and  $V$  is a neighborhood of  $y$ , there is a neighborhood  $W$  of  $y$  contained in  $U \cap V$  and such that

$$H_n(X, X - V; \mathbb{C})|_{(X, X - W)} \subset S|_{(X, X - W)}$$

and for  $k \neq n$ ,

$$H_k(X, X - V; \mathbb{C})|_{(X, X - W)} = 0.$$

REMARK 1. If  $(U, S)$  is a fundamental  $(\mathbb{C}, n)$ -pair and  $Q$  is a non-null open subset of  $U$ , then  $(Q, S|_{(X, X - Q)})$  is also a fundamental  $(\mathbb{C}, n)$ -pair.

DEFINITION. Let  $X$  be a locally compact Hausdorff space. At a point  $x$  of  $X$ ,  $X$  possesses the *property  $P_n(\mathbb{C})$*  if there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, S)$  with  $U$  containing  $x$ .  $X$  is of *type  $P(\mathbb{C})$*  if  $X$  possesses the property  $P_n(\mathbb{C})$  at every point  $x$ ,  $n$  depending on  $x$ . If in particular  $n$  is a constant over  $X$ ,  $X$  is of *type  $P_n(\mathbb{C})$* .

For  $\mathbb{C} = \mathbb{C}_q$  these two definitions are due to Smith [5].

REMARK 2. If  $X$  is connected and of type  $P(\mathbb{C})$ , then  $X$  is of type  $P_n(\mathbb{C})$  for some integer  $n$ .

As a consequence of these definitions and Lemma 2, we have

LEMMA 3. *Let  $X$  be a locally compact Hausdorff space and let  $R^t$  be the euclidean  $t$ -space. Let  $U$  be a non-null bounded open subset of  $X$  and let  $V$  be a bounded open  $t$ -cell in  $R^t$ . Let  $S$  be a subgroup of  $H_n(X, X - U; \mathbb{C})$  and let  $T$  be a natural isomorphic image of  $S$  in  $H_{n+t}(X \times R^t, (X \times R^t) - (U \times V); \mathbb{C})$ ; then  $(U, S)$  is a fundamental  $(\mathbb{C}, n)$ -pair if and only if  $(U \times V, T)$  is a fundamental  $(\mathbb{C}, n+t)$ -pair. Hence  $X$  is of type  $P_n(\mathbb{C})$  if and only if  $X \times R^t$  is of type  $P_{n+t}(\mathbb{C})$ .*

DEFINITION. A locally compact Hausdorff space  $X$  is of dimension  $\leq k$  if the Lebesgue covering dimension of every compact subset of  $X$  is  $\leq k$ .

A characterization of the dimension is [6]

LEMMA 4. *A finite-dimensional locally compact Hausdorff space  $X$  is of dimension  $k$  if and only if  $k$  is the largest integer such that for some compact subsets  $M, N$  of  $X$ ,  $M \supset N$  and  $H_k(M, N; \mathbb{F}) \neq 0$ .*

REMARK 3. Because of Lemma 1, the compact subsets  $M, N$  in Lemma 4 can be so chosen that for some nonzero element  $e$  of  $H_k(M, N; \mathbb{F})$ ,  $e|_{(M, K \cup N)} \neq 0$  whenever  $K$  is a proper compact subset of  $(M - N)^-$ .

COROLLARY. *In a finite-dimensional locally compact Hausdorff space  $X$  there is a point  $x$  such that every neighborhood of  $x$  is of the same dimension as  $X$ .*

Because of Lemma 4, we have

LEMMA 5. *Let  $X$  be a finite-dimensional locally compact Hausdorff space and let  $R^t$  be the euclidean  $t$ -space. Then*

$$\dim(X \times R^t) = (\dim X) + t.$$

LEMMA 6. *Let  $X$  be a finite-dimensional locally compact Hausdorff space and let  $Y$  be a closed subset of  $X$ . Then*

$$\dim X = \max (\dim Y, \dim (X - Y)).$$

Lemma 6 is an immediate consequence of some results of [9].

LEMMA 7. *Let  $A_\alpha$  be a collection of open subsets of a non-null finite-dimensional locally compact Hausdorff space  $X$  indexed by a well-ordered set  $\{\alpha\}$ . If  $\bigcup_\alpha A_\alpha = X$  and  $\alpha' > \alpha$  implies  $A_{\alpha'} \supset A_\alpha$ , then there is some index  $\beta$  such that*

$$\dim X = \dim \left( A_\beta - \bigcup_{\alpha < \beta} A_\alpha \right).$$

**Proof.** Let  $K$  be a compact subset of  $X$  which is of the same dimension as  $X$ . Since  $K \subset X = \bigcup_\alpha A_\alpha$ , it follows that  $K$  is covered by a finite number of the sets  $A_\alpha$ . But  $A_\alpha$  is ordered by inclusion; we infer that there is some  $A_\gamma$  which contains  $K$  and then is of the same dimension as  $X$ . Since  $A_\alpha$  is well-

ordered, there is a minimal  $\beta$  such that  $A_\beta$  and  $X$  are of the same dimension. Hence the conclusion of our lemma follows from Lemma 6 if we can show that the dimension of  $\bigcup_{\alpha < \beta} A_\alpha$  is less than that of  $X$ .

If there is an index  $\gamma$  preceding to  $\beta$ , then  $\bigcup_{\alpha < \beta} A_\alpha = A_\gamma$  and hence our assertion follows from the minimality of  $\beta$ . Suppose now that there is no index preceding to  $\beta$ . If  $\beta$  is the first index, then our assertion is trivial. If  $\beta$  is not the first index, then as shown in the preceding paragraph,  $\bigcup_{\alpha < \beta} A_\alpha$  and  $A_\gamma$  are of the same dimension for some  $\gamma < \beta$ . Hence again our assertion follows from the minimality of  $\beta$ .

LEMMA 8. *Let  $X$  be a finite-dimensional locally compact Hausdorff space of type  $P_n(\mathbb{C})$ . Then for any bounded open subset  $Q$  of  $X$  and any integer  $k > n$ ,*

$$H_k(X, X - Q; \mathbb{C}) = 0.$$

*If in particular  $\mathbb{C} = \mathfrak{R}$ ,  $X$  is of dimension  $n$ .*

**Proof.** Since  $X$  is finite-dimensional, there is a greatest integer  $m$  such that for some bounded open subset  $Q$  of  $X$

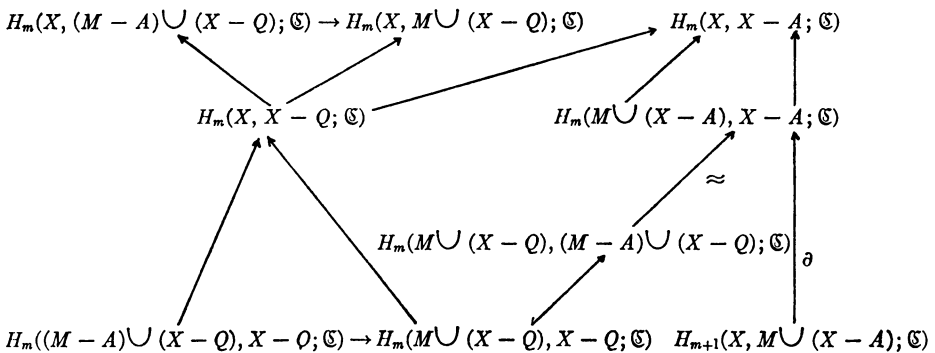
$$H_m(X, X - Q; \mathbb{C}) \neq 0.$$

Let  $e$  be a nonzero element of  $H_m(X, X - Q; \mathbb{C})$ . By Lemma 1 there exists a minimal closed set  $M$  contained in  $\bar{Q}$  and such that for some  $e' \in H_m(M \cup (X - Q), X - Q; \mathbb{C})$ ,

$$e = e' | (X, X - Q).$$

Since  $e \neq 0$ ,  $M \cap Q \neq \emptyset$  and then we may take a point  $x$  of  $M \cap Q$ . Suppose that  $m > n$ ; then there exists, by the definition of the property  $P_n(\mathbb{C})$ , a neighborhood  $A$  of  $x$  contained in  $Q$  and such that

$$H_m(X, X - Q; \mathbb{C}) | (X, X - A) = 0.$$



Because of our choice of  $M$ ,

$$e' \in H_m((M - A) \cup (X - Q), X - Q; \mathbb{C}) | (M \cup (X - Q), X - Q);$$

therefore  $e'|(M \cup (X - Q), (M - A) \cup (X - Q)) \neq 0$  and hence

$$e'|(M \cup (X - A), X - A) \neq 0.$$

Since  $e'|(X, X - A) = e|(X, X - A) = 0$ , it follows that

$$e'|(M \cup (X - A), X - A) \in \partial H_{m+1}(X, M \cup (X - A); \mathfrak{C}).$$

Hence  $H_{m+1}(X, M \cup (X - A); \mathfrak{C}) \neq 0$ , contrary to our assumption on  $m$ . This proves that  $m \leq n$ .

On the other hand, there is a fundamental  $(\mathfrak{C}, n)$ -pair  $(U, S)$ . For such a bounded open set  $U$ ,  $H_n(X, X - U; \mathfrak{C}) \neq 0$ . Hence  $m \geq n$  and consequently  $m = n$ .

Suppose now that  $\mathfrak{C} = \mathfrak{B}$ . If  $(U, S)$  is a fundamental  $(\mathfrak{B}, n)$ -pair, then  $H_n(\bar{U}, \bar{U} - U; \mathfrak{B}) \neq 0$  as it is isomorphic to  $H_n(X, X - U; \mathfrak{B})$ . Hence by Lemma 4,  $X$  is of dimension  $\geq n$ .

For any compact subsets  $M, N$  of  $X$  with  $M \supset N$ , and any integer  $k > n$ ,  $H_k(M, N; \mathfrak{B}) = 0$ . In fact, let  $Q$  be a bounded open subset of  $X$  such that  $M - Q = N$ . Then the composition

$$H_k(M, N; \mathfrak{B}) \rightarrow H_k(M \cup (X - Q), N \cup (X - Q); \mathfrak{B}) \rightarrow H_k(X, X - Q; \mathfrak{B})$$

is an isomorphism into and hence our assertion follows. This proves that  $X$  is of dimension  $\leq n$ .

**LEMMA 9.** *Let  $X$  be a finite-dimensional locally compact Hausdorff space of type  $P_n(\mathfrak{C})$ . Let  $M, N$  be compact subsets of  $X$  with  $M \supset N$ . If there is a nonzero element  $e$  of  $H_n(M, N; \mathfrak{C})$  such that for no proper closed subset  $K$  of  $(M - N)$ ,  $e|(M, K \cup N) = 0$ , then  $M - N$  is open in  $X$ . Hence if  $\mathfrak{C} = \mathfrak{B}$ , a closed subset of  $X$  is of dimension  $n$  if and only if it is somewhere dense.*

**Proof.** Let  $x$  be a point of  $M - N$  and let  $(U, S)$  be a fundamental  $(\mathfrak{C}, n)$ -pair with  $U$  containing  $x$ . Then  $x$  has a neighborhood  $V$  contained in  $U - N$  and such that for some  $u \in S$ ,  $e|(X, X - V) = u|(X, X - V)$ . By hypothesis,  $e|(M, M - V) \neq 0$  and then  $e|(M \cup (X - V), X - V) \neq 0$ . Since

$$H_{n+1}(X, M \cup (X - V); \mathfrak{C}) = 0$$

(Lemma 8), it follows that  $e|(X, X - V) \neq 0$ . Hence  $u \neq 0$ .

Suppose that  $V \not\subset M$ . Then  $W = V - M$  is a non-null open subset of  $U$ . Therefore

$$u|(X, X - W) \neq 0.$$

On the other hand,

$$u|(X, X - W) = e|(X, X - W) = 0.$$

Hence we have arrived at a contradiction. This proves that  $x$  is an inner point of  $M - N$ . Since  $x$  is arbitrary, it follows that  $M - N$  is open.

Suppose now that  $\mathfrak{C} = \mathfrak{B}$  and let  $X'$  be a closed subset of  $X$ . If  $X'$  is  $n$ -

dimensional, there exist, by Remark 3, compact subsets  $M, N$  of  $X'$  such that  $M \supset N$  and for some nonzero element  $e$  of  $H_n(M, N; \mathfrak{F})$ ,  $e|(M, K \cup N) \neq 0$  whenever  $K$  is a proper compact subset of  $(M - N)^-$ . It follows that  $M - N$  is open. Hence  $X'$  is somewhere dense. Conversely, if  $X'$  is somewhere dense, then the interior of  $X'$  is not null and therefore is  $n$ -dimensional. Hence  $X'$  is  $n$ -dimensional.

LEMMA 10. *Let  $X$  be a finite-dimensional locally compact Hausdorff space of type  $P_n(\mathbb{C})$  and let  $Q$  be a connected non-null bounded open subset of  $X$ . Then for any non-null open subset  $Q'$  of  $Q$ , the natural homomorphism of*

$$H_n(X, X - Q; \mathbb{C})$$

*into  $H_n(X, X - Q'; \mathbb{C})$  is one-one. If in particular there is a fundamental  $(\mathbb{C}, n)$ -pair  $(Q, S)$ , then  $S = H_n(X, X - Q; \mathbb{C})$  and for any connected non-null open subset  $Q'$  of  $Q$ , the natural homomorphism of  $H_n(X, X - Q; \mathbb{C})$  into  $H_n(X, X - Q'; \mathbb{C})$  is one-one and onto.*

**Proof.** Let  $e$  be a nonzero element of  $H_n(X, X - Q; \mathbb{C})$ . By Lemma 1, there is a minimal compact subset  $M$  of  $\bar{Q}$  such that for some

$$e' \in H_n(M \cup (X - Q), X - Q; \mathbb{C}), e = e'|(X, X - Q).$$

Clearly there is no proper compact subset  $K$  of  $(M \cap Q)^-$  such that

$$e'|(M \cup (X - Q), K \cup (X - Q)) = 0.$$

Since the natural homomorphism of  $H_n(M, M - Q; \mathbb{C})$  into

$$H_n(M \cup (X - Q), X - Q; \mathbb{C})$$

is one-one and onto, we may apply Lemma 9. Therefore  $M \cap Q$  is open in  $X$ . But  $M \cap Q$  is clearly non-null and closed in  $Q$ . It follows from the connectedness of  $Q$  that  $M \cap Q = Q$ . Hence

$$M = \bar{Q}.$$

It is an immediate consequence of the minimality of  $M$  that whenever  $Q'$  is a non-null open subset of  $Q$ ,

$$e|(X, X - Q') \neq 0.$$

Since  $e$  is an arbitrary nonzero element of  $H_n(X, X - Q; \mathbb{C})$ , the natural homomorphism of  $H_n(X, X - Q; \mathbb{C})$  into  $H_n(X, X - Q'; \mathbb{C})$  is one-one.

Suppose that there is a fundamental  $(\mathbb{C}, n)$ -pair  $(Q, S)$ . For any connected non-null open subset  $Q'$  of  $Q$  we take a point  $x$  of  $Q'$ . Then there is a neighborhood  $A$  of  $x$  contained in  $Q'$  and such that

$$H_n(X, X - Q'; \mathbb{C})|(X, X - A) \subset S|(X, X - A).$$

Since the natural homomorphism of  $H_n(X, X - Q; \mathbb{C})$  into  $H_n(X, X - A; \mathbb{C})$

and that of  $H_n(X, X - Q'; \mathbb{C})$  into  $H_n(X, X - A; \mathbb{C})$  are both one-one, it follows that the natural homomorphism of  $H_n(X, X - Q; \mathbb{C})$  into  $H_n(X, X - Q'; \mathbb{C})$  maps  $S$  isomorphically onto  $H_n(X, X - Q'; \mathbb{C})$ . Let  $Q' = Q$ ; then we have

$$S = H_n(X, X - Q; \mathbb{C}).$$

4.  $(\mathbb{C}, n)$ -manifolds.

DEFINITION. A  $(\mathbb{C}, n)$ -manifold is a connected, finite-dimensional, locally compact Hausdorff space of type  $P_n(\mathbb{C})$ .

It is easily seen that every  $(\mathbb{C}, n)$ -manifold is locally connected [7].

REMARK 4. An  $n$ -dimensional manifold in the ordinary sense is a  $(\mathbb{C}, n)$ -manifold. The dimension of a  $(\mathbb{C}, n)$ -manifold is not less than  $n$  and is equal to  $n$  if  $\mathbb{C} = \mathbb{P}$  (Lemma 8).

On a  $(\mathbb{C}, n)$ -manifold the fundamental  $(\mathbb{C}, n)$ -pairs  $(U, S)$  with  $U$  connected are of particular importance. As shown in Lemma 10, the group  $S$  of such a fundamental  $(\mathbb{C}, n)$ -pair  $(U, S)$  is equal to  $H_n(X, X - U; \mathbb{C})$  which is determined by  $U$  and is isomorphic to  $\mathbb{C}$ . Therefore we may use  $(U, [\mathbb{C}])$  to denote such a fundamental  $(\mathbb{C}, n)$ -pair.

LEMMA 11. Let  $X$  be a  $(\mathbb{P}, n)$ -manifold and let  $\mathbb{C}$  be an arbitrary compact abelian group. If  $(U, [\mathbb{P}])$  is a fundamental  $(\mathbb{P}, n)$ -pair with  $U$  connected, then there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, [\mathbb{C}])$ . Hence every  $(\mathbb{P}, n)$ -manifold is a  $(\mathbb{C}, n)$ -manifold.

Proof. Let  $X$  be a  $(\mathbb{P}, n)$ -manifold. Whenever  $Q$  is a bounded open subset of  $X$ , we denote by  $H^k(X, X - Q)$  the  $k$ th integral Čech cohomology group of  $(X, X - Q)$ . It is known [10] that

$$H_k(X, X - Q; \mathbb{C}) = \text{Hom}(H^k(X, X - Q), \mathbb{C}) \oplus \text{Ext}(\mathbb{C}, H^{k+1}(X, X - Q)).$$

Let  $(U, [\mathbb{P}])$  be a fundamental  $(\mathbb{P}, n)$ -pair with  $U$  connected. Since  $X$  is of dimension  $n$  (Lemma 8),  $H^{n+1}(X, X - U) = 0$ . Therefore  $H^n(X, X - U)$  is isomorphic to the additive group of integers and hence  $H_n(X, X - U; \mathbb{C})$  is isomorphic to  $\mathbb{C}$ . Now we claim that  $(U, H_n(X, X - U; \mathbb{C}))$  is a fundamental  $(\mathbb{C}, n)$ -pair.

For any connected non-null open subset  $W$  of  $U$ , the natural homomorphism of  $H_n(X, X - U; \mathbb{P})$  into  $H_n(X, X - W; \mathbb{P})$  is one-one and onto (Lemma 10) and then so is the natural homomorphism of  $H^n(X, X - W)$  into  $H^n(X, X - U)$ . Therefore the natural homomorphism of  $H_n(X, X - U; \mathbb{C})$  into  $H_n(X, X - W; \mathbb{C})$  is one-one and onto. Hence for any non-null open subset  $W$  of  $U$  the natural homomorphism of  $H_n(X, X - U; \mathbb{C})$  into  $H_n(X, X - W; \mathbb{C})$  is one-one.

Let  $y$  be a point of  $U$  and let  $V$  be a neighborhood of  $y$ . Then there is a neighborhood  $W$  of  $y$  contained in  $U \cap V$  and such that  $H_n(X, X - V; \mathbb{P}) \mid (X, X - W) \subset H_n(X, X - U; \mathbb{P}) \mid (X, X - W)$  and for  $k \neq n$ ,  $H_k(X, X - V; \mathbb{P}) \mid (X, X - W) = 0$ . Since  $X$  is locally connected, we may assume  $W$  connected.



Therefore  $H_n(X, X - U; \mathbb{C}) | (X, X - W) = H_n(X, X - W; \mathbb{C})$  (Lemma 10). Hence

$$H_n(X, X - V; \mathbb{C}) | (X, X - W) \subset H_n(X, X - U; \mathbb{C}) | (X, X - W).$$

Since  $H^n(X, X - W)$  is isomorphic to the additive group of integers,  $\text{Ext}(\mathbb{C}, H^n(X, X - W)) = 0$ . Therefore the natural homomorphism of  $\text{Ext}(\mathbb{C}, H^n(X, X - V))$  into  $\text{Ext}(\mathbb{C}, H^n(X, X - W))$  is trivial. Let  $k \neq n$ . Since  $H_k(X, X - V; \mathbb{P}) | (X, X - W) = 0$ , the natural homomorphism of  $H^k(X, X - W)$  into  $H^k(X, X - V)$  is trivial. Therefore the homomorphism of

$$\text{Hom}(H^k(X, X - V), \mathbb{C}) \text{ into } \text{Hom}(H^k(X, X - W), \mathbb{C})$$

and that of  $\text{Ext}(\mathbb{C}, H^k(X, X - V))$  into  $\text{Ext}(\mathbb{C}, H^k(X, X - W))$  are both trivial. Hence for  $k \neq n$ ,

$$H_k(X, X - V; \mathbb{C}) | (X, X - W) = 0.$$

**5.  $\mathbb{C}$ -orientability.** Let  $X$  be a  $(\mathbb{C}, n)$ -manifold and let  $\mathfrak{F}$  be the totality of fundamental  $(\mathbb{C}, n)$ -pairs  $(U, [\mathbb{C}])$  with  $U$  connected. An ordered pair of elements  $(U, [\mathbb{C}]), (V, [\mathbb{C}])$  of  $\mathfrak{F}$  is called a *step* if either  $U \subset V$  or  $V \subset U$ . A finite sequence of elements of  $\mathfrak{F}$

$$(U_1, [\mathbb{C}]), \dots, (U_t, [\mathbb{C}])$$

is called a *path* from  $(U_1, [\mathbb{C}])$  to  $(U_t, [\mathbb{C}])$  if every pair of adjacent terms is a step. For each  $i = 1, \dots, t - 1$ , we have either  $U_i \subset U_{i+1}$  or  $U_i \supset U_{i+1}$ ; it follows from Lemma 10 that there is a natural isomorphism of  $H_n(X, X - U_i; \mathbb{C})$  onto  $H_n(X, X - U_{i+1}; \mathbb{C})$ . The composition of these isomorphisms is an isomorphism of  $H_n(X, X - U_1; \mathbb{C})$  onto  $H_n(X, X - U_t; \mathbb{C})$ , called the isomorphism induced by the path.

**DEFINITION.** A  $(\mathbb{C}, n)$ -manifold is  $\mathbb{C}$ -orientable if there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, [\mathbb{C}])$  with  $U$  connected and such that every path from  $(U, [\mathbb{C}])$  to itself induces the identity isomorphism on  $H_n(X, X - U; \mathbb{C})$ .

**REMARK 5.** The following are equivalent.

- (i)  $X$  is  $\mathbb{C}$ -orientable.
- (ii) For every fundamental  $(\mathbb{C}, n)$ -pair  $(U, [\mathbb{C}])$  with  $U$  connected, every path from  $(U, [\mathbb{C}])$  to itself induces the identity isomorphism on  $H_n(X, X - U; \mathbb{C})$ .
- (iii) There exist two fundamental  $(\mathbb{C}, n)$ -pairs  $(U, [\mathbb{C}])$  and  $(V, [\mathbb{C}])$  such that  $U$  and  $V$  are connected and all the paths from  $(U, [\mathbb{C}])$  to  $(V, [\mathbb{C}])$  induces the same isomorphism of  $H_n(X, X - U; \mathbb{C})$  onto  $H_n(X, X - V; \mathbb{C})$ .
- (iv) The statement (iii) holds for any two fundamental  $(\mathbb{C}, n)$ -pairs  $(U, [\mathbb{C}]), (V, [\mathbb{C}])$  with  $U, V$  connected.

**REMARK 6.** Every point of a  $(\mathbb{C}, n)$ -manifold has a  $\mathbb{C}$ -orientable neighborhood.

**REMARK 7.** An orientable  $n$ -dimensional manifold in the ordinary sense is always  $\mathbb{C}$ -orientable. A  $(\mathbb{C}_2, n)$ -manifold is always  $\mathbb{C}_2$ -orientable.

LEMMA 12. *Let  $X$  be a  $\mathbb{C}$ -orientable  $(\mathbb{C}, n)$ -manifold and let  $Q$  be a non-null bounded open subset of  $X$ . Then there is a subgroup  $S$  of  $H_n(X, X - Q; \mathbb{C})$  such that  $(Q, S)$  is a fundamental  $(\mathbb{C}, n)$ -pair.*

**Proof.** We first prove that, if  $Q$  is connected, then  $(Q, H_n(X, X - Q; \mathbb{C}))$  is a fundamental  $(\mathbb{C}, n)$ -pair.

For each point  $x$  of  $\bar{Q}$  there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, [\mathbb{C}])$  with  $U$  connected and containing  $x$ . Since  $\bar{Q}$  is compact, there exist a finite number of fundamental  $(\mathbb{C}, n)$ -pairs

$$(U_1, [\mathbb{C}]), (U_2, [\mathbb{C}]), \dots, (U_t, [\mathbb{C}])$$

such that each  $U_i$  is connected and  $U_1 \cup \dots \cup U_t \supset \bar{Q}$ . Since  $\bar{Q}$  is connected, we may assume that for each  $i=1, \dots, t-1$ ,

$$(U_1 \cup \dots \cup U_i) \cap U_{i+1} \neq \emptyset.$$

Since  $X$  is  $\mathbb{C}$ -orientable, all the paths from  $(U_1, [\mathbb{C}])$  to  $(U_2, [\mathbb{C}])$  induces the same isomorphism  $h$  of  $H_n(X, X - U_1; \mathbb{C})$  onto  $H_n(X, X - U_2; \mathbb{C})$ . Therefore whenever  $e \in H_n(X, X - U_1; \mathbb{C})$ ,

$$e| (X, X - V) = h(e)| (X, X - V)$$

for all components  $V$  of  $U_1 \cap U_2$  and so

$$e| (X, X - (U_1 \cap U_2)) - h(e)| (X, X - (U_1 \cap U_2)) = 0.$$

By the exactness of the Mayer-Vietoris sequence of the triad  $(X; X - U_1, X - U_2)$ , there is some  $e' \in H_n(X, X - (U_1 \cup U_2); \mathbb{C})$  such that

$$e'| (X, X - U_1) = e, \quad e'| (X, X - U_2) = h(e).$$

Since  $U_1 \cup U_2$  is connected, it follows from Lemma 10 that the natural homomorphism of  $H_n(X, X - (U_1 \cup U_2); \mathbb{C})$  into  $H_n(X, X - U_1; \mathbb{C})$  is one-one. Hence it is an isomorphism onto. This isomorphism followed by  $h$  is the natural homomorphism of  $H_n(X, X - (U_1 \cup U_2); \mathbb{C})$  into  $H_n(X, X - U_2; \mathbb{C})$  which is also an isomorphism onto. Then we can easily see that  $(U_1 \cup U_2, H_n(X, X - (U_1 \cup U_2); \mathbb{C}))$  is a fundamental  $(\mathbb{C}, n)$ -pair.

Repeating this process, we can finally have a fundamental  $(\mathbb{C}, n)$ -pair  $(U_1 \cup \dots \cup U_t, H_n(X, X - (U_1 \cup \dots \cup U_t); \mathbb{C}))$ . Hence  $(Q, H_n(X, X - Q; \mathbb{C}))$  is a fundamental  $(\mathbb{C}, n)$ -pair.

Now let  $Q$  be any non-null bounded open subset of  $X$ . Since  $X$  is connected, there is a connected bounded open subset  $Q'$  of  $X$  containing  $Q$ . We have just shown that  $(Q', H_n(X, X - Q'; \mathbb{C}))$  is a fundamental  $(\mathbb{C}, n)$ -pair. It follows that  $(Q, H_n(X, X - Q'; \mathbb{C})| (X, X - Q))$  is a fundamental  $(\mathbb{C}, n)$ -pair.

COROLLARY. *For any connected non-null bounded open subset  $U$  of a  $\mathbb{C}$ -orientable  $(\mathbb{C}, n)$ -manifold there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, [\mathbb{C}])$ . Hence for any  $\mathbb{C}$ -orientable  $(\mathbb{C}, n)$ -manifold  $X$ ,  $H_n(X; \mathbb{C})$  is isomorphic to  $\mathbb{C}$ .*

6. Local separation.

LEMMA 13. Let  $X$  be a  $(\mathbb{C}, n)$ -manifold and let  $M, N$  be compact subsets of  $X$  with  $M \supset N$ . If there is a nonzero element  $e$  of  $H_{n-1}(M, N; \mathbb{C})$  such that for no proper compact subset  $K$  of  $(M - N)^-$ ,  $e|(M, K \cup N) = 0$ , then every point  $x$  of  $M - N$  has a neighborhood  $Q$  such that every neighborhood of  $x$  meets at least two components of  $Q - M$ .

**Proof.** Let  $x$  be a point of  $M - N$  and let  $(U, [\mathbb{C}])$  be a fundamental  $(\mathbb{C}, n)$ -pair such that  $U$  is connected and  $x \in U \subset X - N$ . Let  $Q$  be a neighborhood of  $x$  contained in  $U$  and such that

$$H_{n-1}(X, X - U; \mathbb{C}) | (X, X - Q) = 0.$$

The closure of  $Q - M$  contains  $Q$ . Otherwise,  $Q' = Q - (Q - M)^-$  is a non-null open set contained in  $M$ . It follows from

$$\begin{array}{ccc} H_{n-1}(M, N; \mathbb{C}) \rightarrow H_{n-1}(X, X - U; \mathbb{C}) \rightarrow H_{n-1}(X, X - Q; \mathbb{C}) \\ \downarrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ H_{n-1}(M, M - Q'; \mathbb{C}) \xrightarrow{\qquad \qquad \qquad \approx \qquad \qquad \qquad} H_{n-1}(X, X - Q'; \mathbb{C}) \end{array}$$

that  $e|(M, M - Q') = 0$ , contrary to our hypothesis.

Suppose that there is a neighborhood of  $x$  which meets only one component  $A$  of  $Q - M$ . Then

$$V = Q - [Q - (M \cup A)]^-$$

is a neighborhood of  $x$ . Both  $V$  and  $V - M$  contain  $A$  and are contained in  $\bar{A}$ ; therefore they are connected and hence the natural homomorphism of  $H_n(X, X - V; \mathbb{C})$  into  $H_n(X, M \cup (X - V); \mathbb{C})$ ; is one-one and onto.

$$\begin{array}{ccccc} H_{n-1}(X, X - V; \mathbb{C}) \leftarrow H_{n-1}(M \cup (X - V), X - V; \mathbb{C}) \xleftarrow{\partial} H_n(X, M \cup (X - V); \mathbb{C}) \\ \uparrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \uparrow \\ H_{n-1}(M, M - V; \mathbb{C}) \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad H_n(X, X - V; \mathbb{C}) \\ \uparrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \uparrow \\ H_{n-1}(X, X - U; \mathbb{C}) \longleftarrow H_{n-1}(M, N; \mathbb{C}) \end{array}$$

By hypothesis,  $e|(M, M - V) \neq 0$ . Therefore  $e|(X, X - V) \neq 0$ . On the other hand,

$$e|(X, X - V) = (e|(X, X - U)) | (X, X - V) = 0.$$

Hence we have arrived at a contradiction.

LEMMA 14. Let  $X$  be a  $(\mathbb{B}, n)$ -manifold and let  $Y$  be a closed subset of  $X$ . Then  $Y$  is  $(n - 1)$ -dimensional if and only if  $Y$  is nowhere dense and there is a point  $x$  of  $Y$  and a neighborhood  $Q$  of  $x$  such that every neighborhood of  $x$  meets at least two components of  $Q - Y$ .

**Proof.** Suppose that  $Y$  is  $(n - 1)$ -dimensional; then it is nowhere dense by

Lemma 9. By Remark 3, there exist compact subsets  $M, N$  of  $Y$  such that  $M \supset N$  and for some nonzero element  $e$  of  $H_{n-1}(M, N; \mathfrak{F})$ ,  $e|(M, K \cup N) \neq 0$  whenever  $K$  is a proper compact subset of  $[M - N]^-$ . It follows from Lemma 13 that there is a point  $x$  of  $M - N$  and a neighborhood  $Q$  of  $x$  such that every neighborhood of  $x$  meets at least two components of  $Q - M$ . Since  $Y$  is nowhere dense, no open set can be contained in  $Y$ . Hence every neighborhood of  $x$  meets at least two components of  $Q - Y$ .

Conversely, suppose that  $Y$  is nowhere dense and that there is a point  $x$  of  $Y$  and a neighborhood  $Q$  of  $x$  such that every neighborhood of  $x$  meets at least two components of  $Q - Y$ . By Lemma 9, the dimension of  $Y$  is  $\leq n - 1$ . Since  $X$  is a  $(\mathfrak{F}, n)$ -manifold, there is a fundamental  $(\mathfrak{F}, n)$ -pair  $(U, [\mathfrak{F}])$  with  $U$  connected and contained in  $Q$ . Clearly  $U - Y$  contains at least two components. Let  $A$  be a component of  $U - Y$ . Let

$$M = [(\bar{A} - A) \cap U]^-, \quad N = M - U.$$

We can easily see that  $M, N$  are compact subsets of  $Y$  and that the natural homomorphism of  $H_{n-1}(M, N; \mathfrak{F})$  into  $H_{n-1}(\bar{A} - A, \bar{A} - U; \mathfrak{F})$  is one-one and onto. Since  $H_n(\bar{A}, \bar{A} - U; \mathfrak{F}) = 0$  and  $H_n(\bar{A}, \bar{A} - A; \mathfrak{F})$  is isomorphic to  $\mathfrak{F}$  (Lemma 10), it follows from

$$\begin{aligned} H_{n-1}(M, N; \mathfrak{F}) &\xrightarrow{\approx} H_{n-1}(\bar{A} - A, \bar{A} - U; \mathfrak{F}) \xleftarrow{\partial} H_n(\bar{A}, \bar{A} - A; \mathfrak{F}) \\ &\xleftarrow{\quad} H_n(\bar{A}, \bar{A} - U; \mathfrak{F}) \end{aligned}$$

that  $H_{n-1}(M, N; \mathfrak{F}) \neq 0$ . Hence the dimension of  $Y$  is  $\geq n - 1$  by Lemma 4. This proves that  $Y$  is  $(n - 1)$ -dimensional.

LEMMA 15. Let  $X$  be a  $(\mathbb{C}_q, n)$ -manifold and let  $T$  be a periodical transformation on  $X$  of period  $q$ , where  $q$  is any prime number. Then the fixed point set  $L$  of  $T$  is closed (and hence locally compact), locally connected, nowhere dense and of type  $P(\mathbb{C}_q)$ . Hence every component of  $L$  is open in  $L$  and is a  $(\mathbb{C}_q, k)$ -manifold, where  $k$  depends on the component and is  $\leq n - 1$ . The equality holds only if  $q = 2$ .

REMARK 8. This statement, though slightly general, is essentially the same as the one given in [5]. Firstly, the conclusion is a local property; therefore the compactness of  $X$  required in [5] can be replaced by the local compactness. Secondly, the assumption that every bounded open set is a countable union of compact sets is not essential and then can be omitted. Finally, the property  $Q$  used in [5] can be removed, as we can prove that a locally compact Hausdorff space which possesses the property  $P_n(\mathbb{C})$  at point  $x$  possesses also the property  $Q(\mathbb{C})$  at  $x$  (see the Appendix).

LEMMA 16. Let  $X$  be a  $(\mathbb{C}_2, n)$ -manifold, let  $T$  be a periodical transformation on  $X$  of period 2 and let  $L$  be the fixed point set of  $T$ . If  $x$  is a point of  $L$  at which  $L$  possesses the property  $P_{n-1}(\mathbb{C}_2)$  (Lemma 15), then  $x$  has a connected neighborhood  $Q$  such that  $Q = T(Q) = \text{Int } \bar{Q}$  (i.e., the interior of  $\bar{Q}$ ) and  $Q - L$  has exactly two components which are mapped into each other by  $T$ .

**Proof.** By hypothesis, there is a fundamental  $(\mathbb{C}_2, n-1)$ -pair  $(U, S)$  of  $L$  with  $U$  containing  $x$ . Since the natural homomorphism of  $H_{n-1}(\bar{U}, \bar{U}-U; \mathbb{C}_2)$  into  $H_{n-1}(L, L-U; \mathbb{C}_2)$  is one-one and onto, there is an element  $e$  of  $H_{n-1}(\bar{U}, \bar{U}-U; \mathbb{C}_2)$  such that for no proper compact subset  $K$  of  $\bar{U}$ ,  $e|(\bar{U}, K \cup (\bar{U}-U))=0$ . It follows from Lemma 13 that there is a neighborhood  $A$  of  $x$  such that every neighborhood of  $x$  meets at least two components of  $A-\bar{U}$ . Since  $L$  is nowhere dense (Lemma 15), every neighborhood of  $x$  meets at least two components of  $A-L$ . Let  $B$  be the component of  $A \cap T(A)$  containing  $x$  and let

$$Q = \text{Int } \bar{B}.$$

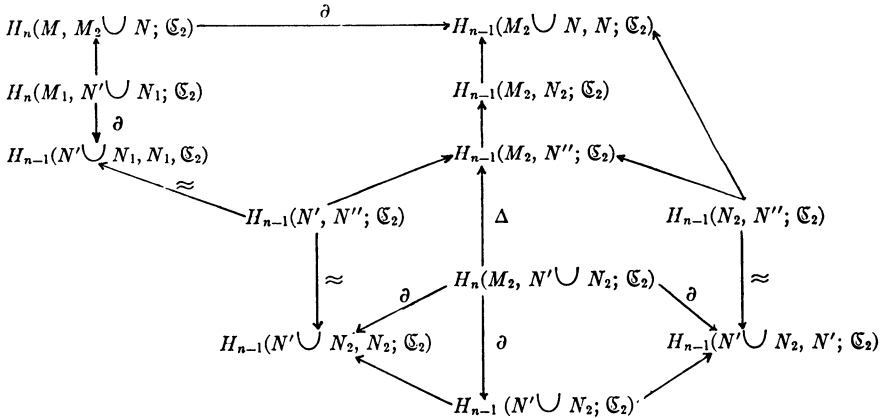
It is clear that  $Q$  is connected,  $Q = T(Q) = \text{Int } \bar{Q}$  and  $Q-L$  has at least two components. We remain to prove that the components of  $Q-L$  are two in number and are mapped into each other by  $T$ .

Let  $C$  be a component of  $Q-L$ . Then  $T(C) \neq C$ , since otherwise, the periodical transformation  $T': Q \rightarrow Q$  defined by

$$T'(x) = \begin{cases} T(x) & \text{if } x \in C, \\ x & \text{if } x \in Q - C \end{cases}$$

has a fixed point set which is not nowhere dense, contrary to Lemma 15.

Now we claim that every point of  $Q \cap \bar{C}$  is contained in  $\text{Int}(\bar{C} \cup T(\bar{C}))$ . Let  $y$  be a point of  $Q \cap \bar{C}$  and let  $D$  be a neighborhood of  $y$  whose closure is contained in  $Q$ . In the diagram



we let

$$\begin{aligned} M_1 &= (C \cap D)^-, & M_2 &= T(M_1), & M &= M_1 \cup M_2; \\ N_1 &= M_1 - D, & N_2 &= T(N_1), & N &= N_1 \cup N_2; \\ N' &= M_1 - C = M_1 \cap M_2; \\ N'' &= M_1 - (C \cup D) = N_1 \cap N_2 = N' \cap N_1 = N' \cap N_2. \end{aligned}$$

Since  $H_n(X, X - Q; \mathbb{C}_2)$  is isomorphic to  $\mathbb{C}_2$  ( Lemmas 10 and 12), there is only one element  $u$  of  $H_n(X, X - Q; \mathbb{C}_2)$  different from 0. Under the natural homomorphisms

$$H_n(X, X - Q; \mathbb{C}_2) \longrightarrow H_n(X, X - (M_i - (N' \cup N_i)); \mathbb{C}_2) \\ \longleftarrow \approx H_n(M_i, N' \cup N_i; \mathbb{C}_2)$$

$u|(X, X - (M_i - (N' \cup N_i))) = u_i|(X, X - (M_i - (N' \cup N_i)))$  determines a unique nonzero element  $u_i$  of  $H_n(M_i, N' \cup N_i; \mathbb{C}_2)$ ,  $i = 1, 2$ .

Every group of the above diagram has an element which is corresponding to either  $u_1$  or  $u_2$  or both. The one in  $H_{n-1}(M_2 \cup N, N; \mathbb{C}_2)$  is 0 as

$$H_{n-1}(N_2, N''; \mathbb{C}_2) | (M_2 \cup N, N) = 0.$$

It follows that for some  $e \in H_n(M, N; \mathbb{C}_2)$ ,

$$u_1 | (M, M_2 \cup N) = e | (M, M_2 \cup N).$$

Let  $K$  be a minimal compact subset of  $M$  such that for some  $e' \in H_n(K \cup N, N; \mathbb{C}_2)$ ,

$$e = e' | (M, N).$$

Let  $D'$  be any neighborhood of  $y$  not intersecting  $N$ . It follows from

$$e' | (X, X - (C \cap D')) = e | (X, X - (C \cap D')) \\ = u_1 | (X, X - (C \cap D')) = u | (X, X - (C \cap D')) \neq 0$$

that  $y \in K$ . By Lemma 9,  $K - N$  is a neighborhood of  $y$  contained in  $M$ . Hence  $y$  is an inner point of  $\bar{C} \cup T(\bar{C})$ .

As a consequence of this result, we have

$$Q = \text{Int} (\bar{C} \cup T(\bar{C})).$$

In fact, it follows from  $Q = \text{Int} \bar{Q}$  that  $Q \supset \text{Int} (\bar{C} \cup T(\bar{C}))$ . Suppose that our assertion is false. Then there exists, by the connectedness of  $Q$ , a point of  $Q \cap (\bar{C} \cap T(\bar{C}))$  which does not belong to  $\text{Int} (\bar{C} \cup T(\bar{C}))$ . But such a point clearly belongs to  $Q \cap \bar{C}$  and then belongs to  $\text{Int} (\bar{C} \cup T(\bar{C}))$  by our result above. Hence we have arrived at a contradiction. This proves that  $Q - L$  has exactly two components  $C$  and  $T(C)$ .

**COROLLARY.** *Let  $X$  be a  $(\mathfrak{F}, n)$ -manifold, let  $T$  be a periodical transformation on  $X$  of period 2 and let  $L$  be a component of the fixed point set of  $T$ . If  $L$  is a  $(\mathbb{C}_2, n - 1)$ -manifold (see Lemmas 11 and 15), then  $L$  is also a  $(\mathfrak{F}, n - 1)$ -manifold.*

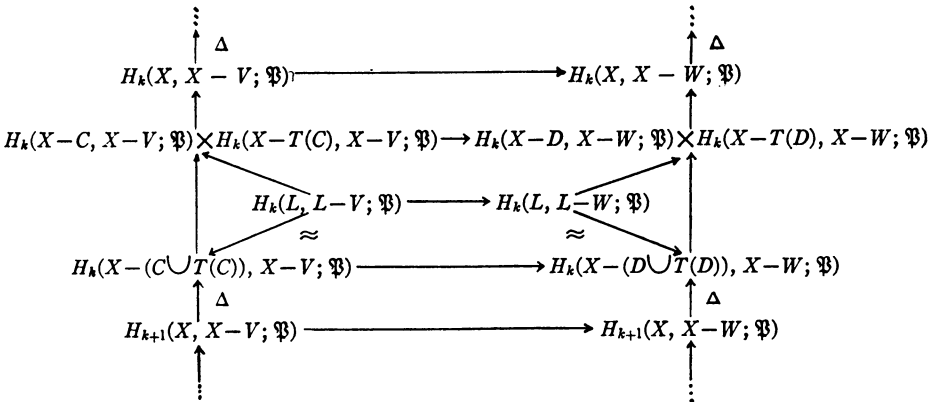
**Proof.** Let  $x$  be a point of  $L$  and let  $(U, [\mathfrak{F}])$  be a fundamental  $(\mathfrak{F}, n)$ -pair with  $U$  connected and containing  $x$ . Since  $X$  is also a  $(\mathbb{C}_2, n)$ -manifold (Lemma 11), it follows from Lemma 16 that  $x$  has a connected neighborhood

$V$  such that  $V \subset U$ ,  $T(V) = V$  and  $V - L$  has exactly two components  $C$  and  $T(C)$ .  $(V, H_n(X, X - V; \mathfrak{P}))$  is clearly a fundamental  $(\mathfrak{P}, n)$ -pair.

For any point  $y$  of  $L \cap V$  and any neighborhood  $Q$  of  $y$  there exists, by Lemma 16 and the definition of the property  $P_n(\mathfrak{P})$ , a connected neighborhood  $W$  of  $y$  such that (i)  $W \subset V \cap Q$ , (ii)  $T(W) = W$ , (iii)  $W - L$  has exactly two components  $D$  and  $T(D)$  with  $D$  contained in  $C$ , (iv) the natural homomorphism of  $H_n(X, X - V; \mathfrak{P})$  into  $H_n(X, X - W; \mathfrak{P})$  is one-one and onto, and (v) whenever  $k \neq n$ ,

$$H_k(X, X - V; \mathfrak{P}) \mid (X, X - W) = 0.$$

Now let us observe the natural homomorphism of the Mayer-Vietoris sequence of  $(X; X - C, X - T(C))$  into that of  $(X; X - D, X - T(D))$ .



For  $k < n$ , the natural homomorphism

$$\begin{aligned} &H_k(X - C, X - V; \mathfrak{P}) \times H_k(X - T(C), X - V; \mathfrak{P}) \\ &\rightarrow H_k(X - D, X - W; \mathfrak{P}) \times H_k(X - T(D), X - W; \mathfrak{P}) \end{aligned}$$

is trivial. In fact, let  $e$  be any element of  $H_k(X - C, X - V; \mathfrak{P})$ . The image of  $(e \mid (X - D, X - W), 0)$  in  $H_k(X, X - W; \mathfrak{P})$  is 0; therefore  $(e \mid (X - D, X - W), 0)$  is the image of some  $e' \in H_k(X - (D \cup T(D)), X - W; \mathfrak{P})$ . Since  $e'$  is invariant under the homomorphism induced by  $T$ , it follows that the isomorphism of  $H_k(X - D, X - W; \mathfrak{P})$  onto  $H_k(X - T(D), X - W; \mathfrak{P})$ , induced by  $T$ , maps  $e \mid (X - D, X - W)$  into 0. Hence  $e \mid (X - D, X - W) = 0$ .

Let  $A$  be a neighborhood of  $y$  contained in  $W$  and such that whenever  $k < n - 1$ ,

$$H_{k+1}(X, X - W; \mathfrak{P}) \mid (X, X - A) = 0.$$

If  $k < n - 1$  and  $e \in H_k(L, L - V; \mathfrak{P})$ , then

$$e | (X - (D \cup T(D)), X - W) = \Delta e'$$

for some  $e' \in H_{k+1}(X, X - W; \mathfrak{P})$ . It follows from  $e' | (X, X - A) = 0$  that  $e | (L, L - A) = 0$ . This proves that whenever  $k < n - 1$ ,

$$H_k(L, L - V; \mathfrak{P}) | (L, L - A) = 0.$$

Since  $L$  is of dimension  $\leq n - 1$ , it follows that whenever  $k > n - 1$ ,  $H_k(L, L - V; \mathfrak{P}) = 0$  and hence

$$H_k(L, L - V; \mathfrak{P}) | (L, L - A) = 0.$$

It is clear that the natural homomorphism

$$H_n(X - C, X - V; \mathfrak{P}) \times H_n(X - T(C), X - V; \mathfrak{P}) \rightarrow H_n(X, X - V; \mathfrak{P})$$

is trivial. Therefore there is a subgroup  $S$  of  $H_{n-1}(L, L - V; \mathfrak{P})$  such that  $\Delta$  maps  $H_n(X, X - V; \mathfrak{P})$  isomorphically onto  $S | (X - (C \cup T(C)), X - V)$ . Similarly there is a subgroup  $S'$  of  $H_n(X, X - W; \mathfrak{P})$  isomorphically  $S' | (X - (D \cup T(D)), X - W)$ . Therefore it is clear that the natural homomorphism of  $H_{n-1}(L, L - V; \mathfrak{P})$  into  $H_{n-1}(L, L - W; \mathfrak{P})$  maps  $S$  isomorphically onto  $S'$ . For any  $e \in H_{n-1}(L, L - V; \mathfrak{P})$ , the image of  $e | (L, L - W)$  in  $H_{n-1}(X - D, X - W; \mathfrak{P}) \times H_{n-1}(X - T(D), X - W; \mathfrak{P})$  is 0; therefore  $e | (X - (D \cup T(D)), X - W)$  belongs to  $\Delta H_n(X, X - W; \mathfrak{P})$ . Hence  $e | (L, L - W) \in S'$ .

From this result it follows that

$$H_{n-1}(L, L - V; \mathfrak{P}) | (L, L - A) = S | (L, L - A).$$

With  $W, A$  in place of  $V, Q$  we can find a neighborhood  $W'$  of  $y$  in  $W$ . Therefore the natural homomorphism of  $H_{n-1}(L, L - W; \mathfrak{P})$  into  $H_{n-1}(L, L - W'; \mathfrak{P})$  maps  $S'$  isomorphically into  $H_{n-1}(L, L - W'; \mathfrak{P})$ . Hence the natural homomorphism of  $H_{n-1}(L, L - V; \mathfrak{P})$  into  $H_{n-1}(L, L - A; \mathfrak{P})$  maps  $S$  isomorphically onto  $S | (L, L - A)$ .

Combining all these results, one can easily see that  $(L \cap V, S)$  is a fundamental  $(\mathfrak{P}, n - 1)$ -pair. This proves that  $L$  possesses that property  $P_{n-1}(\mathfrak{P})$  at  $x$ . But  $x$  is an arbitrary point of  $L$ ; hence  $L$  is a  $(\mathfrak{P}, n - 1)$ -manifold.

REMARK 9. From this Corollary the following general question is raised. If  $X$  is a  $(\mathfrak{P}, n)$ -manifold and  $T$  is a periodical transformation on  $X$  of prime period, is every component of the fixed point set of  $T$  a homological manifold with respect to  $\mathfrak{P}$ ?

7. Theorem 1.

THEOREM 1. *Let  $X$  be a  $(\mathfrak{P}, n)$ -manifold and let  $H$  be a finite group acting effectively on  $X$ . If  $H$  is not trivial, the dimension of the fixed point set of  $H$  is  $\leq n - 1$ . The equality holds only if the order of  $H$  is 2.*

Proof. Let  $T$  be an element of  $H$  of prime order  $q$ . Let  $L$  be the fixed point set of  $T$  and let  $L'$  be the fixed point set of  $H$ .



Since a  $(\mathfrak{P}, n)$ -manifold is of dimension  $n$ ,  $L$  is of dimension  $\leq n$ . By Lemma 11,  $X$  is also a  $(\mathfrak{C}_q, n)$ -manifold; it follows from Lemmas 15 and 9 that  $L$  is of dimension  $\leq n-1$ .

[Since  $L' \subset L$ , the dimension of  $L'$  is  $\leq n-1$ . Now we assume that  $L'$  is of dimension  $n-1$ ; then  $L$  is also of dimension  $n-1$ .

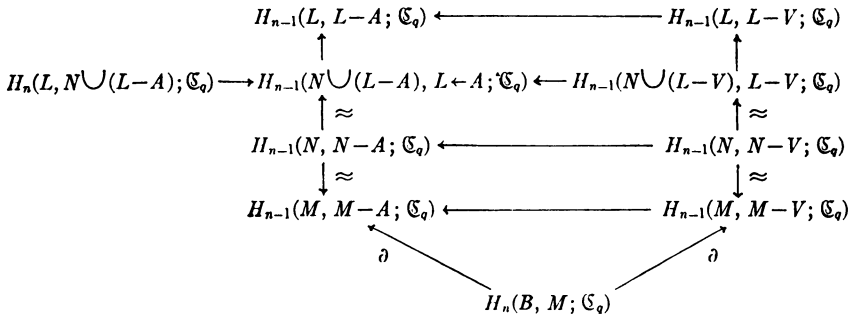
By Lemma 14, there is a point  $x$  of  $L'$  and a neighborhood  $Q$  of  $x$  such that every neighborhood of  $x$  meets at least two components of  $Q-L$ .

Let  $(U, S)$  be a fundamental  $(\mathfrak{P}, n)$ -pair with  $U$  containing  $x$ . Since both  $X$  and  $L$  are locally connected, there is a connected neighborhood  $V$  of  $x$  contained in  $Q \cap U$  and such that  $L \cap V$  is connected. It is clear that  $(V, [\mathfrak{P}])$  is a fundamental  $(\mathfrak{P}, n)$ -pair and that  $V-L$  has at least two components.

Let  $B$  be a component of  $V-L$ . Then  $(B, [\mathfrak{P}])$  is a fundamental  $(\mathfrak{P}, n)$ -pair. By Lemma 11,  $(B, [\mathfrak{C}_q])$  is a fundamental  $(\mathfrak{C}_q, n)$ -pair; therefore  $H_n(\bar{B}, \bar{B}-B; \mathfrak{C}_q)$  has a nonzero element  $u$ . Let  $M = \bar{B}-B$ ; then  $V \cap M$  is not null and is contained in  $L$ . Let  $y$  be a point of  $V \cap M$  and let  $N = (V \cap M)^-$ . For any neighborhood  $A$  of  $y$  contained in  $V$  there is a unique element  $e_A$  of  $H_{n-1}(N, N-A; \mathfrak{C}_q)$  such that

$$e_A \mid (M, M-A) = \partial u,$$

where  $\partial$  is the boundary operator of the homology sequence of the triple  $(\bar{B}, M, M-A)$ .



Since the dimension of  $L$  is  $n-1$ ,  $H_n(L, N \cup (L-A); \mathfrak{C}_q) = 0$ . Therefore

$$e'_A = e_A \mid (L, L-A) \neq 0.$$

Now  $e'_V$  is an element of  $H_{n-1}(L, L-V; \mathfrak{C}_q)$  such that for every neighborhood  $A$  of  $y$  contained in  $V$ ,

$$e'_V \mid (L, L-A) = e'_A \neq 0.$$

It follows that  $L$  possesses the property  $P_{n-1}(\mathfrak{C}_q)$  at  $y$  (Lemma 15). Since  $L \cap V$  is connected and contains both  $x$  and  $y$ ,  $L$  possesses the property  $P_{n-1}(\mathfrak{C}_q)$  at  $x$  (Lemma 15). Moreover,  $q$  must be equal to 2 (Lemma 15).

By Lemma 16, there is a connected neighborhood  $W$  of  $x$  contained in  $V$  and such that  $W = T(W)$  and  $W - L$  contains exactly two components  $C$  and  $T(C)$ .

Since  $W$  is contained in  $Q$ ,  $W - L'$  has at least two components. Therefore  $W \cap L' = W \cap L$  and then  $W - L'$  has exactly two components  $C$  and  $T(C)$ . Let  $Q'$  be a neighborhood of  $x$  invariant under  $H$  and contained in  $W$ . Then every element of  $H$  maps  $Q' \cap C$  into  $Q' \cap T(C)$  or itself.

Suppose that the order of  $H$  is  $> 2$ . Then there is an element other than the identity maps  $Q' \cap C$  into itself. Certain power of this element, denoted by  $T'$ , is of prime order  $q$ . Just as we have seen above,  $q = 2$  and there is a neighborhood  $W'$  of  $x$  contained in  $Q'$  and such that  $T'(W') = W'$  and  $W' - L'$  has exactly two components  $C'$  and  $T'(C')$ .  $C'$  is contained in either  $C$  or  $T(C)$ ; we may assume that  $C' \subset C$ . Since  $T'(Q' \cap C) = Q' \cap C$ , it follows that  $T'(C') \subset C$ . Therefore  $C \cup L'$  contains  $W'$  and then contains  $x$  as an inner point. This is clearly impossible.

**8. Slices.**

**DEFINITION.** Let  $X$  be a Hausdorff space and let  $G$  be a compact Lie group acting topologically on  $X$ . Let  $p$  be a point of  $X$  and  $G_p$  the isotropic subgroup of  $G$  at  $p$ . A *slice* at  $p$  is a subset  $Y$  of  $X$  containing  $p$  and satisfying the following conditions:

- (i) Whenever  $g \in G$  and  $x \in Y$ ,  $g(x) \in Y$  if and only if  $g \in G_p$ .
- (ii) If  $Q$  is a small open cell which is a local cross-section to the cosets of  $G_p$  at the identity of  $G$ , then  $(g, x) \rightarrow g(x)$  defines a homeomorphism of  $Q \times Y$  onto  $Q(Y)$ . Moreover,  $Q(Y)$  is open in  $X$ .

**LEMMA 17.** *If  $X$  is completely regular Hausdorff space and  $G$  is a compact Lie group acting on  $X$ , then at every point of  $X$  there exists a slice.*

This Lemma was first proved by Montgomery-Yang [3] for complete metric spaces and then extended to completely regular spaces by Mostow [12].

**9. Theorem 2.**

**THEOREM 2.** *Let  $X$  be a  $(\mathfrak{P}, n)$ -manifold and let  $G$  be a compact Lie group acting topologically on  $X$  such that the highest dimension of any orbit is  $r$ . Then for any integer  $k$ ,  $0 \leq k < r$ , the union of all the orbits of dimension  $\leq k$  is a closed set of dimension  $\leq n - r + k - 1$ .*

**Proof.** Let  $X$  be a  $(\mathfrak{P}, n)$ -manifold and let  $G$  be a compact Lie group acting on  $X$ . For each point  $x$  of  $X$ , we denote by  $G_x$  the isotropic subgroup of  $G$  at  $x$  and by  $G_x^*$  the identity component of  $G_x$ . Then the order of the quotient group  $G_x/G_x^*$  is finite and will be denoted by  $m(x)$ .

For any integers  $u \geq 0, v \geq 1$  we let

$$X_{u,v} = \{x : x \in X, \dim G(x) = u, m(x) = v\}, \quad X_u = \bigcup_{v=1}^{\infty} X_{u,v}.$$

Since every point  $x$  of  $X$  has a neighborhood  $U$  such that whenever  $y \in U, G_y$  is conjugate to a subgroup of  $G_x$  [11], it follows that whenever  $u \geq 0$  and  $v \geq 0,$

$$Z_{u,v} = X_r \cup X_{r-1} \cup \dots \cup X_{u+1} \cup X_{u,1} \cup X_{u,2} \cup \dots \cup X_{u,v}$$

is open. Hence all  $Z_{u,v}$  and consequently all  $X_{u,v}$  are locally compact Hausdorff.

Each  $X_{u,v}$  intersects  $\bar{X}_r - X_r$  at a set of dimension  $\leq n - 2$ . In fact, let  $p$  be a point of  $X_{u,v} \cap (\bar{X}_r - X_r)$  and let  $Y$  be a slice at  $p$ . Clearly  $Y$  may be assumed to be connected and then is a  $(\mathfrak{B}, n - u)$ -manifold (Lemma 3). Since  $p \in \bar{X}_r,$  there is a point  $x$  of  $Y$  belonging to  $X_r.$  It follows from  $\dim G_p > \dim G_x$  that there is a finite subgroup  $H$  of  $G_p$  in which the index of  $H \cap G_x$  is  $> 2.$  Making use of Theorem 1, one can easily show that the fixed point set of  $H$  is of dimension  $\leq n - u - 2.$  Hence  $X_{u,v} \cap Y$  is of dimension  $\leq n - u - 2,$  as it is contained in the fixed point set of  $H.$  Let  $Q$  be an open  $u$ -cell in  $G$  which contains the identity and is such that  $(g, x) \rightarrow g(x)$  defines a homeomorphism of  $Q \times Y$  onto  $Q(Y)$  and that  $Q(Y)$  is a neighborhood of  $p.$  Then  $Q(X_{u,v} \cap Y)$  is a neighborhood of  $p$  in  $X_{u,v}$  and is of dimension  $\leq n - 2$  (Lemma 5). Since  $p$  is an arbitrary point of  $X_{u,v} \cap (\bar{X}_r - X_r),$  the dimension of  $X_{u,v} \cap (\bar{X}_r - X_r)$  is  $\leq n - 2$  (Corollary to Lemma 4). From this result and Lemma 7, it follows that the dimension of  $\bar{X}_r - X_r$  is  $\leq n - 2.$  Hence  $\bar{X}_r = X$  and consequently  $X - X_r,$  i.e., the union of all the orbits not of the highest dimension, is a closed set of dimension  $\leq n - 2,$  since otherwise  $\bar{X}_r - X_r$  would be of dimension  $n - 1$  by Lemma 9.

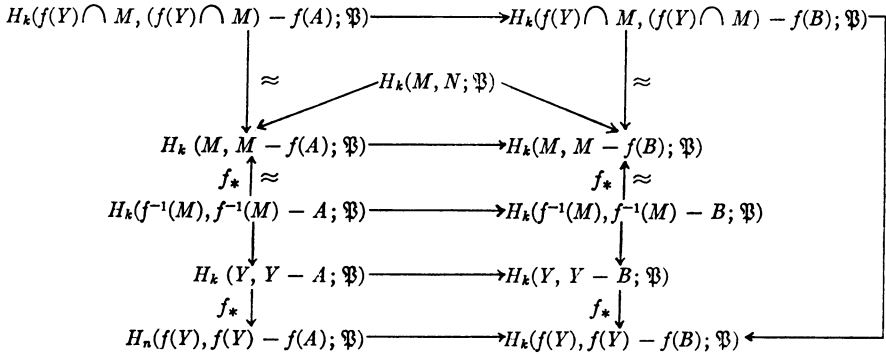
Let  $X^*$  be the orbit space of  $X$  and let  $f$  be the natural map of  $X$  onto  $X^*.$  Then all  $f(Z_{u,v})$  are open in  $X^*.$  Hence all  $f(Z_{u,v})$  and consequently all  $f(X_{u,v})$  are locally compact Hausdorff. Let

$$\dim f(X_{u,v}) = k_{u,v}, \quad 0 \leq u < r.$$

Let  $p, Y, Q$  be as in the preceding paragraph. Then  $f$  maps  $X_{u,v} \cap Y$  onto a neighborhood of  $f(p)$  in  $f(X_{u,v}).$  Therefore  $X_{u,v} \cap Y$  is of dimension  $\leq k_{u,v}.$  Hence  $Q(X_{u,v} \cap Y)$  is a neighborhood of  $p$  in  $X_{u,v}$  whose dimension is  $\leq k_{u,v} + u.$  Since  $p$  is an arbitrary point of  $X_{u,v},$  it follows that the dimension of  $X_{u,v}$  is  $\leq k_{u,v} + u.$  Hence our conclusion that  $X_0 \cup \dots \cup X_k, 0 \leq k < r,$  is a closed set of dimension  $\leq n - r + k - 1$  follows from Lemma 7 if we can show that

$$k_{u,v} \leq n - r - 1, \quad 0 \leq u < r.$$

Suppose that our assertion is false. Then there is some  $X_{\alpha,\beta}$  such that (i)  $0 \leq \alpha < r$ , (ii)  $k = k_{\alpha,\beta} > n - r - 1$  and (iii) if  $u > \alpha$  or if  $u = \alpha$  and  $v < \beta$ , then  $k_{u,v} \leq n - r - 1$ . By Remark 3, there exist compact subsets  $M, N$  of  $X_{\alpha,\beta}$  such that  $M \supset N$  and for some nonzero element  $e$  of  $H_k(M, N; \mathfrak{P})$ ,  $e|_{(M, K \cup N)} \neq 0$  whenever  $K$  is a proper compact subset of  $(M - N)^-$ . Let  $p$  be a point of  $f^{-1}(M - N)$  and let  $Y$  be a connected slice at  $p$ . Clearly  $G_p$  acts as a transformation group on  $Y$  with  $f(Y)$  as the orbit space. Let  $A$  be a neighborhood of  $p$  in  $Y$  such that  $G_p(A) = A$  and  $f(A) \cap N = \emptyset$ . Clearly for any neighborhood  $B$  of  $p$  in  $Y$  contained in  $A$ ,  $e_B = e|_{(M, M - f(B))} \neq 0$ .



Let  $e'_B = f_*^{-1}(e_B)|_{(Y, Y - B)}$ ; it is clear that  $e'_B = e'_A|_{(Y, Y - B)}$ .  $k$  is  $\leq n - u - 2$ , as  $X_{u,v}$  is of dimension  $\leq n - 2$ . Since  $Y$  is a  $(\mathfrak{P}, n - u)$ -manifold,  $B$  can be chosen such that  $e'_B = 0$ . Moreover, we may assume that  $G_p(B) = B$ . Let  $e''_B$  be the element of  $H_k(f(Y) \cap M, (f(Y) \cap M) - f(B); \mathfrak{P})$  such that  $e''_B|_{(M, M - f(B))} = e_B$ . Then

$$e''_B \neq 0$$

and

$$e''_B|_{(f(Y), f(Y) - f(B))} = f_*(e'_B) = 0.$$

Since  $f(Y)$  is contained in  $f(Z_{\alpha,\beta})$  which is of dimension  $k$  by Lemma 7, it follows that the natural homomorphism of  $H_k(f(Y) \cap M, (f(Y) \cap M) - f(B); \mathfrak{P})$  into  $H_k(f(Y), f(Y) - f(B); \mathfrak{P})$  is one-one. Hence we have arrived at a contradiction. This completes the proof of Theorem 2.

10. Appendix.

DEFINITION. Let  $X$  be a locally compact Hausdorff space and let  $\mathfrak{C}$  be a compact abelian group.  $X$  is said to possess the property  $Q(\mathfrak{C})$  at  $x$  if for every neighborhood  $A$  of  $x$  there is a neighborhood  $B$  of  $x$  contained in  $A$  and with the property that whenever  $y$  is a point of  $B$  and  $C$  is a neighborhood of  $y$

contained in  $B$ , there is a neighborhood  $D$  of  $y$  contained in  $C$  and such that for every  $k$ ,

$$H_k(X - C, X - A; \mathbb{C}) \mid (X - D, X - B) = 0.$$

**PROPOSITION.** *Let  $X$  be a locally compact Hausdorff space and let  $x$  be a point of  $X$ . If  $X$  possesses the property  $P_n(\mathbb{C})$  at  $x$ , then  $X$  possesses the property  $Q(\mathbb{C})$  at  $x$ .*

**Proof.** By hypothesis, there is a fundamental  $(\mathbb{C}, n)$ -pair  $(U, S)$  with  $U$  containing  $x$ . Let  $A$  be a fixed neighborhood of  $x$ . Then there is a neighborhood  $B$  of  $x$  contained in  $A \cap U$  and such that

$$(1) \quad H_n(X, X - A; \mathbb{C}) \mid (X, X - B) \subset S \mid (X, X - B)$$

and for every  $k \neq n$ ,

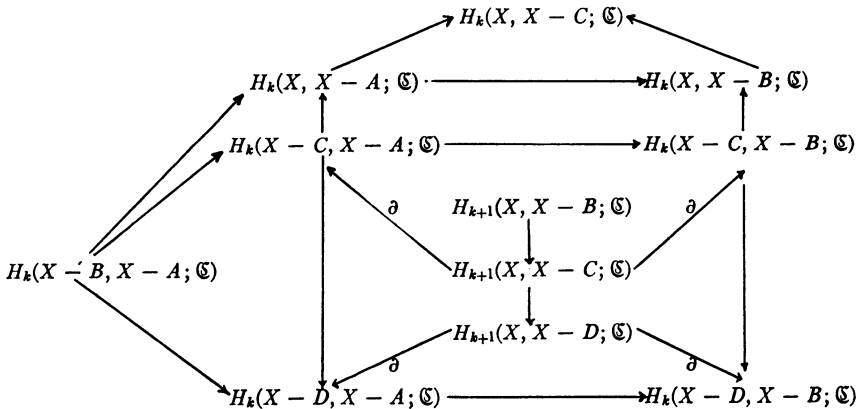
$$(2) \quad H_k(X, X - A; \mathbb{C}) \mid (X, X - B) = 0.$$

Let  $y$  be a point of  $B$  and let  $C$  be a neighborhood of  $y$  contained in  $B$ . Then there is a neighborhood  $D$  of  $y$  contained in  $C$  and such that

$$(3) \quad H_n(X, X - C; \mathbb{C}) \mid (X, X - D) \subset S \mid (X, X - D)$$

and for every  $k \neq n$ ,

$$(4) \quad H_k(X, X - C; \mathbb{C}) \mid (X, X - D) = 0.$$



Let  $e$  be an element of  $H_k(X - C, X - A; \mathbb{C})$ ; then

$$e \mid (X, X - B) = 0.$$

In fact, if  $k \neq n$ , our assertion follows from (2). If  $k = n$ , it follows from (1) that for some  $u \in S$ ,

$$e \mid (X, X - B) = u \mid (X, X - B).$$

Since  $e \mid (X, X - C) = 0$ ,  $u \mid (X, X - C) = 0$  and then  $u = 0$ . Hence  $e \mid (X, X - B) = 0$ .

From this result, it follows that for some  $e' \in H_k(X - B, X - A; \mathbb{C})$ ,

$$e \mid (X, X - A) = e' \mid (X, X - A).$$

Let

$$e'' = e - e' \mid (X - C, X - A).$$

Then

$$e'' \mid (X, X - A) = e \mid (X, X - A) - e' \mid (X, X - A) = 0.$$

Therefore for some  $e''' \in H_{k+1}(X, X - C; \mathbb{C})$ ,

$$\partial e''' = e''.$$

Now we claim that

$$\partial(e''' \mid (X, X - D)) = 0,$$

where  $\partial$  is the boundary operator of the homology sequence of  $(X, X - D, X - B)$ . If  $k \neq n - 1$ , then  $e''' \mid (X, X - D) = 0$  by (4) and therefore our assertion holds. If  $k = n - 1$ , then by (3) there is some  $u \in S$  such that

$$\begin{aligned} e''' \mid (X, X - D) &= u \mid (X, X - D) \\ &= (u \mid (X, X - B)) \mid (X, X - D). \end{aligned}$$

Hence our assertion again holds.

From this result, we have

$$\begin{aligned} 0 &= \partial(e''' \mid (X, X - D)) \\ &= e'' \mid (X - D, X - B) \\ &= e \mid (X - D, X - B). \end{aligned}$$

This proves our proposition.

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