REMARKS ON LATTICE ORDERED GROUPS AND VECTOR LATTICES(1)
I. CARATHÉODORY FUNCTIONS

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1. Although the literature on the representation of ordered groups, [1; 2; 3](2), and in particular of vector lattices, [4; 5; 6](3), as spaces of functions, or "functionoids," [7], is vast, the central fact that every archimedean lattice ordered group has such a representation does not seem to have been published. The first purpose here is to furnish such a representation as a space of Carathéodory functions [8; 9](4).

An important biproduct of this representation is the fact that every archimedean lattice ordered group is abelian, a fact which was first proved by Iwasawa [10] and then by Birkhoff [11] in different ways. The proof given here conforms more closely to the proof for the totally ordered case, as given by Baer [12] and Cartan [13] and, in general, seems to be appropriate.

Our main tool is a lattice of carriers (these were introduced by Jaffard [14] under the name of filets) associated with a given ordered group. Our preference for carriers to bands [15] is dictated by subsequent studies in the nonarchimedean case, bands being seemingly inadequate for our needs there, as well as being somewhat less suitable for our present purpose.

For a distributive lattice L with a minimal element, the lattice L^* of carriers is the image of L under a certain lattice homomorphism. This homomorphism is discussed in §2. §3 is concerned with special properties of L^* when L is the positive cone in a Banach lattice B. Under certain conditions on B, L^* is conditionally complete, σ complete, and relatively complemented.

The Carathéodory functions are defined in §4, and in §5 every archimedean lattice ordered group G is shown to be isomorphic to a subspace of the space of Carathéodory functions generated by its lattice of carriers.

Finally, §6 is devoted to the existence of compatible semi norms in an archimedean vector lattice. It is known that there are such vector lattices which cannot be normed. Here we seek general criteria for the existence and

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(2) Numbers in brackets refer to the bibliography at the end of the article.

(3) The references are by no means complete, but are intended as a representation of the literature which exists.

(4) Continuous functions, finite on residual sets, on disconnected locally compact spaces, for which the compact neighborhoods are Stone spaces, may perhaps also be used, but there seems to be little advantage in this.

nonexistence of compatible semi norms. A criterion is obtained for their existence in an archimedean vector lattice, and another for their nonexistence in the space of all Carathéodory functions based on a boolean algebra. These criteria are in terms of properties of the carriers.

2. This section deals with the lattice of carriers. For this, let L be a distributive lattice with a minimal element 0; i.e., $x \geqq 0$ for every $x \in L$. An example is the positive cone in a lattice ordered group. An ideal $J \subset L$ is a set such that $x, y \in J$ implies $x \cup y \in J$, and $x \in J$, $y \in L$ implies $x \cap y \in J$. Krishnan [16] has shown that, for every ideal J, the homomorphism α, for which $\alpha(x) = \alpha(y)$ if and only if there are $z, w \in J$ for which $x \cup z = y \cup w$, is the minimal homomorphism whose kernel is J. Pierce [17] has shown that the homomorphism β, for which $\beta(x) = \beta(y)$ if and only if $x \cap z \in J$ for exactly those $z \in L$ for which $y \cap z \in J$, is the maximal homomorphism whose kernel is J. It follows from the results of Krishnan and Pierce that, for a given L and $J \subset L$, the homomorphisms α and β are the same if and only if the image of L under α is disjunctive.

Our interest is in the case $J = [0]$. In this case, the homomorphism β has as its cosets the equivalence sets in L given by:

$$x \sim y \quad \text{if } z \cap x = 0 \quad \text{if and only if } z \cap y = 0.$$

Thus, if for every $x \in L$, we define $D(x)$ as the set of elements in L which are disjoint with x, then $x \sim y$ if and only if $D(x) = D(y)$. If L^* is the set of equivalence classes $x^*, y^*, \cdots$ in L then L^* is also a distributive lattice with minimal element $0^* = [0]$, since β is a lattice homomorphism. Moreover, L^* is disjunctive. We call the elements of L^* the *carriers* of the elements of L. Thus, if $\beta(x) = x^*$, then x^* is the carrier of x. Now, α and β are the same if and only if L is disjunctive. Also, β is an isomorphism if and only if L is disjunctive. On the other hand, while L^* is disjunctive it need not always be relatively complemented. For, if L is disjunctive without being relatively complemented then $L = L^*$. The lattice L whose elements are unions of finite numbers of closed intervals and of points is an example.

For a lattice ordered group G, we consider the lattice G^* of carriers of the positive cone in G, but refer to G^* as the *lattice of carriers of* G.

The carriers are like "sets" on which the members of a lattice of "functions" do not vanish. Indeed, if L is the set of all non-negative real functions on a set S, then L^* is the lattice of subsets of S. If the elements of L are equivalence classes of functions, those of L^* may, instead of being sets, be equivalence classes of sets. Moreover, we point out in the next section, that even for a lattice of real functions, the carriers may not be sets, but are equivalence classes of sets.

3. We wish to remark on the lattice of carriers for a Banach lattice. A vector lattice X is a vector space which is also a lattice such that $x > 0$, $y > 0$, $a > 0$ imply $ax > 0$, $x + y > 0$. Then, for every $x \in X$, we write $x^+ = x \cup 0$,

$x^- = x \cap 0$, and $|x| = x^+ - x^-$. Moreover, it is true that $x = x^+ + x^-$. A compatible norm for X is one such that $|x| \geqq |y|$ implies $\|x\| \geqq \|y\|$. A vector lattice with a compatible norm is called a normed vector lattice, and if it is a Banach space, it is called a Banach lattice. Some authors also assume X is a conditionally σ complete lattice, but we do not make this assumption since it excludes some interesting cases.

Let X be a Banach lattice and $\{x_n^*\}$ a sequence of carriers in X. Then $\{x_n^*\}$ has a least upper bound in X^*. Indeed, for every n, there is an $x_n \in x_n^*$, $x_n > 0$, such that $\|x_n\| \leqq 2^{-n}$. (We have used the fact that $(ax)^* = x^*$ for every $a > 0$). Now, for every n, let

$$y_n = \sum_{m=1}^{n} x_m.$$

Then $\bigcup_{m=1}^n x_m^*$ is the carrier of y_n. Moreover, $\{y_n\}$ is a Cauchy sequence so that it has a limit x with carrier x^*. Evidently, $x^* \geqq x_n^*$, for every n, so that x^* is an upper bound for $\{x_n^*\}$. Suppose $y^* < x^*$ is also an upper bound. Then, since X^* is disjunctive, there is a $z^* > 0^*$, $y^* \cap z^* = 0^*$, $z^* < x^*$. There is a $z \in z^*$ such that $0 < z < x - x_n$ for every n. Then $\|x - x_n\| \geqq \|z\|$, for every n. Since $z > 0$, this contradicts the fact that $\lim_{n \to \infty} \|x - x_n\| = 0$. We call a lattice *upper σ complete* if every countable set has a least upper bound, and have

PROPOSITION 1. *If X is a Banach lattice, its lattice X^* of carriers is upper σ complete.*

We do not know whether, for every Banach lattice X, the associated lattice X^* of carriers is relatively complemented. However, this fact is easily established in a special case.

We suppose X satisfies:

(a) The norm is strictly increasing; i.e.,

$$|x| > |y| \quad \text{implies} \quad \|x\| > \|y\|.$$

(b) Every bounded monotone sequence in X is a Cauchy sequence.

Let x^* and y^* be carriers, $y^* < x^*$, and let $x > 0$, $y > 0$ be such that their carriers are x^* and y^*, respectively. Let Z be the set of all $z > 0$, $z < x$, for which $z^* \cap y^* = 0^*$. That Z is nonempty follows since X^* is disjunctive. For every $z \in Z$, $\|z\| \leqq \|x\|$, so that the norm is bounded on Z. Let

$$m = \sup [\|z\| \mid z \in Z],$$

and let $\{z_n\}$ be an increasing sequence in Z such that $\lim_{n \to \infty} \|z_n\| = m$. By Condition (b), $\{z_n\}$ is a Cauchy sequence. We let $\zeta = \lim_{n \to \infty} z_n$. Then $\|\zeta\| = m$. Suppose $\zeta^* \cup y^* < x^*$. Then, there is a $w^* < x^*$ such that $w^* \cap (\zeta^* \cup y^*) = 0^*$, and there is a $w < x$ whose carrier is w^*. Now, $\zeta + w \in Z$, and by condition (a),

$\|\zeta+w\|>m$, which is impossible. Hence, $\zeta^*\cup y^*=x^*$, so that X^* is relatively complemented.

More is true; X^* is conditionally complete. Let S be any set in X^*, let $x^*\in S$, and let $x>0$ have x^* as carrier. We consider the set Z of all $z<x$, $z>0$, whose carriers are lower bounds of S and then proceed as before to find a greatest lower bound. That every bounded set in X^* has a least upper bound follows by complementation. We thus have:

PROPOSITION 2. *If X is a Banach lattice satisfying conditions* (a) *and* (b), *then its lattice X^* of carriers is a conditionally complete and σ complete relatively complemented lattice.*

We consider now the Banach lattice C of continuous functions $x=x(t)$ on the closed interval [0, 1] with

$$\|x\| = \max |x(t)| .$$

In the first place, this is a vector lattice of real functions. Secondly, it satisfies neither conditions (a) nor (b). We notice, however, that its lattice C^* of carriers is a complete boolean algebra whose elements are not subsets of [0, 1]. Indeed, x and y are equivalent if and only if the sets on which they do not vanish differ by nowhere dense sets. The carriers are accordingly equivalence classes of open sets modulo nowhere dense sets. The complement of a carrier x^* is the equivalence class to which the interior of the complement of any set belonging to x^* belongs. Moreover, the supremum of a set S of carriers is the carrier to which the open set belongs which is the union of open sets, one from each carrier in S. We thus have:

PROPOSITION 3. *The lattice C^* of carriers of the Banach lattice C of continuous functions on* [0, 1] *is the complete boolean algebra of open sets modulo nowhere dense sets in the interval* [0, 1].

On the other hand, we know that there are archimedean vector lattices for which the lattice of carriers is not relatively complemented. For example, we consider the smallest vector lattice of real functions on [0, 1] which contains the function x^2 and the characteristic functions of the intervals $[0, a]$ for all $0<a<1$. The complement of $[0, a]$ with respect to [0, 1] is not a carrier.

Finally, if the lattice L (positive cone of a lattice ordered group G) is conditionally complete, we shall show in §5 that its lattice L^* of carriers is also conditionally complete. Moreover, if Λ is the conditional completion of L then Λ^* is the conditional completion of L^*.

4. Our version of the Carathéodory functions is a cross between the original definition of Carathéodory and the one given by Kappos [18] and is based on a slightly more general boolean system.

We start with a relatively complemented distributive lattice L. Then, for every $\alpha \in L$, the set

$$L_\alpha = [\beta \leqq \alpha \mid \beta \in L]$$

is a boolean algebra, so that L may be called a local boolean algebra.

For every finite set (the empty set is allowed) of pairwise disjoint elements $\alpha_1, \alpha_2, \cdots, \alpha_n$ in L and real $a_1, a_2, \cdots, a_n$, all different from zero, we consider the form

$$f = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n.$$

Two forms $f = \sum_{k=1}^n a_k\alpha_k$ and $g = \sum_{k=1}^m b_k\beta_k$ are considered equal if $\bigcup_{k=1}^n \alpha_k = \bigcup_{k=1}^m \beta_k$ and if $\alpha_i \cap \beta_j \neq 0$ implies $a_i = b_j$. We add forms according to the rule

$$f + g = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j)(\alpha_i \cap \beta_j) + \sum_{i=1}^n a_i\left(\alpha_i - \bigcup_{j=1}^m \beta_j\right) + \sum_{j=1}^m b_j\left(\beta_j - \bigcup_{i=1}^n \alpha_i\right)$$

for $f = \sum_{i=1}^n a_i\alpha_i$a nd $g = \sum_{j=1}^m b_j\beta_j$, where the sum

$$\sum_{i=1}^n \sum_{j=1}^m (a_i + b_j)(\alpha_i \cap \beta_j)$$

is restricted to those terms for which $a_i + b_j \neq 0$, and we multiply by real scalars according to the rule $af = \sum_{i=n}^n aa_i\alpha_i$, where $f = \sum_{i=1}^n a_i\alpha_i$ and a is real. If the set $\alpha_1, \alpha_2, \cdots, \alpha_n$ is empty; i.e., $n=0$, we have the zero form. A form $f = \sum_{i=1}^n a_i\alpha_i$ is positive if $a_i > 0$, $i = 1, \cdots, n$. The set E of all these forms is then an archimedean vector lattice and we call its members *elementary Carthéodory functions generated by L*. We note that $E^* = L$.

We define the vector lattice B of *bounded Carathéodory functions generated by L* as the conditional completion of E. Then B^* is the conditional completion of L, so that B^* is a conditionally complete local boolean algebra. For $f = \sum_{i=1}^n a_i\alpha_i \in E$ and $\alpha \in B^*$, we define the restriction of f to α as $f^\alpha = \sum_{i=1}^n a_i(\alpha_i \cap \alpha)$. Now, let $f \in B$ and $\alpha \in B^*$. Let $A \subset E$ be the lower segment of the Dedekind cut defining f, and let A^α be the set of restrictions of elements of A to α. We define the restriction f^α of f to α as the least upper bound of A^α in B.

We now define the *vector lattice C of all Carathéodory functions generated by L*. For this, let $\alpha \in B^*$ and let $\{\alpha_n\}$ be a countable decomposition of α such that $\alpha = \bigcup_{n=1}^\infty \alpha_n$ and $\alpha_n \cap \alpha_m = 0$ for $n \neq m$. Then, for every n, let $f_n \in B$ have carrier α_n. We let f be the formal sum

$$f = \sum_{n=1}^\infty f_n.$$

C consists of all these forms f. Two forms $f = \sum_{n=1}^\infty f_n$, with carriers $\alpha = \bigcup_{n=1}^\infty \alpha_n$, and $g = \sum_{m=1}^\infty g_m$, with carriers $\beta = \bigcup_{m=1}^\infty \beta_m$, are considered equal if $\alpha = \beta$ and if $\alpha_i \cap \beta_j \neq 0$ implies $f_i^{\alpha_i \cap \beta_j} = g_j^{\alpha_i \cap \beta_j}$. If $f = \sum_{n=1}^{\infty} f_n$, with carriers $\alpha = \bigcup_{n=1}^{\infty} \alpha_n$, and $g = \sum_{m=1}^{\infty} g_m$, with carriers $\beta = \bigcup_{m=1}^{\infty} \beta_m$, then $f+g$ is defined as

$$f + g = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f_i^{\alpha_i \cap \beta_j} + g_j^{\alpha_i \cap \beta_j}) + \sum_{i=1}^{\infty} f_i^{\alpha_i - \bigcup_{j=1}^{\infty} \beta_j} + \sum_{j=1}^{\infty} g_j^{\beta_j - \bigcup_{i=1}^{\infty} \alpha_i},$$

and $f \geqq g$ if $f_i^{\alpha_i \cap \beta_j} \geqq g_j^{\alpha_i \cap \beta_j}$ for every i, j. If $f = \sum_{n=1}^{\infty} f_n$ and a is real, then $af = \sum_{n=1}^{\infty} af_n$. It is then an easy matter to show that C is a conditionally complete vector lattice, and that $C^* = B^*$.

A related vector lattice associated with L was used by Carathéodory and will also be needed in this paper. For this purpose, we suppose L is a relatively complemented distributive lattice endowed with a completely additive, non-negative, possibly infinite valued, real function μ. We now consider the elementary functions $f = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ for which $\mu(\alpha_i) < \infty$, $i = 1, \cdots, n$. These form a vector lattice I which is a subspace of E. Now, the norm

$$\|f\| = \sum_{i=1}^{n} |a_i| \cdot \mu(\alpha_i)$$

is compatible with the order relation in I. The *Carathéodory space of summable functions on* L is the Banach space completion of I.

5. We now proceed to the proof that every archimedean lattice ordered group G is isomorphic to a subgroup of a vector space C of Carathéodory functions. It is clear that we may suppose G to be conditionally complete. We consider a generalized weak unit u associated with the group G whose existence is assured by:

PROPOSITION 4. *There is a set* e_α^*, $\alpha \in \mathfrak{A}$, *contained in* G^* *such that*
(a) *if* $\alpha \neq \beta$, *then* $e_\alpha^* \cap e_\beta^* = 0^*$,
(b) *for every* $x^* \in G^*$,

$$\bigcup_{\alpha \in \mathfrak{A}} x^* \cap e_\alpha^* = x^*.$$

The existence of this set follows by a standard application of Zorn's lemma or of transfinite induction.

For every $\alpha \in \mathfrak{A}$, we consider an $e_\alpha \in G$, $e_\alpha > 0$, whose carrier is e_α^*. The collection e_α, $\alpha \in \mathfrak{A}$, we designate by u, and call a *generalized weak unit* for G.

We now give several statements which we shall need.

PROPOSITION 5. *If* $x \cap y = 0$, *then* $x + y = x \cup y$ *so that, since* $x \cup y = y \cup x$, *we have* $x + y = y + x$.

For the proof, see [11, p. 220]. (It is convenient to use the additive notation even though G is not assumed to be abelian.)

PROPOSITION 6. *If* $x>0$ *and* $y^*\in G^*$ *then* $x=w+v=v+w$, *where* $w^*\leqq y^*$ *and* $v^*\cap y^*=0^*$. *In particular,* G^* *is relatively complemented.*

PROOF. We let

$$w = \sup [z \mid z \geqq 0, z \leqq x, z^* \leqq y^*]$$

and

$$v = \sup [z \mid z \geqq 0, z \leqq x, z^* \cap y^* = 0^*].$$

Then $w\leqq x$, $v\leqq x$, and $w\cap v=0$, so that $x\geqq w\cup v=w+v=v+w$. Suppose $x>v+w$. Then $x-(w+v)>0$. There is then a $t<x-(w+v)$, $t>0$, so that either $t^*<y^*$ or $t^*\cap y^*=0^*$. Now $t+w\leqq x$ and $t+v\leqq x$ and either $(t+w)^*\leqq y^*$ or $(t+v)^*\cap y^*=0^*$ so that one of w and v is not the specified supremum.

The decomposition of Proposition 6 is unique, and w is called the *projection of* y^* on x. This will be designated by

$$w = (x, y^*).$$

COROLLARY 1. *If* G *is conditionally complete then so is* G^*.

PROOF. Let x_α^*, $\alpha\in\mathcal{J}$, have an upper bound y^* in G^*. Let y have carrier y^* and consider the set of all (y, x_α^*), $\alpha\in\mathcal{J}$. This set has y as an upper bound. Let $x=\sup (y, x_\alpha^*)$. Then $x\leqq y$. We show that $x_\alpha=\sup x_\alpha^*$. In the first place, $x^*\geqq x_\alpha^*$, for all $\alpha\in\mathcal{J}$. Suppose $z^*<x^*$ and $z^*\geqq x_\alpha^*$ for all $\alpha\in\mathcal{J}$. Since G^* is disjunctive there is $w^*\leqq x^*$, $w^*\cap z^*=0^*$ and there is $w<x$ whose carrier is w^*. It follows that $x-w\geqq (y, x_\alpha^*)$, for all $\alpha\in\mathcal{J}$, so that $x\neq\sup (y, x_\alpha^*)$. But this contradicts our assumption regarding x. That G^* is relatively complemented follows from Proposition 6 with $y^*\leqq x^*$.

PROPOSITION 7. *If* $x>0$, $y>0$, *and* $x^*=y^*$, *then*

$$y = \sup_n (nx \cap y).$$

PROOF. Suppose $y-\sup_n (nx\cap y)=z>0$. Then $z^*>0^*$. But this implies that $n(x, z^*)<y$, for all n, which contradicts the fact that G is archimedean.

PROPOSITION 8. *If* $0<w<v$, *then for every* $z>0$ *there is* k *such that* $kw+z \not> kv$.

The proof, which uses the archimedean character of G, is left to the reader.

Let C be the space of Carathéodory functions generated by G^*. We define a mapping ϕ of the positive elements of G into C which is one-one, order preserving, and operation preserving. Then ϕ may be extended to all of G by letting $\phi(x-y)=\phi(x)-\phi(y)$.

For every pair m, n of positive integers and every α let

$$\overset{*}{y}_{mn,\alpha} = \left[(me_\alpha - 2^n x)^+\right]^*$$

and let

$$\overset{*}{y}_{mn} = \sup\left[\overset{*}{y}_{mn,\alpha} \mid \alpha \in \mathfrak{A}\right].$$

Then

$$\overset{*}{y}_{m+1,n} \geqq \overset{*}{y}_{mn}, \qquad \text{for every } m,$$

and, for every n,

$$\overset{*}{x} = \sup_m \overset{*}{y}_{mn},$$

for, evidently, $x^* \geqq \sup_m y^*_{mn}$, and if $x^* > \sup_m y^*_{mn}$, and we write

$$\overset{*}{z} = \overset{*}{x} - \sup_m \overset{*}{y}_{mn},$$

there is an $\alpha \in \mathfrak{A}$ such that

$$\overset{*}{z}_\alpha = \overset{*}{z} \cap \overset{*}{e}_\alpha > \overset{*}{0}.$$

This implies that

$$\sup_m (me_\alpha \cap 2^n x) < (2^n x, \overset{*}{e}_\alpha),$$

which contradicts Proposition 7.

If we let

$$\overset{*}{x}_{mn} = \overset{*}{y}_{m+1,n} - \overset{*}{y}_{mn}, \qquad m = 1, 2, \cdots$$

then

$$\overset{*}{x} = \bigcup_{m=1}^{\infty} \overset{*}{x}_{mn}$$

where the x^*_{mn}, $m=1, 2, \cdots$, are pairwise disjoint. Moreover, for every n, $x^*_{m,n+1}$, $m=1, 2, \cdots$, is a refinement of x^*_{mn}, $m=1, 2, \cdots$. The forms

$$f_n = \sum_{m=1}^{\infty} m 2^{-n} \overset{*}{x}_{mn}, \qquad n = 1, 2, \cdots,$$

belong to the vector lattice of all Carathéodory functions generated by G^* and

$$f_n \geqq f_{n+1}, \qquad n = 1, 2, \cdots.$$

We let

$$f = \inf_n f_n$$

and define the mapping

$$\phi: x \to f.$$

In order to show that this mapping is one-one, we suppose $x>0$, $y>0$, $x\neq y$. Then there is a z^* such that $(x, z^*)>(y, z^*)$ or $(y, z^*)>(x, z^*)$. Thus we need only show that $x\cap y=0$ implies $\phi(x\cup y)=\phi(x)+\phi(y)$ and that $x<y$ implies $\phi(x)\neq\phi(y)$. We omit the easy proof of the first statement. For the second statement, there is an $\alpha\in\mathfrak{A}$ such that $(x, e_\alpha^*)<(y, e_\alpha^*)$, so that there are m, n such that

$$\overset{*}{z}_{mn,\alpha} = \overset{*}{x}_{mn,\alpha} - \overset{*}{y}_{mn,\alpha} > \overset{*}{0},$$

since, by Proposition 8, $w<v$ implies, for every z, the existence of a k such that $kw+z \not> kv$. It follows that

$$\phi(y) - \phi(x) \geqq (\phi(y), \overset{*}{z}_{mn,\alpha}) - (\phi(x), \overset{*}{z}_{mn,\alpha}) \geqq 2^{-n}\overset{*}{z}_{mn,\alpha},$$

thus completing the proof.

In similar fashion, we may show that if $0<x<y$ then $\phi(x)<\phi(y)$ and that if x, $y>0$, then

$$\phi(x \cup y) = \phi(x) \cup \phi(y)$$

where the last operation is in all of C.

It remains only to show that

$$\phi(x + y) = \phi(x) + \phi(y).$$

We may restrict our attention to the case $x^*=y^*$. We write $h=\phi(x+y)$, $f=\phi(x)$ and $g=\phi(y)$, and consider the x_{mn}^* and y_{mn}^* which correspond to x and y, respectively, according to the above description. For every m_1, m_2, and n, we let

$$\overset{*}{z}_{m_1m_2n} = \overset{*}{x}_{m_1n} \cap \overset{*}{y}_{m_2n}.$$

Then $x^*=y^*=\bigcup_{m_1,m_2=1}^\infty z_{m_1m_2n}^*$, for every n. It follows that for all m_1, m_2, n

$$2^{-n}(m_1 + m_2 - 2)\overset{*}{z}_{m_1m_2n} \leqq (h, \overset{*}{z}_{m_1m_2n}) \leqq 2^{-n}(m_1 + m_2)\overset{*}{z}_{m_1m_2n}$$

and

$$2^{-n}(m_1 - 1)\overset{*}{z}_{m_1m_2n} \leqq (f, \overset{*}{z}_{m_1m_2n}) \leqq 2^{-n}m_1\overset{*}{z}_{m_1m_2n},$$
$$2^{-n}(m_2 - 1)\overset{*}{z}_{m_1m_2n} \leqq (g, \overset{*}{z}_{m_1m_2n}) \leqq 2^{-n}m_2\overset{*}{z}_{m_1m_2n}$$

so that, since

$$\overset{*}{x} = \overset{*}{y} = \overset{*}{(x+y)} = \bigcup_{m_1,m_2=1}^{\infty} \overset{*}{z}_{m_1m_2n}, \quad \text{for every } n,$$

we have

$$|(f+g) - h| \leqq 2^{-n}x^*.$$

Hence $f+g=h$. This proves

THEOREM 1. *Every conditionally complete lattice ordered group G is isomorphic, as a lattice ordered group, to a subgroup of the vector lattice of Carathéodory functions generated by its lattice G^* of carriers.*

COROLLARY 1. *Every archimedean lattice ordered group has a representation such as the one described in the theorem.*

COROLLARY 2. *Every archimedean lattice ordered group is abelian.*

As an example, let X be an abstract L space. By this, we mean a Banach lattice for which $x>0$, $y>0$, implies

$$\|x+y\| = \|x\| + \|y\|.$$

We notice that X satisfies conditions (a) and (b) of Proposition 2, so that X^* is conditionally complete and σ complete.

We fix a generalized weak unit $u=[e_\alpha, \alpha\in\mathfrak{A}]$ in X. For every α, we let

$$m(\overset{*}{e_\alpha}) = \|e_\alpha\|,$$

and for every $x^*\in X^*$, we let

$$m(x^*) = \sum_{\alpha\in\mathfrak{A}} \|(e_\alpha, x^*)\|$$

where the sum is infinite if an uncountable number of summands differ from zero.

We thus have a completely additive measure on X^* which is not necessarily σ finite. Thus, X is isomorphic to a subspace of the space C of Carathéodory functions generated by the measure algebra X^*. In order to determine precisely which functions in C are in the image space, we observe first that the elements

$$f = a_1\overset{*}{x}_1 + a_2\overset{*}{x}_2 + \cdots + a_n\overset{*}{x}_n$$

of C for which $m(x_i^*) < \infty$, $i=1, 2, \cdots, n$, are surely there. This is the subspace I of C. It is not difficult to see that the image of X in C is the Banach space completion S of I with the norm

$$\|f\| = \sum_{i=1}^{n} |a_i| \cdot m(x_i^*).$$

In order words, X is isomorphic with the Carathéodory space S of summable functions generated by the measure algebra X^*. This is Kakutani's theorem, [19], for the general case; i.e., the nonseparable case.

We remark that the measure algebra X^* is σ finite if and only if X has a weak unit or, equivalently, if X^* has a maximal element so that it is a boolean algebra. Whether or not the measure for X^* is finite depends upon the choice of the weak unit.

6. We turn now to a slightly different, but related, topic in which the Carathéodory functions are involved. A difficult problem in the theory of vector lattices is that of determining when compatible norms exist. Not so well known, but perhaps equally difficult, is the existence of a set of compatible semi norms p_α, $\alpha \in \mathfrak{A}$, for a vector lattice X such that for every $x \neq 0$ there is an $\alpha \in \mathfrak{A}$ for which $p_\alpha(x) \neq 0$. By a compatible semi norm we understand a semi norm p such that $|x| \geqq |y|$ implies $p(x) \geqq p(y)$.

Our purpose now is to discuss the existence of compatible semi norms for the space of Carathéodory functions generated by a local boolean algebra. Our results are in the form of two criteria, the first for the existence of compatible semi norms and the second for their nonexistence.

Criterion 1. Let X be an archimedean vector lattice, $x \in X$, $x > 0$, and x^*, the carrier of x, such that if $\mathfrak{D} = [D]$ is the set of all countable subdivisions of x^* then, for every $D \in \mathfrak{D}$, there is an $x_D^* \in D$ such that for every D, $D' \in \mathfrak{D}$ there is a $D'' \in \mathfrak{D}$ for which $x_D^* \cap x_{D'}^* \supset x_{D''}^*$. There is then a compatible semi norm p on X such that $p(x) \neq 0$.

Remark 1. It is the second criterion, not this one, in which the Carathéodory functions are involved.

Remark 2. By a countable subdivision of x^* we mean a countable set $\{x_n^*\}$ of pairwise disjoint carriers whose union is x^*. Then x_D^* is a member of the countable subdivision $D = \{x_n^*\}$.

Proof. Let $x > 0$ have carrier x^*. We show there is a compatible semi norm p on X such that $p(x) = 1$. It is necessary only to define p for all $y > 0$. For every $D \in \mathfrak{D}$, let

$$c(y, D) = \inf \left[c \mid c \cdot (x, x_D^*) > (y, x_D^*) \right]$$

and let

$$p(y) = \inf \left[c(y, D) \mid D \in \mathfrak{D} \right].$$

It is evident that $p(x) = 1$.

Moreover, for every $y > 0$, $p(y) < \infty$. For, if we let

$$y_n = (nx - y)^+, \qquad n = 1, 2, \cdots,$$

then $\{y_n^*\}$ is an increasing sequence of carriers whose union is x^*. We consider the subdivision D whose members are

$$\overset{*}{x}_n = \overset{*}{y}_{n+1} - \overset{*}{y}_n, \qquad n = 1, 2, \cdots.$$

Then $x_D^* = x_n^*$, for some n, so that

$$p(y) \leqq c(y, D) \leqq n.$$

That $p(ky) = kp(y)$, for every $k > 0$, is obvious.

We finally show that $p(y) + p(z) \geqq p(y+z)$. For, if $\epsilon > 0$, we may choose D and D' so that

$$c(y, D) < p(y) + \epsilon/2 \quad \text{and} \quad c(z, D') < p(z) + \epsilon/2.$$

Now, let D'' be such that $x_{D''}^* \subset x_D^* \cap x_{D'}^*$. Then $c(y, D'') < p(y) + \epsilon/2$ and $c(z, D'') < p(z) + \epsilon/2$. We thus have

$$\begin{aligned} p(y + z) \leqq c(y + z, D'') &\leqq c(y, D'') + c(z, D'') \\ &< p(y) + p(z) + \epsilon. \end{aligned}$$

We give two examples in which this criterion holds.

(a) Suppose x^* contains an atom $e^* \leqq x^*$. Then every subdivision D of x^* has a member which contains e^*. We let x_D^* be this element. It is clear that this selection satisfies Criterion 1.

(b) Let H be a totally ordered abelian group for which the interval topology [20] does not have a countable base at the identity 0. We consider a fixed interval I and consider finite unions of subintervals of I, modulo finite sets, as the elements of a boolean algebra A. Now, for every subdivision D of I one of the members of the subdivision must contain an interval with 0 as left end point. For, otherwise, the left end points of all the intervals of the members of the subdivision would form a countable set with 0 as limit point. We let this member of the subdivision be I_D. It is obvious that this selection satisfies Criterion 1. Hence, for every Carathéodory function f, with carrier I, there is a semi norm p on the space $C(A)$ of Carathéodory functions generated by A, such that $p(f) \neq 0$. It is evident that I may be replaced here by any element in A.

Proposition 8. *The vector lattice of Carathéodory functions generated by an algebra every member of which contains an atom admits compatible locally convex Hausdorff topologies. The vector lattice of Carathéodory functions generated by the algebra of finite sets of non degenerate intervals, modulo finite sets, in a totally ordered abelian group for which no countable sequence converges to* 0, *admits compatible locally convex Hausdorff topologies.*

Our second criterion assures the nonexistence of compatible semi norms in certain Carathéodory spaces.

CRITERION 2. Let X be a space of all Carathéodory functions generated by a local boolean algebra, let $x \in X$ and let x^*, the carrier of x, be such that it may be split as follows:

$$\overset{*}{x} = \overset{*}{x}_1 \cup \overset{*}{x}_2, \quad \text{where} \quad \overset{*}{x}_1 \cap \overset{*}{x}_2 = \overset{*}{0},$$

$$\overset{*}{x}_i = \overset{*}{x}_{i1} \cup \overset{*}{x}_{i2}, \quad \text{where} \quad \overset{*}{x}_{i1} \cap \overset{*}{x}_{i2} = \overset{*}{0}, \qquad i = 1, 2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overset{*}{x}_{i_1 \cdots i_{n-1}} = \overset{*}{x}_{i_1 \cdots i_{n-1}1} \cup \overset{*}{x}_{i_1 \cdots i_{n-1}2},$$

where

$$\overset{*}{x}_{i_1 \cdots i_{n-1}1} \cap \overset{*}{x}_{i_1 \cdots i_{n-1}2} = \overset{*}{0}, \qquad i_1, \cdots, i_{n-1} = 1, 2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

so that, for every sequence $i_1, i_2, \cdots, i_n, \cdots, i_n = 1, 2$, we have

$$\overset{*}{x}_{i_1} \cap \overset{*}{x}_{i_1 i_2} \cap \cdots \cap \overset{*}{x}_{i_1 i_2 \cdots i_n} \cap \cdots = \overset{*}{0}.$$

There is then no compatible semi norm on X with $p(x) \neq 0$.

PROOF. Suppose there is a semi norm p such that $p(x) > 0$. Then there is a chain

$$x_{i_1}, x_{i_1 i_2}, \cdots, x_{i_1 \cdots i_n}, \cdots$$

such that, for every n, the carrier of $x_{i_1 \cdots i_n}$ is $\overset{*}{x}_{i_1 \cdots i_n}$, and such that

$$p(x_{i_1 \cdots i_n}) > 0.$$

For, otherwise, there would be x, y with $p(x) = p(y) = 0$ and $p(x+y) > 0$. We consider the function

$$f = \sum_{n=1}^{\infty} \frac{n}{p(x_{i_1 \cdots i_n})} (\overset{*}{x}_{i_1 \cdots i_{n-1}} - \overset{*}{x}_{i_1 \cdots i_n}).$$

It then follows that $p(f) > n$, for every n, which is impossible.

As an example, we consider the boolean algebra of measurable sets, modulo sets of measure 0, on the interval $[0, 1]$. The space of Carathéodory functions is then the set of all equivalence classes of measurable functions. Clearly, every carrier satisfies Criterion 2, so that we have:

COROLLARY 1. *The vector lattice of equivalence classes of measurable functions on* $[0, 1]$ *admits no nontrivial compatible semi norms.*

It would be interesting to know whether there are carriers which satisfy neither criterion, but we have not been able to settle this matter.

Added in proof (April 11, 1958): The question posed after Proposition 1 has been answered by M. Henriksen. If $C(S)$ is the Banach lattice of continuous functions on the ordered set $S=\omega+1+\Omega^*$ with the interval topology, the lattice of carriers of $C(S)$ is not relatively complemented.

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