

UNIQUENESS AND THE FORCE FORMULAS FOR PLANE SUBSONIC FLOWS

BY

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Introduction. The first proof of uniqueness of a plane subsonic flow of a compressible fluid past an obstacle was given by Bers [1]. This proof utilizes an elaborate mathematical apparatus encompassing some of the most advanced tools of modern function theory. A conceptually simpler proof under somewhat weaker hypotheses and a proof of the Joukowski force formula, due to the authors [3], make essential use of a general existence theorem and hence cannot properly be called elementary. In this note we show that the uniqueness of a compressible flow and the Joukowski force formula can be obtained directly from a simple geometrical property of the velocity field defined by the flow. Specifically, we base our proof on the facts that the velocity components u, v of the flow define a quasi-conformal⁽¹⁾ mapping of the (x, y) plane, and that a quasi-conformal mapping with dilatation ratio not exceeding $\kappa = 1/\mu$ satisfies at each point a Hölder condition with exponent μ . Until recently we would have considered this proof more difficult than our original one. We are, however, now able to refer to the preceding paper [5], in which a simple demonstration is given for the needed lemma. We note, further, that a simpler form of the arguments in [5] suffices for the present paper.

The problem of proving uniqueness of a flow past an obstacle P can be divided into two parts; to prove the uniqueness if the velocity at infinity and circulation are both prescribed, and to prove that if P has a corner or cusp T the circulation is uniquely determined by the condition that the speed is finite at T . We shall settle both points here, but for the second we need the additional assumption, not used in [1] or [3], that the velocity components have bounded derivatives up to P . (This assumption can be avoided by a discussion analogous to that in [3, §6.2].) By the same method we shall also obtain a new proof, simpler than that of [3], for the force formula of gas dynamics.

1. Uniqueness with prescribed circulation. As in [3] we define the *velocity potential* of a flow past P to be a solution $\phi(x, y)$ of

$$(1) \quad (\rho\phi_x)_x + (\rho\phi_y)_y = 0$$

where $\rho = f(\phi_x^2 + \phi_y^2) = f(q^2)$, and we call the flow *subsonic* if $\rho + 2f'q^2 > 0$. We assume this condition satisfied for all q such that $0 \leq q < q_{cr} \leq \infty$. The functions

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⁽¹⁾ For the definition of this and other terms appearing here we refer the reader to [3] and [5].

$u = \phi_x$ and $v = \phi_y$ are called the *velocity components* of the flow. We assume the flow to be directed tangentially ($\partial\phi/\partial n = 0$) on P .

Equation (1) is an integrability condition for a *stream function* $\psi(x, y)$ such that

$$(2) \quad \psi_y = \rho\phi_x, \quad -\psi_x = \rho\phi_y.$$

For a flow past P , $\psi = \text{const.}$ on P .

Suppose we are given two subsonic flows (ϕ, ψ) , $(\bar{\phi}, \bar{\psi})$ past P for which the velocities (u, v) and (\bar{u}, \bar{v}) tend to a common subsonic limit (u_0, v_0) at infinity and for which the *circulations* are the same,

$$(3) \quad \Gamma = \oint d\phi = \oint d\bar{\phi}$$

for any closed curve surrounding P . For a region \mathcal{E}_r bounded by P and by a circumference C_r of radius r enclosing P , we consider the identity

$$(4) \quad \int_{\mathcal{E}_r} \begin{pmatrix} \Phi & \Psi \\ x & y \end{pmatrix} dx dy = \int_{C_r} \Phi d\Psi$$

where $\Phi = \phi - \bar{\phi}$, $\Psi = \psi - \bar{\psi}$, and the integrand on the left is the Jacobian of the mapping (Φ, Ψ) . (Note that by (3) this mapping is 1-valued.)

The Jacobian has constant sign in \mathcal{E}_r and vanishes only if $\Phi_x^2 + \Phi_y^2 = 0$. For set $\rho\phi_x = \mathcal{A}(u, v)$, $\rho\phi_y = \mathcal{B}(u, v)$ and consider the function

$$F(t) = \Phi_x \{ \mathcal{A}[\bar{u} + t(u - \bar{u}), \bar{v} + t(v - \bar{v})] - \mathcal{A}[\bar{u}, \bar{v}] \} \\ + \Phi_y \{ \mathcal{B}[\bar{u} + t(u - \bar{u}), \bar{v} + t(v - \bar{v})] - \mathcal{B}[\bar{u}, \bar{v}] \}, \quad 0 \leq t \leq 1.$$

Then $F(0) = 0$ and $F(1)$ is the given Jacobian. But $F'(t)$ is, by the assumption that both flows are subsonic, a definite quadratic form in Φ_x, Φ_y .

Our problem is thus to show that the line integral in (4) tends to zero as $r \rightarrow \infty$. To do this we consider first the behavior at infinity of the velocity fields of each of the given flows. Using (1) we find that (u, v) appears as a solution of the system

$$au_x + 2bu_y + cv_y = 0, \quad u_y - v_x = 0$$

where $a = \rho + 2\rho'u^2$, $b = 2\rho'uv$, $c = \rho + 2\rho'v^2$. From this follows that for any subsonic flow, the complex function $w = u - iv$ defines a quasi-conformal mapping of the (x, y) plane. By a linear transformation of (x, y) we may arrange that the dilatation ratio tends to one at infinity, and by continuity of a, b, c , this ratio will be arbitrarily close to one in the exterior of C_r if r is chosen sufficiently large. Hence, using Theorem 1 of [5], we find the existence of a constant $C(\epsilon)$ such that $|w - w_0| < Cr^{\epsilon-1}$ for any given $\epsilon > 0$. Therefore $|w - \bar{w}| < 2Cr^{\epsilon-1}$ as $r \rightarrow \infty$. Integrating this inequality, we find $|\Phi| = |\phi - \bar{\phi}| < C_1 r^\epsilon$ for some constant C_1 . But we have seen that

$$\begin{pmatrix} \Phi & \Psi \\ x & y \end{pmatrix}$$

can be expressed as a definite quadratic form in Φ_x, Φ_y . Hence $\Phi + i\Psi$ defines a quasi-conformal mapping in the exterior of P . Let κ be the maximum dilatation ratio of this mapping in some neighborhood of infinity. Applying Theorem 4 of [5], we see that if ϵ is chosen smaller than $1/\kappa$ the singularity of $\Phi + i\Psi$ at infinity must be removable. In particular Φ tends to a limit at infinity and since we are free to adjust Φ by an additive constant we may assume $\Phi \rightarrow 0$. Further, for any $\epsilon > 0$ there is a constant $C(\epsilon)$ such that $\Phi < Cr^{\epsilon-1}$. Thus,

$$(5) \quad \left| \oint_{C_r} \Phi d\Psi \right| \leq \text{const.} \cdot \oint_{C_r} (\Psi_x^2 + \Psi_y^2)^{1/2} ds \cdot r^{\epsilon-1}.$$

But $\Psi_x = \rho u - \bar{\rho} \bar{u} = \rho(u - \bar{u}) + (\rho - \bar{\rho})\bar{u} = (u - \bar{u})(\rho + \bar{u}\partial\rho/\partial u) + (v - \bar{v})\bar{u}\partial\rho/\partial v$, where $\partial\rho/\partial u$ and $\partial\rho/\partial v$ are to be evaluated at certain intermediate values of their arguments, and a similar relation holds for Ψ_y . Thus, using the above estimate for $|w - \bar{w}|$, we see that for any prescribed $\epsilon > 0$ there is a constant $C(\epsilon)$ such that $(\Psi_x^2 + \Psi_y^2)^{1/2} < Cr^{\epsilon-1}$. Inserting this result into (5) leads directly to the desired estimate on the line integral and completes the uniqueness proof. We formulate our result:

THEOREM 1. *A subsonic flow past a profile is uniquely determined by its free stream velocity and circulation.*

2. The Kutta-Joukowski condition. We assume now that P has a continuously turning tangent with the exception of a *trailing edge* T at which the direction of the tangent has a simple discontinuity of amount $k\pi$, $0 < k \leq 1$. We assume further that there is a neighborhood N_T of T in which each point of P can be contacted from the exterior by a circle which meets no other point of P and that the radius of all such circles can be chosen bounded from zero in N . Finally we suppose that the velocity components of the flow have uniformly bounded derivatives in N_T . Under these conditions we obtain the analogue of the result of Kutta [8] and Joukowski [9]:

THEOREM 2. *The circulation is uniquely determined by the velocity at infinity.*

Proof. Suppose to the contrary, that there exist two flows $(\phi, \psi), (\bar{\phi}, \bar{\psi})$ with the same limiting velocity and different circulation $\Gamma \neq \bar{\Gamma}$. For the function $\Phi + i\Psi = (\phi - \bar{\phi}) + i(\psi - \bar{\psi})$ we find easily that after a linear transformation of the (x, y) plane (we do not change notation),

$$(6) \quad \Phi_x^2 + \Phi_y^2 + \Psi_x^2 + \Psi_y^2 \leq 2K \begin{pmatrix} \Phi & \Psi \\ x & y \end{pmatrix}$$

where $K \rightarrow 1$ as $O(r^{\epsilon-1})$ when $r \rightarrow \infty$ for any $\epsilon > 0$. If we set $\bar{\phi} + i\bar{\psi} = \Phi + i\Psi + i(\Gamma - \bar{\Gamma}) \log z / 2\pi$ then $\bar{\phi} + i\bar{\psi}$ will be a single valued function of (x, y) in a neighborhood of infinity, hence also in a neighborhood of the origin in the $\xi + i\eta$ plane, where $\xi + i\eta = 1/(x + iy)$. The inequality (6) transforms to

$$\bar{\phi}_\xi^2 + \bar{\phi}_\eta^2 + \bar{\psi}_\xi^2 + \bar{\psi}_\eta^2 \leq 2K \begin{pmatrix} \bar{\phi} & \bar{\psi} \\ \xi & \eta \end{pmatrix} + O(r^{-1+\epsilon})$$

where r now denotes $(\xi^2 + \eta^2)^{1/2}$. Following the proof of Theorem 1 we see that $\bar{\phi} + i\bar{\psi} = O(r^{-\epsilon})$ as $r \rightarrow 0$. Thus $\bar{\phi} + i\bar{\psi}$ satisfies the hypotheses of Theorem 4 in [5] and we conclude that $\bar{\phi} + i\bar{\psi}$ tends to a limit at infinity in the (x, y) plane. But this means $\Psi \rightarrow -\infty$ as $x^2 + y^2 \rightarrow \infty$. On the other hand, an application of the mean value theorem (cf. [2, pp. 274–277]) shows that Ψ satisfies a partial differential equation of the form

$$\alpha\Psi_{xx} + 2\beta\Psi_{xy} + \gamma\Psi_{yy} + \sigma\Psi_x + \tau\Psi_y = 0$$

where α, β, γ are continuous up to P , $\alpha\gamma - \beta^2 > 0$, and σ, τ are bounded in N_T and continuous throughout the exterior of P . It is known [6] that a solution of such an equation admits no maximum or minimum interior to its region of definition. Since $\Psi = 0$ on P , we conclude $\Psi < 0$ in the exterior of P .

We may now apply a lemma of E. Hopf [7]: at all points of P in N_T except at the point T , $\partial\Psi/\partial n < -\delta < 0$, where n denotes the exterior directed normal. Since $\partial\phi/\partial n = \partial\bar{\phi}/\partial n = 0$ on P , this means $\rho\partial\phi/\partial s > \bar{\rho}\partial\bar{\phi}/\partial s + \delta$ for an appropriate choice of positive sense for the arc length s on P . Because both flows are *subsonic* we conclude $\partial\phi/\partial s > \partial\bar{\phi}/\partial s + \delta'$ for some $\delta' > 0$. But by assumption the velocity of each flow is continuous up to T , hence on opposite sides of T on P each velocity must be directed in *opposite* senses with respect to s (if $k = 1$) or else tend to zero at T . This contradicts the preceding inequality and establishes Theorem 2.

3. The force on P . The classical expressions for the components of the net force exerted by the fluid on P are

$$(7) \quad \begin{aligned} X &= - \int_0^{2\pi} [p \cos \theta + \rho u(v \sin \theta + u \cos \theta)] r d\theta, \\ Y &= - \int_0^{2\pi} [p \sin \theta + \rho v(v \sin \theta + u \cos \theta)] r d\theta \end{aligned}$$

where the integrations are performed over a circumference C_r of radius r enclosing p and the pressure p has the form

$$p = \text{const.} - \int \rho q dq.$$

We shall take (7) as definition.

From the proof of Theorem 1 we know that

$$u = u_0 + O(r^{-1+\epsilon}), \quad v = O(r^{-1+\epsilon})$$

where we have introduced a rotation of coordinates so that $v_0 = 0$. From these estimates we find

$$p = \text{const.} - \rho u(u - u_0) + O(r^{-2+2\epsilon}), \quad \rho u = u_0 \rho + \rho_0 u - u_0 \rho_0 + O(r^{-2+2\epsilon}).$$

Thus,

$$X = -u_0 \int_0^{2\pi} \rho(v \sin \theta + u \cos \theta) r d\theta + O(r^{-1+2\epsilon}).$$

The integral appearing here represents the flux of fluid crossing C_r in unit time and hence vanishes (this is easily proved formally, using (1)). Since ϵ is an arbitrary positive quantity, we may choose $\epsilon < 1/2$ and let $r \rightarrow \infty$ to obtain

$$X = 0.$$

Similarly

$$Y = -\rho_0 u_0 \int_0^{2\pi} (v \cos \theta - u \sin \theta) r d\theta + O(r^{-1+2\epsilon}),$$

which implies

$$Y = -\rho_0 u_0 \Gamma.$$

These formulas give the extension to plane compressible flows of the D'Alembert paradox and the Joukowski lift formula of incompressible flow. For a corresponding result in three dimensions we refer the reader to [4].

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