L_p FOURIER TRANSFORMS ON LOCALLY COMPACT UNIMODULAR GROUPS

BY

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1. **Introduction.** In this paper extensions to arbitrary locally compact unimodular groups of some classical results in analysis are obtained which have their origin in the following theorem of Hausdorff and Young:

THEOREM. *Let* $1<p\leqq 2$, $p'=p/p-1$, $f\in L_p(0, 1)$ *and* $c_n=\int f(x)e^{-2\pi inx}dx$. *Then*

(1) $(\sum_n |c_n|^{p'})^{1/p'}\leqq(\int|f(x)|^p dx)^{1/p}$, $n=0, 1, 2, \cdots$.

(2) *If* $\{c_n\}$ *is a sequence such that* $(\sum_n |c_n|^{p'})^{1/p'}$ *is finite, then there is an* $f\in L_{p'}(0, 1)$ *such that* $c_n=\int f(x)e^{-2\pi inx}dx$ *and* $(\int|f(x)|^p dx)^{1/p}\leqq(\sum_n |c_n|^p)^{1/p}$.

We show among other things that (1) may be generalized as follows: Let G be a compact group with Haar measure 1 and let $\{\phi_\lambda\}$ be any collection of inequivalent continuous irreducible unitary representations of G where ϕ_λ has degree d_λ. Let C_λ be the matrix $\int_G f(x)\phi_\lambda(x)dx$. For any matrix A let $\|A\|_p=\text{trace }(|A|^p)^{1/p}$ where $|A|=(A^*A)^{1/2}$. Then $(\sum_\lambda \|C_\lambda\|_{p'}^{p'} d_\lambda)^{1/p'}\leqq(\int_G|f(x)|^p dx)^{1/p}$.

As is well known analogous results hold for Fourier integrals. For example, a function f in $L_p(-\infty, \infty)$, $1<p<2$, has a Fourier transform F in $L_{p'}(-\infty, \infty)$ and $((2\pi)^{-1/2}\int|F(y)|^{p'}dy)^{1/p'}\leqq((2\pi)^{-1/2}\int|f(x)|^p dx)^{1/p}$. Our main result is an extension of this theorem to the noncompact and nonabelian case.

If G is a locally compact abelian group the Fourier transform F of an integrable function f on G is generally defined as the function on the character group $\hat{G}$ of G given by $F(\chi)=\int_G\chi(a)f(a)da$; when f is in $L_1(G)\cap L_2(G)$, $T: f\rightarrow F$ is an isometric map into $L_2(\hat{G})$ which can be extended to an isometry of $L_2(G)$ with $L_2(\hat{G})$. Moreover the operation L_f of convolution by f in $L_2(G)$ is unitarily equivalent via T to multiplication, M_F by F in $L_2(\hat{G})$. In fact $M_F=TL_fT^{-1}$. If G is abelian, any closed densely defined operator in $L_2(G)$ which commutes with the group translations is equivalent via T to a multiplication in $L_2(\hat{G})$ by a measurable function on $\hat{G}$, (Segal [4]). In particular if f is measurable on the abelian group G and L_f is closed and densely defined with $TL_fT^{-1}=M_F$ it is natural to call F the Fourier transform of f. Since F and M_F are essentially equivalent it makes sense to define the Fourier

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transform of f as the operator L_f. Now for a general locally compact unimodular group we say that a measurable function f on G has a Fourier transform whenever L_f is a measurable operator, that is, is closed, densely defined and satisfies certain additional conditions which we state later. In this setting, the Hausdorff-Young Theorem takes the form $\|L_f\|_{p'} \leqq \|f\|_p$ where f belongs to $L_p(G)$, $1 \leqq p \leqq 2$ and $\|L_f\|_{p'}$ is defined, without the use of reduction theory, in terms of noncommutative integration. We note that in this type of integration the objects to be integrated are operators rather than functions.

In the abelian case theorems of this type generally depend upon certain convexity properties and are most easily proved by using results on the interpolation of linear operations. We show that the Riesz-Thorin interpolation theorem can be extended to the noncommutative situation and use it as the main tool for proving our results.

The paper is organized as follows: Part 2 is devoted principally to definitions and a summary of the basic facts about noncommutative integration, as developed by Segal [4]. In §3 we develop the basic properties of noncommutative L_p spaces. Dixmier [2] has done much of this, but whereas his L_p spaces are defined as abstract completions of certain sets of operators of a relatively simple type we prefer to identify points in the completion. In the present treatment a general element of L_p is a measurable operator, just as a general element of L_p in the usual measure-theoretic case is a measurable function. §4 treats linear mappings between L_p spaces and gives an extension of the Riesz-Thorin interpolation theorem, similar to that given by Zygmund and Calderón in [1]. In §5 we apply these results to groups.

2. **Noncommutative integration.** A not necessarily bounded linear operator T in a Hilbert space $\mathcal{H}$ is essentially measurable with respect to a ring $\mathcal{A}$ of operators on $\mathcal{H}$ if T has a closure, T is affiliated with $\mathcal{A}$ and if there exists an increasing sequence $\mathcal{K}_1, \mathcal{K}_2, \cdots$, of closed linear manifolds in the domain of T such that the restriction $T^{(i)}$ of T to $\mathcal{K}_i$ is bounded for each i, $\mathcal{K}_i^\perp$ is algebraically finite, $\mathcal{K}_i^\perp \downarrow 0$ and the projections P_i on $\mathcal{K}_i$ are affiliated with $\mathcal{A}$. T is measurable if T is essentially measurable and closed. If S and T are essentially measurable, then so are S^*, $S+T$, and ST. When S, T are measurable the closures $(S+T)^-$ and $\overline{ST}$ are called the strong sum and strong product respectively and the collection of all operators measurable with respect to a given ring is an algebra with involution relative to adjunction and the strong operations. Throughout the rest of the paper all sums and products of measurable operators will be taken in the strong sense and when S, T are measurable, we will write $S+T$ for $(S+T)^-$ and ST for $\overline{ST}$; however, algebraic operations on operators not known to be measurable will be taken in the ordinary sense. If T is measurable and $U|T|$ is the canonical polar decomposition of T then U belongs to $\mathcal{A}$ and $|T|$ is measurable. A sequence $\{T_n\}$ of operators on a Hilbert space is said to converge nearly everywhere (n.e.) relative to a measurable operator T if for $\epsilon > 0$ there exist projec-

tions $P_n(\epsilon)$ such that $\|(T_n-T)P_n(\epsilon)\|<\epsilon$ and $I-P_n(\epsilon)$ is algebraically finite for all sufficiently large n.

If $S_n \rightarrow S$ and $T_n \rightarrow T$ n.e. then $S_n+T_n \rightarrow S+T$ n.e.; $T_n \rightarrow T$ n.e. and S is measurable implies $T_nS \rightarrow TS$ n.e.

By a regular gage space Γ we mean a system $(\mathfrak{H}, \mathfrak{A}, m)$ composed of a complex Hilbert space $\mathfrak{H}$, a ring of operators $\mathfrak{A}$ on $\mathfrak{H}$ and a measure or gage m on the projections in $\mathfrak{A}$ with the properties:

1. $m(P)>0$ if $P\neq 0$ and $m(0)=0$.
2. m is completely additive. (i.e. if $\{P_\alpha\}$ is any mutually orthogonal collection of projections in $\mathfrak{A}$ then $m(\bigcup P_\alpha)=\sum_\alpha m(P_\alpha)$.)
3. Every projection is the l.u.b. of projections on which m is finite.
4. If U is unitary in $\mathfrak{A}$ then $m(U^*PU)=m(P)$.

We mention that a measure space M is technically not a special instance of a gage space; however, M is essentially equivalent to a regular commutative gage space constructed from the multiplication algebra of M [4] and in the future we will not distinguish carefully between a measure space and its associated gage space.

A measurable operator T on a gage space Γ is said to be elementary if it is everywhere defined and lives on an m-finite manifold in the sense that T and T^* are 0 on the orthogonal complement of a subspace of finite gage. The collection $\mathcal{E}$ of elementary operators is a 2-sided ideal in $\mathfrak{A}$ on which there exists a unique linear functional m' such that

(1) $m'(P)=m(P)$ if P is a projection in $\mathfrak{A}$.

(2) T belongs to $\mathcal{E}$ and $T>0$ implies $m'(T)>0$.

(3) If S belongs to $\mathfrak{A}$ and T is in $\mathcal{E}$ then $m'(ST)=m'(TS)$.

(4) m' is strongly continuous on the operators in the unit sphere of $\mathcal{E}$ that live on a fixed m-finite manifold.

We now replace the symbol m' by m and say that a measurable operator T is integrable if there is a sequence $\{T_n\}$ of elementary operators converging to T n.e. which satisfies the additional condition that $m(|T_n-T_k|)\rightarrow 0$ as $n, k\rightarrow\infty$. The integral of T which is denoted by $m(T)$ is defined as $\lim_n m(T_n)$ and is single valued. The functional m thus defined has the usual properties and when T is measurable and positive with spectral resolution $\int\lambda dE_\lambda$, $m(T) =\lim_{\epsilon\rightarrow+0}\int_\epsilon \lambda dm(E_\lambda)$. The integral is extended to not necessarily measurable operators as follows: If T is a positive hyperhermitian operator affiliated with the gage space then $m(T)$ is defined as l.u.b. of $m(X)$ as X ranges over the positive elementary operators bounded by T; when $m(T)$ is finite, T is measurable and is in fact integrable.

3. L_p spaces.

DEFINITION 3.1. Let $\Gamma=(\mathfrak{H}, \mathfrak{A}, m)$ be a regular gage space and let B be a positive hyperhermitian operator affiliated with $\mathfrak{A}$. Set $\|B\|_\infty$ equal to the bound of B if B is bounded and otherwise put $\|B\|_\infty=\infty$. For $1\leq p<\infty$ put $\|B\|_p=(m(B^p))^{1/p}$ if $m(B^p)$ is finite and if not, set $\|B\|_p=\infty$. When $\|B\|_p$ is finite we note that B is necessarily measurable. For an arbitrary measurable operator T put $\|T\|_p=\||T|\|_p$, and let $L_p(\Gamma)$ be the collection of all measurable operators T with $\|T\|_p<\infty$.

DEFINITION 3.1. An elementary operator S is said to be simple if $S = U\sum_{i=1}^n a_iE_i$ where U is measurable partially isometric and $E_1, E_2, \cdots, E_n$ are nonzero mutually orthogonal m-finite projections in $\mathfrak{A}$.

THEOREM 1. *For any regular gage space* Γ, $L_p(\Gamma)$, $1\leqq p\leqq\infty$, *is a self adjoint normed linear space over the complex numbers in which the norm of an operator* T *is given by* $\|T\|_p$. *Furthermore,*

(a) *if* T *belongs to* $L_p(\Gamma)$ *and* S *is in* $L_{p'}(\Gamma)$, *then* TS *is integrable and* $\|TS\|_1\leqq\|T\|_p\|S\|_{p'}$. *And if* T *is an arbitrary measurable operator*

(b) $\|T^*\|_p=\|T\|_p$,

(c) $\|T\|_p=\text{Sup}\ \{\|TS\|_1: S \textit{ is simple}, \|S\|_{p'}\leqq 1, \textit{and } TS \textit{ is integrable}\}$.

With the exception of (c), Segal [4] has shown that the theorem is true for the cases $p=1$, 2, and ∞. Since there is no difficulty in verifying (c) when $p=1$ and as the case $p=\infty$ will be treated somewhat more generally later on we will assume in the succeeding lemmas that $1<p<\infty$. Before proceeding, it is instructive to consider an

EXAMPLE. Let Λ be a set and let $\mathfrak{F}$ be the collection of all operator valued functions $T=\{T_\lambda\}$ defined on Λ such that T_λ is a linear operator on a fixed complex finite dimensional space of dimension d_λ. Then $\mathfrak{F}$ is a linear algebra over the complex numbers with involution $*$, the algebraic operations and $*$ being defined in the obvious fashion. For $1\leqq p<\infty$ set $\|T\|_p=(\sum_\lambda\|T_\lambda\|_p^p d_\lambda)^{1/p}$ where $\|T_\lambda\|_p=[\text{trace }(T_\lambda^*T_\lambda)^{p/2}]^{1/p}$ and put $\|T\|_\infty=\text{Sup}\ \{\|T_\lambda\|_\infty:\lambda\in\Lambda\}$. Now for $1\leqq p\leqq\infty$ put $L_p(\Lambda)$ for the collection of all T in $\mathfrak{F}$ such that $\|T\|_p<\infty$. Then it turns out that $\|T\|_p$ is a monotone decreasing function of p for each fixed T in $\mathfrak{F}$; from this we see that $L_p(\Lambda)\subset L_\infty(\Lambda)$ for $1\leqq p\leqq\infty$, and furthermore it is also true that if T belongs to $L_\infty(\Lambda)$ and S is in $L_p(\Lambda)$ then $\|TS\|_p \leqq\|T\|_\infty\|S\|_p\leqq\|T\|_p\|S\|_p$. In fact $L_p(\Lambda)$ is a complex Banach algebra with involution $*$. The case $p=2$ is of special interest in that $L_2(\Lambda)$ is a Hilbert algebra, the inner product of two elements T, S in $L_2(\Lambda)$ being given by $(T, S) =\sum_\lambda \text{trace }(T_\lambda S_\lambda^*)d_\lambda$. $L_\infty(\Lambda)$ is algebraically isomorphic to a ring $\mathfrak{A}$ of operators on $L_2(\Lambda)$, the element T in $L_\infty(\Lambda)$ corresponding to the operator L_T where $L_T(S)=TS$ for all S in $L_2(\Lambda)$. If L_Q is a projection in $\mathfrak{A}$, put $n(L_Q) =\|Q\|_2^2$; then n is a regular gage on $\mathfrak{A}$ and the triple $\Gamma=(L_2(\Lambda), \mathfrak{A}, n)$ is a regular gage space. In addition $L_p(\Gamma)$ and $L_p(\Lambda)$ are algebraically isomorphic under an appropriate restriction of the correspondence $T\leftrightarrow L_T$.

We now return to the proof of Theorem 1. The essential ideas in the proofs of the first two lemmas are taken from Dixmier [2].

LEMMA 1.1. *If* T *and* S *are elementary operators then* $\|TS\|_1\leqq\|T\|_p\|S\|_{p'} <\infty$.

Let $T=U|T|$ and $S=V|S|$ be the canonical polar decompositions of T and S. Let $\int\lambda dE_\lambda$ and $\int\lambda dF_\lambda$ be the spectral resolutions of $|T|$ and $|S|$ respectively. For arbitrary bounded complex valued, Borel measurable functions f, g defined on the line put $T_f=U\int f(\lambda)dE_\lambda$ and $S_g=V\int g(\lambda)dF_\lambda$. Then since $\int f(\lambda)dE_\lambda$ is elementary and U is bounded and in $\mathfrak{A}$, it follows that T_f is elementary. Now $T_f^*T_f=\int\bar{f}(\lambda)dE_\lambda U^*U\int f(\lambda)dE_\lambda$ and U^*U is the projection on the closure of the range of $|T|$. Putting $U^*U=Q$ it results that $Q|T| =|T|=|T|Q$; hence Q commutes with $\int\bar{f}(\lambda)dE_\lambda$ and $\int f(\lambda)dE_\lambda$. Thus $T_f{}^*T_f =Q\int|f(\lambda)|^2dE_\lambda Q$ and $(T_f^*T_f)^{p/2} = Q\int|f(\lambda)|^pdE_\lambda Q\leqq\int|f(\lambda)|^pdE_\lambda$. Setting $(\int|f(\lambda)|^pdm(E_\lambda))^{1/p}=\|(f, E)\|_p$ it follows that $\|T_f\|_p\leqq\|(f, E)\|_p<\infty$ and if we put $(\int|g(\lambda)|^qdm(F_\lambda))^{1/q}=\|(g, F)\|_q$, a similar argument shows that S_g is elementary and $\|S_g\|_q\leqq\|(g, F)\|_q<\infty$. We put $\|(f, E)\|_\infty=\|f\|_\infty$ and $\|(g, F)\|_\infty =\|g\|_\infty$ and observe that the above inequalities hold for $1\leqq p,\ q\leqq\infty$. Now let $TS=W|TS|$ be the polar decomposition of TS and set $B(f, g)=m(W^*T_fS_g)$. By results known for $L_1(\Gamma)$ and $L_\infty(\Gamma)$ (part (a) of Theorem 1) it follows that $|B(f, g)|\leqq\|W^*T_fS_g\|_1\leqq\|W^*\|_\infty\|T_fS_g\|_1\leqq\|T_fS_g\|_1\leqq\|T_f\|_r\|S_g\|_s\leqq\|(f, E)\|_r \cdot\|(g, F)\|_s$ where $(r, s)=(1, \infty), (\infty, 1)$. Since B is bilinear, it follows by the Riesz-Thorin interpolation theorem as stated in [1, Theorem F], that $|B(f, g)|\leqq\|(f, E)\|_p\|(g, F)\|_{p'}$ for $1\leqq p\leqq\infty$. Setting $f(\lambda)=\lambda$ if $0\leqq\lambda\leqq\|T\|_\infty$ and putting $f(\lambda)=0$ for other values of λ we see that $T=T_f$ and that $\|T\|_p =\|(f, E)\|_p$. Let g be defined similarly. Then $|B(f, g)|=m(W^*TS)=\|TS\|_1 \leqq\|T\|_p\|S\|_{p'}$ for $1\leqq p\leqq\infty$.

LEMMA 1.2. *If T is measurable $\|T\|_p=\|T^*\|_p$, and if T belongs to $L_p(\Gamma)$ there exists an operator S in $L_{p'}(\Gamma)$ such that $\|TS\|_1=\|T\|_p$ and $\|S\|_{p'}=1$. If, in addition, T is bounded (elementary) then S is also bounded (elementary).*

Let $T=U|T|$. Then $|T^*|=U|T|U^*$ and $|T^*|^p=U|T|^pU^*$ so that $m(|T^*|^p)=m(U|T|^pU^*)$. Thus if $m(|T|^p)$ is finite, $m(|T|^p)=m(U^*U|T|^p) =m(U|T|^pU^*)=m(|T^*|^p)$; hence $\|T\|_p=\|T^*\|_p$. Applying the same argument to T^* we see that when $\|T^*\|_p$ is finite, $\|T^*\|_p=\|T^{**}\|_p=\|T\|_p$. Thus $\|T^*\|_p=\|T\|_p$ for any measurable operator T. Suppose now that T belongs to $L_p(\Gamma)$. Then $|T^*|^p=U|T|U^*U|T|^{p-1}U^*=TR$ where $R=P|T|^{p-1}U^*$ and P is the projection U^*U on the closure of the range of $|T|$. $R^*R = U|T|^{p-1}PP|T|^{p-1}U^*=U|T|^{2(p-1)}U^*=(U|T|U^*)^{2(p-1)}=|T^*|^{2(p-1)}$ and from this follows the fact that $|R|=|T^*|^{p-1}$; on the other hand since $(p-1)p'=p$, $|R|^{p'}=|T^*|^p$ and therefore $\|R\|_{p'}=\|T^*\|_p^{p/p'}=\|T\|_p^{p-1}<\infty$. We may assume without loss of generality that $\|T\|_p\neq0$ and setting $S=\|T\|_p^{1-p}R$ it is easily verified that $\|S\|_{p'}=1$. Now $TS=\|T\|_p^{1-p}TR=\|T\|_p^{1-p}|T^*|^p$ so that $\|TS\|_1=\|T\|_p^{1-p}m(|T^*|^p)=\|T\|_p$. Finally from the construction of S, it is clear that the last statement of the lemma is valid.

LEMMA 1.3. *If T is a measurable operator there exists a sequence $\{S_n\}$ of simple operators such that $\|S_n\|_{p'}\leqq1$, TS_n is integrable, and $\|TS_n\|_1\to\|T\|_p$.*

We show first of all that it suffices to prove the lemma for T positive. Suppose then that $R_1, R_2, \cdots$, is a sequence of simple operators satisfying the conclusion of the lemma for $|T^*|$. Let $T=U|T|$ be the canonical polar decomposition of T and put $S_n=U^*R_n$. Then $|T^*|R_n=U|T|U^*R_n=TS_n$ and $\||T^*|R_n\|_1=\|TS_n\|_1\to\|T^*\|_p=\|T\|_p$. Clearly S_n is simple, and to verify that $\|S_n\|_{p'}\leqq 1$ choose, by the preceding lemma, an elementary operator S_n' such that $\|S_nS_n'\|_1=\|S_n\|_{p'}$ and $\|S_n'\|_p=1$. Then $\|S_n\|_{p'}=\|U^*R_nS_n'\|_1 \leqq \|U^*\|_\infty\|R_nS_n'\|_1$ and applying Lemma 1.1, it follows that $\|R_nS_n'\|_1 \leqq\|R_n\|_{p'}\|S_n'\|_p=\|R_n\|_{p'}\leqq 1$; thus $\|S_n\|_{p'}\leqq 1$, and we may suppose that $T\geqq 0$. Let $\int\lambda dE_\lambda$ be the spectral resolution of T and set $Q_n=\int_{2^{-n}}^{2^n}dE_\lambda$ and $T_n =\int_{2^{-n}}^{2^n}\lambda dE_\lambda$. We now consider two cases: (a) $\|T_n\|_p<\infty$ for all n; and (b) there exists n_0 such that $\|T_{n_0}\|_p=\infty$. If (a) holds let f_k be the function defined for each positive integer $k\geqq n$ by $f_k(\lambda)=i/2^k$ if $i/2^k\leqq\lambda<(i+1)/2^k$ where $i=0, 1, 2, \cdots$. Putting $F_k=Q_n\int f_k(\lambda)dE_\lambda$ it results that $F_k=\int_{2^{-n}}^{2^n}f_k(\lambda)dE_\lambda$ and as $\|T_n\|_p<\infty$, that Q_n is elementary and that F_k is simple. There is no difficulty in verifying that $0\leqq T_n^p-F_k^p\leqq p(2^n+1)^{p-1/2^k}Q_n$ so that $m(F)_k^p\uparrow m(T_n^p)$ as $k\to\infty$. Choose j such that $\|T_n\|_p-1/n\leqq\|F_j\|_p$ and set $S_n=\|F_j\|_p^{1-p}F_j^{p-1}$ (it is no loss of generality to assume $\|F_j\|_p\neq 0$). Then S_n is simple, $TS_n =T(Q_nS_n)=T_nS_n$ is integrable, and $T_nS_n\geqq F_jS_n\geqq 0$. Hence $\|TS_n\|_1\geqq\|F_jS_n\|_1 =\|F_j\|_p\geqq\|T_n\|_p-1/n$. Now by an argument given in Lemma 1.2, $\|S_n\|_{p'}=1$ and since $TS_n=T_nS_n$ we can apply Lemma 1.1 to get the inequality $\|TS_n\|_1 \leqq\|T_n\|_p\|S_n\|_{p'}=\|T_n\|_p$. Thus $\|T_n\|_p-1/n\leqq\|TS_n\|_1\leqq\|T_n\|_p$ and using the fact that $\|T_n\|_p\uparrow\|T\|_p$ we see that $\|TS_n\|_1\to\|T\|_p$. We now consider case (b). Choose for each $n\geqq n_0$ an m-finite projection J_n such that $J_n\leqq Q_n$ and $\|J_n\|_p\geqq n$. Set $S_n=\|J_n\|_p^{1-p}J_n^{p-1}=\|J_n\|_p^{1-p}J_n$. Then S_n is simple, $\|S_n\|_{p'}=1$, and $TS_n=T(Q_nS_n)=T_nS_n$ is integrable. Since $T_n\geqq Q$, $S_nT_nS_n\geqq S_nQ_nS_n=\|J_n\|_p^{1-p}S_n \geqq 0$ and $\|S_n\|_\infty\|T_nS_n\|_1\geqq\|S_nT_nS_n\|_1\geqq\|J_n\|_p^{1-p}\|S_n\|_1$. Because $\|S_n\|_\infty=\|J_n\|_p^{1-p}$ it follows that $\|T_nS_n\|_1\geqq\|S_n\|_1=\|J_n\|_p^{1-p}\|J_n\|_1=\|J_n\|_p^{1-p}\|J_n\|_k^p=\|J_n\|_p\geqq n$. Thus for $n\geqq n_0$, $n\leqq\|TS_n\|_1<\infty$ and $\|TS_n\|_1\to\|T\|_p=\infty$.

Lemma 1.4. *If T is a non-negative measurable operator in $L_p(\Gamma)$ there exists a sequence $T_1, T_2, \cdots$, of elementary operators such that $T_n\to T$ n.e., $\|T_n-T_k\|_p\to 0$, $\|T_n\|_p\uparrow\|T\|_p$ and $\|T-T_n\|_p\to 0$.*

Let $T=\int\lambda dE_\lambda$ be the spectral resolution of T. Set $T_n=\int_{n^{-1}}^n\lambda dE_\lambda$ and put $R_n=\int_0^n dE_\lambda$. Then $\|(T-T_n)R_n\|_\infty=n^{-1}$ and because $\int_n dE_\lambda$ is m-finite, it is algebraically finite (as Γ is a regular gage space) so that $T_n\to T$ n.e.. Putting $Q_n=\int_{n^{-1}}^n dE_\lambda$ it follows that $Q_n\leqq T^p$ so that Q_n is elementary and hence $T_n=T_nQ_n$ is also. Now $\|T_n\|_p^p=\int_{n^{-1}}^n\lambda^p dm(E_\lambda)\uparrow\int_0^\infty\lambda^p dm(E_\lambda)=\|T\|_p^p<\infty$ so that $\|T_n-T_k\|_p^p=\int_{n^{-1}}^{k^{-1}}\lambda^p dm(E_\lambda)+\int_k^n\lambda^p dm(E_\lambda)$ $(n>k)\to 0$ as n, $k\to\infty$ and also

$$\|T-T_n\|_p^p=\int_0^{n^{-1}}\lambda^p dm(E_\lambda)=\int_n^\infty\lambda^p dm(E_\lambda)\to 0 \text{ as } n\to\infty.$$

Lemma 1.5 (Holder's inequality). *If T is in $L_p(\Gamma)$ and if S belongs to $L_{p'}(\Gamma)$ then TS is integrable and $\|TS\|_1\leqq\|T\|_p\|S\|_{p'}$.*

Suppose first of all that T is elementary and that $S \geqq 0$. Choose elementary operators $S_1, S_2, \cdots$, satisfying the conclusion of the preceding lemma, which we apply to S. Since T is bounded $TS_n \to TS$ n.e. and by Lemma 1.1, $\|TS_n - TS_k\|_1 \leqq \|T\|_p \|S_n - S_k\|_{p'} \to 0$. Thus $\{TS_n\}$ is a Cauchy sequence of elementary operators and it follows that TS is integrable. Furthermore $\|TS\|_1 = \lim_n \|TS_n\|_1 \leqq \lim_n \|T_n\|_p \|S_n\|_{p'} = \|T\|_p \|S\|_{p'}$. Now let S be an arbitrary element of $L_{p'}(\Gamma)$ and let $S = V|S|$ be the polar decomposition of S. Then $TS = (TV)|S|$ is integrable. For TV is elementary, and in addition we have the inequality, $\|TS\|_1 \leqq \|TV\|_p \|S\|_{p'}$. To show that $\|TS\|_1 \leqq \|T\|_p \|S\|_{p'}$ it therefore suffices to show that $\|TV\|_p \leqq \|T\|_p$. For this purpose choose by Lemma 1.2 an elementary operator T' such that $\|T'\|_{p'} = 1$ and $\|V^*T^*T'\|_1 = \|V^*T^*\|_p$. Then

$$\|TV\|_p = \|(TV)^*\|_p = \|V^*T^*\|_p = \|V^*T^*T'\|_1 \leqq \|V^*\|_\infty \|T^*T'\|_1$$
$$\leqq \|T^*\|_p \|T'\|_{p'} = \|T\|_p.$$

Hence the lemma is valid for T elementary and S arbitrary in $L_{p'}(\Gamma)$. Now suppose that T is an arbitrary non-negative element of $L_p(\Gamma)$. Choose elementary operators T_n as in Lemma 1.4. Then $T_nS \to TS$ n.e. and by what we have just established, $\|T_nS - T_kS\|_1 \leqq \|T_n - T_k\|_p \|S\|_{p'} \to 0$ so that $\{T_nS\}$ is a Cauchy sequence in $L_1(\Gamma)$. Since $L_1(\Gamma)$ is complete there exists K in $L_1(\Gamma)$ such that $\|T_nS - K\|_1 \to 0$. By passing to a subsequence we can assume that $T_nS \to K$ n.e. and as limits n.e. are unique it follows that $TS = K$. Hence TS is integrable and by continuity $\|TS\|_1 = \lim_n \|T_nS\|_1 \leqq \lim_n \|T_n\|_p \|S\|_{p'} = \|T\|_p \|S\|_{p'}$. Finally, for an arbitrary operator T in $L_p(\Gamma)$ we can write $T = U|T|$ and therefore $TS = U(|T|S)$ is integrable and $\|TS\|_1 = \|U(|T|S)\|_1 \leqq \|U\|_\infty \|TS\|_1 \leqq \|T\|_p \|S\|_{p'}$.

Proof of the theorem. Parts (a), (b), and (c) have already been established in the preceding lemmas, and it is clear that $\|\alpha T\|_p = \alpha \|T\|_p$ when α is a complex number and T belongs to $L_p(\Gamma)$. So we must show that if R and T belong to $L_p(\Gamma)$ then $R+T$ does also, and that $\|R+T\|_p \leqq \|R\|_p + \|T\|_p$. By Lemma 1.3, there is a sequence $S_1, S_2, \cdots$, of simple operators such that $(R+T)S_i$ is integrable, $\|(R+T)S_i\|_1 \to \|R+T\|_p$ and $\|S_i\|_{p'} \leqq 1$. Now $\|(R+T)S_i\|_1 = \|RS_i + TS_i\|_1 \leqq \|RS_i\|_1 + \|TS_i\|_1$, since RS_i and TS_i are integrable and applying Holder's inequality we get $\|RS_i\|_1 + \|TS_i\|_1 \leqq \|R\|_p \|S_i\|_{p'} + \|T\|_p \|S_i\|_{p'} \leqq \|R\|_p + \|T\|_p$. Hence $\|R+T\|_p \leqq \|R\|_p + \|T\|_p$.

COROLLARY 1.1. *If K is bounded and measurable and if T belongs to $L_p(\Gamma)$, $1 \leqq p \leqq \infty$, then $\|KT\|_p \leqq \|K\|_\infty \|T\|_p$. Moreover, $\|K\|_\infty$ is the bound of the operation L_K of left multiplication by K on $L_p(\Gamma)$.*

For $p = 1, \infty$ the inequality $\|KT\|_p \leqq \|K\|_\infty \|T\|_p$ is known and if $1 < p < \infty$ there exists a sequence $S_1, S_2, \cdots$, of simple operators such that KTS_n is integrable, $\|KTS_n\|_1 \to \|KT\|_p$, and $\|S_n\|_{p'} \leqq 1$. Now $\|KTS_n\|_1 \leqq \|K\|_\infty \|TS_n\|_1 \leqq \|K\|_\infty \|T\|_p \|S_n\|_{p'} \leqq \|T\|_p$ so that passing to the limit we get $\|KT\|_p \leqq \|K\|_\infty \cdot \|T\|_p$.

To show that $\|K\|_\infty$ is the least number satisfying the inequality for all T in $L_p(\Gamma)$ it suffices to show that there exists a sequence $T_1, T_2, \cdots$, of simple operators such that $\|T_n\|_p \leqq 1$ and $\|KT_n\|_p \to \|K\|_\infty$. We note that this result includes part (c) of Theorem 1 for the case $p=\infty$. Now writing $K=U|K|$ and considering $|K^*|=KU^*$ it is easy to see that it is sufficient to consider the case when K is positive. Suppose then that $K\geqq 0$ and that K has the spectral resolution $\int_0^a \lambda dE_\lambda$ where $a=\|K\|_\infty$. Put $Q_n=\int_0^{a-n^{-1}} dE_\lambda$ for all positive integers n such that $a-n^{-1}\geqq 0$ and set $K_n=KQ_n$. Let J_n be a m-finite projection such that $0<J_n\leqq Q_n$ and let T_n be a nonzero multiple of J_n such that $\|T_n\|_p=1$. Then T_n is simple $\|KT_n\|_p\leqq\|K\|_\infty$ and $KT_n=K(QT_n)=K_nT_n$. Hence $\|KT_n\|_p=\|K_nT_n\|_p$. Now

$$
\begin{aligned}
a-n^{-1} = \|(a-n^{-1})T_n\|_p &\leqq \|(a-n^{-1})Q_nT_n - K_nT_n\|_p + \|K_nT_n\|_p \\
&\leqq \|(a-n^{-1})Q_n - K_n\|_\infty \|T_n\|_p + \|KT_n\|_p \leqq n^{-1} + \|KT_n\|_p
\end{aligned}
$$

so that $\|K\|_\infty - 2n^{-1} \leqq \|KT_n\|_p \leqq \|K\|_\infty$. Hence $\|KT\|_p \to \|K\|_\infty$.

Corollary 1.2. *The simple operators are dense in* $L_p(\Gamma)$ *for* $1\leqq p<\infty$.

Let T belong to $L_p(\Gamma)$. Using the polar decomposition of T and the preceding corollary we see that it suffices to consider the case when T is positive. Now let $\epsilon>0$ and choose by Lemma 1.4 an elementary operator T_ϵ such that $\|T-T_\epsilon\|_p<\epsilon/2$. Since T_ϵ is positive and elementary, an argument similar to one used in the proof of Lemma 1.3 shows that there exists a simple operator S_ϵ such that $\|T_\epsilon - S_\epsilon\|_p<\epsilon/2$. As we have shown that $L_p(\Gamma)$ has a norm it follows that $\|T-S_\epsilon\|_p<\epsilon$.

Corollary 1.3. *If* $T_n\to T$ *in* $L_p(\Gamma)$ *and* $S_n\to S$ *in* $L_{p'}(\Gamma)$ *then* $\|TS-T_nS_n\|_1 \to 0$ *and* $m(T_nS_n)\to m(TS)$.

As the integral is continuous on $L_1(\Gamma)$ it suffices to prove $\|TS-T_nS_n\|_1\to 0$. Now $\|TS-T_nS_n\|_1 \leqq \|TS-T_nS\|_1 + \|T_nS-T_nS_n\|_1 \leqq \|T-T_n\|_p\|S\|_{p'} + \|T_n\|_p\|S-S_n\|_{p'}$, and because $\{T_n\}$ is convergent the norms $\|T_n\|_p$ are bounded and the result follows.

Theorem 2. $L_p(\Gamma)$ *is complete relative to the norm* $\|T\|_p$.

Lemma 2.1. *Suppose* $\{T_k\}$ *is a Cauchy sequence in* $L_p(\Gamma)$ *and that* T_k *converges n.e. to a measurable operator* T. *Suppose also that* $\|T_k\|_p\leqq M$ *for all* k. *Then* T *belongs to* $L_p(\Gamma)$ *and* $\|T\|_p\leqq M$.

Let S belong to $L_{p'}(\Gamma)$ and suppose $\|S\|_{p'}\leqq 1$. Then $T_kS\to TS$ n.e.; moreover $\{T_kS\}$ is a Cauchy sequence in $L_1(\Gamma)$. For $\|T_kS-T_iS\|_1\leqq\|T_k-T_i\|_p\|S\|_{p'} \leqq \|T_k-T_i\|_p\to 0$. Since $L_1(\Gamma)$ is complete there exists R belonging to $L_1(\Gamma)$ such that $\|R-T_kS\|_1\to 0$. By passing to a subsequence we can assume $T_kS\to R$ n.e. But $T_kS\to TS$ n.e. and as limits n.e. are unique, it results that $TS=R$. Thus TS is in $L_1(\Gamma)$ and $\|(TS)\|_1=\lim_k \|(T_kS)\|_1$. Because $\|(T_kS)\|_1\leqq\|T_k\|_p \cdot\|S\|_{p'}\leqq\|T_k\|_p\leqq M$, $\|(TS)\|_1\leqq M$. So by Lemma 1.10, $\|T\|_p\leqq M$.

LEMMA 2.2. *If* $\{T_n\}$ *is Cauchy in* $L_p(\Gamma)$ *and* $T_n \to T$ *n.e. then* T *belongs to* $L_p(\Gamma)$ *and* $\|T-T_n\|_p \to 0$.

For each fixed n we can assume $\|T_k-T_n\|_p \leqq a_n$ for all k where $a_n \to 0$. Now $T_k-T_n \to T-T_n$ n.e. as $k\to\infty$, and is Cauchy in $L_p(\Gamma)$. By the previous Lemma $T-T_n$ belongs to $L_p(\Gamma)$ and $\|T-T_n\|_p \leqq a_n \to 0$. As the sum of two elements of $L_p(\Gamma)$ is again in $L_p(\Gamma)$, $T=(T-T_n)+T_n$ is in $L_p(\Gamma)$.

Proof of the theorem. Now let $\{T_n\}$ be a Cauchy sequence in $L_p(\Gamma)$. We wish to prove there exists T in $L_p(\Gamma)$ such that $\|T-T_n\|_p \to 0$. Observe first that we can assume $T_n=T_n^*$. For the L_p norms of the real and imaginary parts of T_n never exceed the L_p norm of T_n. It is also no loss of generality to suppose that T_1 is bounded and that $\|T_{n+1}-T_n\|_p \leqq 4^{-n}$. Let $\int \lambda dE_\lambda$ be the spectral resolution of $T_{n+1}-T_n$ and put $J_n=\int_{-2^{-n}}^{2^{-n}} dE_\lambda$. Then $\|(T_{n+1}-T_n)J_n\|_\infty \leqq 2^{-n}$ and $m(I-J_n) \leqq 2^{-np}$. Put $Q_k=\bigcup_{n\geqq k}(I-J_n)$. Then $m(Q_k) \leqq 2^{1-k}$ and $Q_k \geqq I-J_n$ so $J_n \geqq I-Q_k$ and $\|(T_{n+1}-T_n)(I-Q_k)\|_\infty \leqq 2^{-n}$ for all $n \geqq k$.

As in the proof that $L_1(\Gamma)$ is complete [4, p. 431] there exists a bounded operator $T^{(k)}$ defined on $(I-Q_k)\mathcal{H}=\mathcal{H}_k$ to which the restrictions $T_n^{(k)}$ of T_n to $\mathcal{H}_k$ converge uniformly, and the $T^{(k)}$ define an operator whose closure T is measurable. $T_n \to T$ n.e.; for $\|(T_n-T)(I-Q_k)\|_\infty \leqq 2^{-n+1}$ when $n \geqq k$, and we can take $P_n(\epsilon)$ equal to $I-Q_n$ for $2^{-n+1}<\epsilon$ and equal to 0 otherwise. Then $P_n(\epsilon)\uparrow I$, $m(I-P_n(\epsilon))<\infty$ and $\|(T_n-T)P_n(\epsilon)\|_\infty<\epsilon$ for large n. By Lemma 2.2, T belongs to $L_p(\Gamma)$ and $\|T-T_n\|_p \to 0$.

COROLLARY 2.1. *If* $T_n \to T$ *in* $L_p(\Gamma)$ *there exists a subsequence* $\{T_{n_j}\}$ *and a strongly dense domain* $\{\mathcal{H}_k\}$ *such that*

(a) $\|T^{(k)}-T_{n_j}^{(k)}\|_\infty \to 0$ *as* $j\to\infty$ *where for an arbitrary operator* S, $S^{(k)}$ *denotes the restriction of* S *to* $\mathcal{H}_k$,

(b) $T_{n_j} \to T$ *n.e.*

The above proof shows this when the T_n are self adjoint. For the general case write $T_n=R_n+iS_n$ with R_n, S_n self adjoint; and if $\{\mathfrak{M}_k\}$, $\{\mathfrak{N}_k\}$ are the strongly dense domains for which (a) is satisfied by R_{n_j}, S_{n_j} respectively let $\mathcal{H}_k=\mathfrak{M}_k \cap \mathfrak{N}_k$.

Because of the completeness just established we can state the following standard result:

COROLLARY 2.2. *Let* $\Gamma_1=(\mathcal{H}_1, \mathfrak{A}_1, m_1)$ *and* $\Gamma_2=(\mathcal{H}_2, \mathfrak{A}_2, m_2)$ *be regular gage spaces. Suppose* $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, *and that* ϕ *is a bounded linear transformation from a dense subset of* $L_p(\Gamma_1)$ *to* $L_q(\Gamma_2)$. *Then* ϕ *can be uniquely extended to a linear transformation from all of* $L_p(\Gamma_1)$ *to* $L_q(\Gamma_2)$ *with preservation of the bound.*

4. **Interpolation.** Zygmund and Calderon have extended the Riesz-Thorin interpolation theorem to a rather general theorem concerning interpolation of multi-linear operations, [1, Theorem F]. For the sake of simplicity and clarity, we will restrict our attention to the bilinear case. Before stating the

theorem we introduce some notation. Let $1 \leqq p_1, p_2, q_1, q_2, r_1, r_2 \leqq \infty$ and set $1/\infty = 0$ and $1/0 = \infty$. Suppose that t is a real number such that $0 < t < 1$ and define p, q and r by the equations

$$1/p = (1 - t)/p_1 + t/p_2,$$
$$1/q = (1 - t)/q_1 + t/q_2,$$
$$1/r = (1 - t)/r_1 + t/r_2.$$

THEOREM 3. *Let* Γ, Γ_1, Γ_2 *be regular gage spaces and let* $\mathbf{D}$ *be the collection of all pairs* (R, S) *of elementary operators on* Γ_1, Γ_2 *respectively. Suppose that* ϕ *is a bilinear mapping from* $\mathbf{D}$ *to measurable operators on* Γ *such that for all* (R, S) *in* $\mathbf{D}$,

(a) $\|\phi(R, S)\|_{r_1} \leqq M_1 \|R\|_{p_1} \|S\|_{q_1}$,
(b) $\|\phi(R, S)\|_{r_2} \leqq M_2 \|R\|_{p_2} \|S\|_{q_2}$.

Then

(c) $\|\phi(R, S)\|_r \leqq M_1^{1-t} M_2^t \|R\|_p \|S\|_q$

for all (R, S) *in* $\mathbf{D}$. *Furthermore if* $p \neq \infty$ *and* $q \neq \infty$, ϕ *has a unique continuous bilinear extension to* $L_p(\Gamma_1) \times L_q(\Gamma_2)$, *preserving* (c).

Proof of the theorem. D could be replaced by the collection of all pairs of simple operators. If this were done a direct proof of the theorem could be presented, analogous to the one given by Zygmund and Calderon; however, it is a relatively straightforward matter to reduce the proof to the commutative case, and we follow this latter course.

Let $T = \phi(R, S)$ and let $R = U|R|$ and $S = V|S|$ be the polar decomposition R and S. Suppose that $|R| = \int \lambda dE_\lambda$ and that $|S| = \int \lambda dF_\lambda$. If f, g are bounded measurable functions on the line, put $R_f = U \int f(\lambda) dE_\lambda$ and $S_g = V \int g(\lambda) dF_\lambda$. Now set $T(f, g) = \phi(R_f, S_g)$ and let X be a fixed elementary operator on Γ. Let $W|X| = X$ be the polar decomposition of X and let $\int \lambda dK_\lambda$ be the spectral resolution of $|X|$. For each bounded measurable function h on the line put $X_h = W \int h(\lambda) dK_\lambda$. Now suppose that $TX = W_1|TX|$ is the polar decomposition of TX and set $\phi(f, g, h) = m(W_1^* T(f, g) X_h)$. Then ϕ is a tri-linear form and

$$|\phi(f, g, h)| \leqq \|W_1^* T(f, g) X_h\|_1 \leqq \|T(f, g) X_h\|_1 \leqq \|T(f, g)\|_{r_i} \|X_h\|_{r'_i}$$

and by assumption, $\|T(f, g)\|_{r_i} \leqq M_i \|R_f\|_{p_i} \|S_g\|_{q_i}$. Putting $\|(f, E)\|_p = (\int |f(\lambda)|^p dm(E_\lambda))^{1/p}$ and defining $\|(g, F)\|_q$ and $\|(h, K)\|_r$ similarly for any p, q, and r such that $1 \leqq p, q, r \leqq \infty$, it results that $|\phi(f, g, h)| \leqq M_i \|(f, E)\|_{p_i} \cdot \|(g, F)\|_{q_i} \|(h, K)\|_{r'_i}$. In this inequality we make the further convention that $\|(f, E)\|_\infty = \|f\|_\infty$, $\|(g, F)\|_\infty = \|g\|_\infty$, and that $\|(h, K)\|_\infty = \|h\|_\infty$. Now since $1/r = (1-t)/r_1 + t/r_2$ it follows that $1/r' = (1-t)/r_1' + t/r_2'$. For $1/r' = 1 - 1/r = 1 + (1-t)(1/r_1' - 1) + t(1/r_2' - 1)$. Hence by Theorem F of [1] specialized to the case of trilinear forms the inequality $|\phi(f, g, h)| \leqq M_1^{1-t} M_2^t \|(f, E)\|_p \cdot \|(g, F)\|_q \|(h, K)\|_{r'}$ holds for all simple functions f, g and h. Because the measure spaces over which the norms $\|(f, E)\|_p, \cdots$, are taken are finite, the simple functions are dense with respect to these norms, even in the case that p, q or $r'=\infty$. Thus the inequality is valid for any bounded measurable functions f, g and h. As in the proof of Lemma 1.1, we may choose f, g, and h so that $R_f=R$, $\|(f, E)\|_p=\|R\|_p, \cdots$, and $T=T(f, g)$. Then $\phi(f, g, h) = m(W_1^*TX)$ and $|\phi(f, g, h)| = m(W_1^*TX) = \|TX\|_1 \leqq M_1^{1-t}M_2^t\|R\|_p\|S\|_q\|X\|_{r'}$. Applying part (c) of Theorem 1, we see that $\|T\|_r = \|\phi(R, S)\|_r \leqq M_1^{1-t}M_2^t\|R\|_p \cdot\|S\|_q$. If $p\neq\infty$ and $q\neq\infty$ and R, S are arbitrary operators in $L_p(\Gamma_1)$, $L_q(\Gamma_2)$ there exist elementary operators R_n, S_n on Γ_1, Γ_2 such that $R_n\to R$ in $L_p(\Gamma_1)$ and $S_n\to S$ in $L_q(\Gamma_2)$. Putting $T_n=\phi(R_n, S_n)$ it results that $\|T_n-T_k\|_r \leqq M_1^{1-t}M_2^t(\|R_n-R_k\|_p\|S\|_q+\|R_k\|_p\|S_n-S_k\|_q)\to 0$ as $n, k\to\infty$. Since $L_r(\Gamma)$ is complete the sequence $\{T_n\}$ has a limit, say T, and $\|T\|_r \leqq \|T-T_k\|_r+\|T_k\|_r \leqq \|T-T_k\|_r+M_1^{1-t}M_2^t\|R_k\|_p\|S_k\|_q$. There is no difficulty in verifying that $\|T-T_k\|_r\to 0$ as $k\to\infty$; so $\|T\|_r\leqq M_1^{1-t}M_2^t\|R\|_p\|S\|_q$. Thus if $R_n\to 0$ in $L_p(\Gamma_1)$ and $S_n\to 0$ in $L_q(\Gamma_2)$, $\phi(R_n, S_n)\to 0$ in $L_r(\Gamma)$. From this we see that the definition $T=\phi(R, S)$ is unambiguous and gives the desired extension of ϕ to $L_p(\Gamma_1)\times L_q(\Gamma_2)$.

A particularly interesting and easy consequence of the theorem is given in the following result.

COROLLARY 3.1. *Suppose ϕ is a linear mapping from elementary operators on a regular gage space Γ_1 to measurable operators on a regular gage space Γ_2 such that for all elementary operators R on Γ_1,*

(a) $\|\phi(R)\|_{q_1}\leqq M_1\|R\|_{p_1}$,

(b) $\|\phi(R)\|_{q_2}\leqq M_2\|R\|_{p_2}$.

Then

(c) $\|\phi(R)\|_q\leqq M_1^{1-t}M_2^t\|R\|_p$

for all elementary operators R on Γ_1. If $p\neq\infty$ then ϕ has a unique continuous linear extension to $L_p(\Gamma_1)$, preserving (c).

COROLLARY 3.2. *If T belongs to $L_p(\Gamma)$ and S belongs to $L_q(\Gamma)$ where $1/p+1/q \leqq 1$ then TS is in $L_r(\Gamma)$ where $1/r=1/p+1/q$ and $\|TS\|_r\leqq\|T\|_p\|S\|_q$.*

Let R_S be the operator defined for all measurable T by $R_S(T)=TS$. Then for T in $L_{q'}(\Gamma)$ we have $\|R_S(T)\|_1\leqq\|S\|_q\|T\|_{q'}$ and for T in $L_\infty(\Gamma)$, $\|R_S(T)\|_q \leqq\|S\|_q\|T\|_\infty$. Now if $0<t<1$, $t(1/q', 1)+(1-t)(0, 1/q)=(t/q', t+(1-t)/q)$ and putting $1/q=t/q'$ and $1/r=t+(1-t)/q$ it follows that $\|R_S(T)\|_r\leqq\|S\|_q \cdot\|T\|_p$ when T is restricted to be an elementary operator. Note that, $1/p+1/q =t/q'+1/q\leqq 1/q'+1/q=1$ and that $1/p+1/q=t(1/q'+1/q)-t/q+1/q =t+1-t/q=1/r$. Since we have already considered the case $p=\infty$ we may assume $p\neq\infty$, so that R_S has a continuous extension to all of $L_p(\Gamma)$; thus we need only verify that $R_S(T)=TS$ for arbitrary T in $L_p(\Gamma)$. Let $\{T_n\}$ be a sequence of elementary operators such that $\|T_n-T\|_p\to 0$ and $T_n\to T$ n.e. Then $\|R_S(T_n)-R_S(T_k)\|_r\to 0$ so that $\{T_nS\}$ is a Cauchy sequence in $L_r(\Gamma)$. There exists therefore an operator K in $L_r(\Gamma)$ such that $\|T_nS-K\|_r\to 0$. By passing to a subsequence we can assume that $T_nS \to K$ n.e. Since $T_nS \to TS$ n.e., $K = TS$, and $\|TS\|_r = \|R_S(T)\|_r \leqq \|S\|_q\|T\|_p = \|T\|_p\|S\|_q$.

5. **The Hausdorff-Young theorem.** In this section we apply the previous theorem to a semi-commutative situation in which one gage space is commutative being the measure space obtained from the group G and Haar measure; the other gage space is, in general, noncommutative, and arises from the Hilbert algebra determined by the square integrable functions on the group.

For the remainder of the paper G is a locally compact unimodular group; all integration over G is with respect to Haar measure. $L_p(G)$, $1 \leqq p < \infty$, is the collection of measurable complex valued functions f on G with $\|f\|_p = (\int |f(x)|^p dx)^{1/p} < \infty$. The collection of measurable, essentially bounded functions is designated by $L_\infty(G)$, and if f is a measurable function $Jf = f^*$ is the function defined by $f^*(x) = \bar{f}(x^{-1})$. The convolution $f*g$ of two measurable functions f, g is said to exist if $\int |f(xy^{-1})g(y)| dy$ exists for almost all x and is defined by $f*g(x) = \int (xy^{-1})g(y)dy$. Because G is unimodular the transformation $y \to y^{-1}x$ is measure preserving and it follows that $f*g$ exists if and only if $\int |f(y)g(y^{-1}x)| dy$ exists for almost all x and when it exists $f*g(x) = \int f(y)g(y^{-1}x)dy$. We will use these two expressions for $f*g$ interchangeably, as convenience dictates. When $f*g$ exists $(f*g)^*$ is easily seen to equal $g^* * f^*$. For a measurable function f, left convolution by f is the operator L_f in $L_2(G)$ whose domain consists of all g in $L_2(G)$ for which $f*g$ exists and $f*g$ belongs to $L_2(G)$; for such g, $L_f(g)$ is defined to be $f*g$. Right convolution by f is designated by R_f and is defined similarly.

Let $\mathfrak{B}_p$ denote the collection of all f in $L_p(G)$, $1 \leqq p \leqq 2$, such that L_f is a bounded everywhere defined operator on $L_2(G)$.

THEOREM 4. *If f is an element of $\mathfrak{B}_p$ then $(L_f)^* = L_{f^*}$, $R_f = JL_f{}^* J$ and $(R_f)^* = R_{f^*}$. Furthermore if g is in $\mathfrak{B}_q$ and $f*g$ belongs to $L_r(G)$ where $1 \leqq p, q, r \leqq 2$ then $L_{f*g} = L_fL_g$. In particular, $\mathfrak{B}_2$ is an algebra under convolution.*

In the proof we will establish some lemmas that are more general than necessary but which will be useful later on.

LEMMA 4.1. *If f is an element of $L_p(G)$, $1 \leqq p \leqq 2$, then L_f is a densely defined operator in $L_2(G)$ and the domain of L_f contains $L_1(G) \cap L_2(G)$.*

Suppose f is an element of $L_p(G)$, that g is in $L_q(G)$, and that $1/p + 1/q \geqq 1$. Then by Young's inequality $f*g$ exists and $\|f*g\|_r \leqq \|f\|_p\|g\|_q$ where $1/r = 1/p + 1/q - 1$. When $1 \leqq p \leqq 2$ there is a q such that $1 \leqq q \leqq 2$ and $r = 2$. Now $L_1(G) \cap L_2(G)$ is dense in $L_2(G)$ and contained in $L_q(G)$ for $1 \leqq q \leqq 2$.

LEMMA 4.2. *If f is an element of $L_p(G)$, $1 \leqq p \leqq 2$, and if $\mathfrak{D}$ is any subset of $L_1(G) \cap L_2(G)$, then $(f^* * g, h) = \int g(y)[f*h(y)]^- dy$ for all g in $\mathfrak{D}$ and all h in $L_2(G)$.*

First of all observe by the preceding lemma, that the left side of the above relation makes sense. Now applying the same lemma we see that

$f^* * g$ belongs to $L_2(G)$ whenever g is in $\mathfrak{D}$, and because the product of two functions in $L_2(G)$ is integrable, $\int |h(x)| dx \int |f^*(xy^{-1})g(y)| dy$ is finite. Hence the Fubini theorem applies and

$$\int \bar{h}(x)dx \int f^*(xy^{-1})g(y)dy = \int g(y)dy \int f^*(xy^{-1})\bar{h}(x)dx = \int g(y)[f * h(y)]^- dy.$$

The next lemma is a slight generalization of a result of Segal's [3].

LEMMA 4.3. *Suppose $f = f_1 + f_2 + \cdots + f_n$ where f_i belongs to $L_{p_i}(G)$, $1 \leqq p_i \leqq 2$, and let $\mathfrak{D} = L_1(G) \cap L_2(G)$. Then the domain of L_{f^*} contains $\mathfrak{D}$ and if L''_{f^*} is the contraction of L_{f^*} to $\mathfrak{D}$, $(L''_{f^*})^* = L_f$.*

By Lemma 4.1, the domain of $L_{f_i^*}$ contains $\mathfrak{D}$ for each i. It follows easily that $L''_{f^*} = L''_{f_1^*} + \cdots + L''_{f_n^*}$ where the sum is taken in the ordinary sense. Furthermore for any h in $L_2(G)$, $(f^* * g, h) = (f_1^* * g, h) + \cdots + (f_n^* * g, h)$ which by Lemma 4.2, equals $\int g(y)[f_1 * h(y)]^- dy + \cdots + \int h(y)[f_n * h(y)]^- dy = \int g(y)[f * h(y)]^- dy$. Thus $(f^* * g, h) = \int g(y)[f * h(y)]dy$ for all g in $\mathfrak{D}$ and all h in $L_2(G)$. Now let h be an arbitrary element of the domain of L_f. Then $f * h$ belongs to $L_2(G)$ and the above relation becomes $(f^* * g, h) = (g, f * h)$. Hence $L_f \subset (L''_{f^*})^*$. To show the converse let h belong to the domain of $(L''_{f^*})^*$ and put $(L''_{f^*})^*(h) = k$. Then $(f^* * g, h) = (g, k)$ for all g in $\mathfrak{D}$. Thus $\int (f * h)\bar{g} = \int k\bar{g}$ for all g in $\mathfrak{D}$ and it results that $f * h = k$ almost everywhere. So h is in the domain of L_f and $L_f(h) = k$, which shows that $(L''_{f^*})^* \subset L_f$. Notice that the Lemma is true without any change in the proof if for $\mathfrak{D}$ we take the collection of simple functions. More generally, we can let $\mathfrak{D}$ be any subset of $L_1(G) \cap L_2(G)$ which is dense in $L_2(G)$ with the property that $(f^* * g, h) = (g, k)$ for all g in D implies $(f^* * C_E, h) = (C_E, k)$ for each set E of finite measure, where C_E is the characteristic function of E.

LEMMA 4.4. *Suppose f, g and $f * g$ are elements of $L_p(G)$, $L_q(G)$, and $L_r(G)$ respectively where $1 \leqq p, q, r \leqq 2$. Let $\mathfrak{D}$ be the collection of simple functions. Then*

(a) $L''_{(f*g)^*} \subset L_{g^*} L_{f^*}$,

(b) $L_{f*g} \supset (L_{f^*})^* (L_{g^*})^*$.

Let h be a simple function. Since h belongs to all Lebesgue classes it follows from Young's inequality that $g * h$ is in $L_{p'}(G)$. Hence the integral $\int |f(y)| dy \int |g(y^{-1}xz^{-1})h(z)| dz$ is finite, and

$$\int f(y)dy \int g(y^{-1}xz^{-1})h(z)dz = \int h(z)dz \int f(y)g(y^{-1}xz^{-1})dy.$$

As a result, $f * (g * h) = (f * g) * h$ and by Lemma 4.1, $g * h$, $(f * g) * h$ are in $L_2(G)$. So the above relation may be written as

$$L_f L_g(h) = L_{f*g}(h).$$

This shows that $L''_{f*g} \subset L_f L_g$ and replacing f by g^* and g by f^* we get $L''_{g^* * f^*}$

$\subset L_g*L_{f^*}$. Now part (a) follows from that fact that $g^**f^*=(f*g)^*$. The inclusion just established shows that $(L_g\cdot L_{f^*})^*$ exists and that $L_{f*g}=(L''_{(f*g)^*})^* \supset (L_g\cdot L_{f^*})^* \supset (L_{f^*})^*(L_{g^*})^*$, which proves (b).

LEMMA 4.5. *If f is in $L_p(G)$, $1\leqq p\leqq 2$, and if the contraction L'_f of L_f to any dense subset $\mathfrak{D}$ of $L_2(G)$ contained in the domain of L_f is bounded then L_f is bounded and everywhere defined.*

Let g be an arbitrary element of $L_2(G)$ and suppose g_i belongs to $\mathfrak{D}$ and that $g_i\to g$ in $L_2(G)$. Then $\|f*g-f*g_i\|_r\leqq\|f\|_p\|g-g_i\|_2\to 0$, and by passing to a subsequence we can assume $f*g_i\to f*g$ almost everywhere. By Fatou's Lemma $\int|f*g(x)|^2dx\leqq\lim\inf_i\int|f*g_i(x)|^2dx\leqq\lim\inf_i\|L'_f\|^2_\infty\|g_i\|^2_2=\|L'_f\|^2_\infty\|g\|^2_2$. It follows that $f*g$ is in $L_2(G)$ and that L_f is bounded.

Proof of the theorem. Suppose f belongs to $\mathfrak{B}_p$, $1\leqq p\leqq 2$, and let $\mathfrak{D}$ be the collection of simple functions. From the inclusion $L''_{f^*}\subset L_{f^*}$ follows the fact that $(L''_{f^*})^*\subset(L_{g^*})^*$; by Lemma 4.3 $(L_f)=(L''_{f^*})^*\supset(L_{f^*})^*$ and applying the same lemma to f^* we see that L_{f^*} is closed, linear and densely defined. Thus $(L_f)^*=(L''_{f^*})^{**}\subset(L_{f^*})^{**}=L_{f^*}$. It is straightforward to verify that $R_f=JL_f^*J$ so that for g, h in $L_2(G)$, $(R_fg, h)=(JL_f{}^*Jg, J^2h)=(Jh, L_f{}^*Jg)=(f*h^*, g^*)=(g, h*f^*)=(g, R_{f^*}h)$; hence $(R_f)^*=R_{f^*}$. Now suppose that g is an element of $\mathfrak{B}_q$ and that $f*g$ belongs to $L_r(G)$ where $1\leqq q$, $r\leqq 2$. By Lemma 4.4, and the preceding results on adjoints it follows that $L_{f*g}\supset(L_{f^*})^*(L_{g^*})^*=L_fL_g$. Since L_fL_g is bounded and everywhere defined $L_{f*g}=L_fL_g$. Finally, if f, g belong to $\mathfrak{B}_2$ then $f*g=L_f(g)$ is in $L_2(G)$ and the result just established applies; hence $L_{f*g}=L_fL_g$ is bounded and $f*g$ belongs to $\mathfrak{B}_2$.

Having established Theorem 4, it is easy to verify that the system $\boldsymbol{H}=(L_2(G), J, \mathfrak{B}_2)$ is a Hilbert algebra in the sense of [4] and that for f in $\mathfrak{B}_2$, L_f as defined here coincides with the $\boldsymbol{H}$ definition of L_f. Now for arbitrary f in $L_2(G)$ put L'_f for the operator with domain $\mathfrak{B}_2$ given by $L'_f(g)=R_g(f)$, g in $\mathfrak{B}_2$. Since $R_g(f)=f*g=L_f(g)$, L'_f is just the restriction of L_f to $\mathfrak{B}_2$, and according to the definition, f is in the bounded algebra of $\boldsymbol{H}$ if L'_f is bounded. Because $\mathfrak{B}_2$ is dense in $L_2(G)$, it follows from Lemma 4.5 that L_f is bounded if and only if L'_f is bounded. Thus we have proved

COROLLARY 4.1. *$\mathfrak{B}_2$ is already the bounded algebra of the Hilbert algebra $\boldsymbol{H}$.*

THEOREM 5. *If f is an element of $L_2(G)$, then $(L'_{f^*})^*=L_f$.*

LEMMA 5.1. *If two essentially measurable operators agree on a strongly dense domain they have identical closures.*

For the proof see [4, Corollary 5.1].

Proof of the theorem. Let $\mathfrak{D}$ be any dense subset of $L_2(G)$ contained in $L_1(G)$. Then $\mathfrak{D}$ is contained in $\mathfrak{B}_2$ and $L''_{f^*}\subset L'_{f^*}$. By Lemma 4.3, $(L'_{f^*})^*\subset L_f$. Now let $\mathfrak{L}$ be the weakly closed ring of operators generated by the L_g for g in B_2. $\mathfrak{L}$ is called the left ring of $\boldsymbol{H}$ and by Theorem 19 and Remark 5.1 of [4] it follows that $(L'_{f^*})^*$ is measurable with respect to $\mathfrak{L}$. $\mathfrak{L}$ is the commutor of the ring generated by the right translations by elements of G; so since L_f commutes with all right translations, L_f is affiliated with $\mathfrak{L}$. Because $L_f \supset (L'_{f^*})^*$, L_f has a strongly dense domain and being closed L_f is therefore measurable. Thus by Lemma 5.1, $(L'_{f^*})^* = L_f$.

By Theorem 17 of [4] a regular gage m can be defined on the projections in $\mathfrak{L}$ as follows: For any projection Q in $\mathfrak{L}$ set $m(Q) = \|f\|_2^2$ if $Q = L_f$ for some f in B_2 and otherwise put $m(Q) = \infty$. The resulting regular gage space $G' = (L_2(G), \mathfrak{L}, m)$ will be called the canonical gage space of G.

According to [4, Corollary 19.1] $(L'_{f^*})^*$ belongs to $L_2(G')$ for each f in $L_2(G)$ and the mapping $f \to (L'_{f^*})^*$ is unitary between $L_2(G)$ and $L_2(G')$. Since $(L'_{f^*})^* = L_f$ it follows that the mapping $f \to L_f$ is unitary between $L_2(G)$ and $L_2(G')$.

From now on, $\mathfrak{D}$ will denote $L_1(G) \cap L_2(G)$.

Theorem 6. *Let G be a locally compact unimodular group and let $G' = (L_2(G), \mathfrak{L}, m)$ be the canonical gage space of G. Suppose that $\mathfrak{M}$ is the smallest linear collection of measurable functions containing $L_p(G)$ for each p with $1 \leqq p \leqq 2$. Then for f in $\mathfrak{M}$*

(1) *L_f is a measurable operator with respect to $\mathfrak{L}$ and the domain of L_f contains $\mathfrak{D}$;*

(2) *if g belongs to $\mathfrak{M}$ and if a is a scalar, then $L_{af+g} = aL_f + L_g$, the sum being taken in the strong sense;*

(3) *$L_f = L_g$, g in $\mathfrak{M}$ implies $f = g$ almost everywhere;*

(4) *$(L_f)^* = L_{f^*}$;*

(5) *$\|L_f\|_{p'} \leqq \|f\|_p$, $1 \leqq p \leqq 2$.*

Lemma 6.1. *If $1 \leqq p \leqq 2$, there is a bounded linear mapping $\boldsymbol{U}$ from $L_p(G)$ to $L_{p'}(G')$ such that*

(1) *$\|\boldsymbol{U}(f)\|_{p'} \leqq \|f\|_p$, for all f in $L_p(G)$;*

(2) *if f is a simple function then $\boldsymbol{U}(f) = L_f$.*

Let f be a simple function. Then L_f is a measurable operator with respect to the ring $\mathfrak{L}$, $\|L_f\|_2 = \|f\|_2$ and $\|L_f\|_\infty \leqq \|f\|_1$. Setting $\boldsymbol{U}(f) = L_f$ we have $\|\boldsymbol{U}(f)\|_\infty \leqq \|f\|_1$ and $\|\boldsymbol{U}(f)\|_2 = \|f\|_2$. Now by Corollary 3.1, $\boldsymbol{U}f$ belongs to $L_{p'}(G')$ and $\|\boldsymbol{U}(f)\|_{p'} \leqq \|f\|_p$. From this inequality, the denseness of the simple functions in $L_p(G)$, and Corollary 2.2, follows the fact that $\boldsymbol{U}$ has an extension to all of $L_p(G)$ having the stated properties.

Lemma 6.2. *If f is an element of $L_p(G)$, $1 \leqq p \leqq 2$, then $\boldsymbol{U}(f) = L_f$.*

Let f belong to $L_p(G)$, $1 \leqq p \leqq 2$, and let f_n be simple functions such that $f_n \to f$ in $L_p(G)$. Let $T = \boldsymbol{U}(f)$. By Corollary 2.1, we can assume that $L_{f_n} \to T$ n.e. and that there exists a strongly dense domain $\{\mathcal{K}_i\}$ such that the restriction $T^{(i)}$ of T to $\mathcal{K}_i$ is bounded and $\|T^{(i)} - L_{f_n}\|_\infty \to 0$, for each fixed i. Let g be an arbitrary element of $\mathcal{K}_i$. By Young's inequality $f*g$ is in L_r where $1/r = 1/p - 1/2$ and $\|f*g - f_n*g\|_r \leqq \|f - f_n\|_p \|g\|_2 \to 0$. Thus $f_n*g \to T_{(g)}$ in $L_2(G)$ and

$f_n*g \to f*g$ in $L_r(G)$. Therefore $T_{(g)}=f*g$ almost everywhere. Thus L_f agrees with T on a strongly dense domain, is closed (Lemma 4.3) and is easily seen to commute with right translations by group elements. Hence $L_f=T$. For L_f and T are essentially measurable and agree on a strongly dense domain, and therefore by Lemma 5.1, have identical closures.

LEMMA 6.3. *If f belongs to $L_p(G)$, $1 \leqq p \leqq 2$, then $(L_f)^* = L_{f^*}$.*

First of all, $L''_{f^*} \subset L_{f^*}$ so that $(L''_{f^*})^* \supset (L_{f^*})^*$. By Lemma 4.3 $L_f=(L''_{f^*})^*$; thus $L_f \supset (L_{f^*})^*$. Now by Lemma 6.2, L_{f^*} is a measurable operator with respect to the ring $\mathfrak{L}$. And as the adjoint of a measurable operator is measurable, $(L_{f^*})^*$ is measurable. Since $L_f \subset (L_{f'})^*$, it follows, by Lemma 5.1, that $(L_{f'})^* = L_f$. Upon replacing f by f^*, we get the indicated result.

Before proceeding we mention that unless stated to the contrary, all sums and products of measurable operators are understood to be strong sums and strong products respectively.

LEMMA 6.4. *If $f=f_1+f_2+\cdots+f_n$, f_i in $L_{p_i}(G)$, $1 \leqq p_i \leqq 2$, then L_f is measurable and $L_f=L_{f_1}+\cdots+L_{f_n}$.*

By Lemma 4.3, L_f is closed, linear and densely defined. Furthermore by Lemma 6.2, L_{f_i} is measurable for each i, and there is no difficulty in showing that $L_f \supset L_{f_1}+\cdots+L_{f_n}$. Since L_f is affiliated with L we can apply Lemma 5.1 to conclude that L_f is measurable and that $L_f=L_{f_1}+\cdots+L_{f_n}$.

Proof of the theorem. If f is in $\mathfrak{M}$ then $f=f_1+f_2+\cdots+f_n$ where f_i belongs to $L_{p_i}(G)$ and $1 \leqq p_i \leqq 2$. So (1) follows from the preceding lemma and (2) follows similarly. Now suppose that f, g belong to $\mathfrak{M}$ and that $L_f=L_g$. Then if h is a simple function $f*h=g*h$ almost everywhere. From the definition of $\mathfrak{M}$ and from the fact that h belongs to all Lebesgue classes it follows that $f*h$ and $g*h$ are sums of continuous everywhere defined functions. Hence by the regularity of Haar measure, $f*h=g*h$ and evaluating at the identity we get $\int f\bar{h}^* = \int g\bar{h}^*$ for each simple function h. So $f=g$ almost everywhere, and this establishes (3). To prove (4), let $L_f=L_{f_1}+\cdots+L_{f_n}$ as above. Then $(L_f)^*=(L_{f_1})^*+\cdots+(L_{f_n})^*$ which by Lemma 6.3 equals $L_{f_1^*}+\cdots+L_{f_n^*}$. Since $f^*=f_1^*+\cdots+f_n^*$ and since f_i^* belongs to $L_{p_i}(G)$, $1 \leqq p_i \leqq 2$, we can apply Lemma 6.4 to conclude that $L_{f^*}=L_{f_1^*}+\cdots+L_{f^*}$. Thus $(L_f)^*=L_{f^*}$. If f is in $\mathfrak{M}$ and if f does not belong to $L_p(G)$, then (5) is trivially true. On the other hand if f is an element of $L_p(G)$, then the inequality follows from Lemma 6.1 and Lemma 6.2.

DEFINITION 5.1. Let $\boldsymbol{U}$ be the linear mapping with domain $\mathfrak{M}$ given for f in $\mathfrak{M}$ by $\boldsymbol{U}(f)=L_f$. $\boldsymbol{U}(f)$ is called the Fourier transform of f and the restriction of $\boldsymbol{U}$ to $L_p(G)$ will be denoted by $\boldsymbol{U}_p$.

COROLLARY 6.1. *If f, g, $f*g$ belong to $L_p(G)$, $L_q(G)$, and $L_r(G)$ respectively, $1 \leqq p,\ q,\ r \leqq 2$, then $L_{f*g}=L_fL_g$.*

L_{f*g}, L_f and L_g are measurable and it follows easily from Lemma 4.4 that $L_{f*g} \supset L_f L_g$; hence by Lemma 5.1, $L_{f*g} = L_f L_g$.

COROLLARY 6.2. *If f belongs to $\mathfrak{M}$ and if g belongs to $\mathfrak{D}$, then $U_2(f*g) = L_f L_g$ and $U_2(g*f) = L_g L_f$.*

Since f is in $\mathfrak{M}$, g is in the domain of L_f so that $f*g$ belongs to $L_2(G)$. Let $f = f_1 + \cdots + f_n$ where f_i is in $L_{p_i}(G)$, $1 \leqq p_i \leqq 2$. Then $f*g = f_1*g + \cdots + f_n*g$ and by the preceding corollary, $U_2(f_i*g) = L_{f_i} L_g$; thus

$$U_2(f*g) = (L_{f_1} + \cdots + L_{f_n}) L_g$$

and by Lemma 6.4, $L_{f_1} + \cdots + L_{f_n} = L_f$. A similar argument shows that $U_2(g*f) = L_g L_f$.

An inverse Fourier transform V mapping certain operators to functions also exists and we begin consideration of this question.

DEFINITION 5.2. Let $\mathfrak{D}' = L_1(G') \cap L_2(G')$ and let V'' be the restriction of U^{-1} to $\mathfrak{D}'$. Note that U^{-1} exists, by (3) of Theorem 6, and, as U_2 is unitary between $L_2(G)$ and $L_2(G')$, that $\mathfrak{D}'$ is included in the range of U.

COROLLARY 6.3. *If F is in $\mathfrak{D}'$ then $\|V''(F)\|_{p'} \leqq \|F\|_p$, $1 \leqq p \leqq 2$.*

First of all observe that F belongs to $L_p(G')$ for $1 \leqq p \leqq 2$. For if F belongs to $L_r(G') \cap L_s(G')$, $r \leqq s$, there is no difficulty in verifying that $\|F\|_u^u \leqq \|F\|_r^r + \|F\|_s^s$ for $r \leqq u \leqq s$. Now taking g to be a simple function, it results that $\int f\bar{g} = m(L_f L_g^*)$. For f, g are in $L_2(G)$ and $(f, g) = (Uf, Ug)$. Since $L_f = F$ and as F belongs to $L_p(G)$, $|\int f\bar{g}| \leqq \|F\|_p \|L_g^*\|_{p'}$. By Theorem 6, $\|L_g\|_{p'} \leqq \|L_g\|_p$ and as $\|L_g^*\|_{p'} = \|L_g\|_{p'}$ it follows that $|\int f\bar{g}| \leqq \|F\|_p \|L_g\|_{p'}$ for all simple functions g; hence $\|f\|_{p'} \leqq \|F\|_p$.

THEOREM 7. *Let G' be the canonical gage space determined by the locally compact unimodular group G. Put $\mathfrak{M}'$ for the smallest linear collection of measurable operators on G' containing $L_p(G')$ for each p with $1 \leqq p \leqq 2$. Then there exists a unique linear mapping V from $\mathfrak{M}'$ to measurable functions* (mod *null functions*) *on G such that*

(1) V extends V'';

(2) $\|V(F)\|_{p'} \leqq \|F\|_p$, $1 \leqq p \leqq 2$.

V necessarily has the further properties,

(3) $V(F^*) = V(F)^*$;

(4) $V(F) = 0$ implies $F = 0$.

LEMMA 7.1. *V'' has a unique extension V' to the set theoretic union of the $L_p(G)$, $1 \leqq p \leqq 2$, such that*

(1) $\|V'(F)\|_{p'} \leqq \|F\|_p$;

(2) $V'(F^*) = V'(F)^*$.

Since $\mathfrak{D}'$ is dense in $L_p(G)$, and in view of the inequality given in Corollary 6.3, V'' has a unique bounded norm decreasing extension V_p taking

$L_p(G')$ into $L_{p'}(G)$. By an easy continuity argument, $V_p(F^*)=V_p(F)^*$. We now define V' as follows; if F belongs to $L_p(G')$ for some p such that $1\leqq p\leqq 2$, put $V'(F)=V_p(F)$. To show that V' is well defined we must show that if F also belongs to $L_q(G')$, $1\leqq q\leqq 2$, then $V_p(F)=V_q(F)$. Suppose then that this is the case. By Lemma 1.4 there exist elementary operators F_n such that $F_n\to F$ in $L_p(G')$ and $F_n\to F$ in $L_q(G')$. Since F_n belongs to $\mathfrak{D}'$, $V''(F_n)\to V_p(F)$ in $L_{p'}(G)$ and $V''(F_n)\to V_q(F)$ in $L_{q'}(G)$. Thus $V_p(F)=V_q(F)$ almost everywhere. Now for a given F and index p either F belongs to $L_p(G')$ in which case (1) is satisfied or $\|F\|_p=\infty$ and again (1) holds. Also $V'(F^*)=V_p(F^*)=V_p(F)^*=V'(F)^*$ for some p so that V' exists. V' is unique. For if W extends V'' and satisfies (1) then the restriction of W to $L_p(G')$ is a continuous extension of V'' and must therefore equal V_p.

Having the transformation V' it is easy to establish a Parseval formula.

LEMMA 7.2. *If f belongs to $L_p(G)$ and H is in $L_p(G')$, $1\leqq p\leqq 2$, then* $\int f[V'(H)]^-=m(L_fH^*)$.

Let f_i be simple functions such that $f_i\to f$ in $L_p(G)$ and choose H_i in $\mathfrak{D}$ such that $H_i\to H$ in $L_p(G')$. Then $(Uf_i, UV'H_i)=(Uf_i, H_i)$. As $V'(H_i)\to V'(H)$ in $L_{p'}(G)$ and $Uf_i\to L_f$ in $L_{p'}(G')$ it follows that $\int f[V'(H)]^-=\lim_i\int f_i[V'(H_i)]^-=\lim_i m(Uf_iH_i^*)$ which by Corollary 1.3 equals $m(L_fH^*)$.

LEMMA 7.3. *If F_i belongs to $L_{p_i}(G')$, $1\leqq p_i\leqq 2$, and if $F_1+F_2+\cdots+F_n=0$ then $V'(F_1)+V'(F_2)+\cdots+V'(F_n)=0$ almost everywhere.*

Let $f_i=V'(F_i)$ and put $f=f_1+\cdots+f_n$. Let h be a simple function. Then $\int f\bar{h}=\int f_1\bar{h}+\cdots+\int f_n\bar{h}$ and applying the Parseval formula we get $\int f\bar{h}=m(F_1L_h^*)+\cdots+m(F_nL_h^*)=m((F_1+\cdots+F_n)L_h^*)=m(0\cdot L_h^*)=0$. Since h was an arbitrary simple function, $f=0$ almost everywhere.

LEMMA 7.4. *If F belongs to $L_p(G')$, $1\leqq p\leqq 2$ and if g is an element of $L_1(G)$, then $V'(L_gF)=g*V'(F)$.*

Since g belongs to $L_1(G)$, L_g is bounded, and L_gF is in $L_p(G')$. Let F_n belong to $\mathfrak{D}'$ be such that $F_n\to F$ in $L_p(G')$. Then $F_n=L_{f_n}$ where f_n is in $L_2(G)$. Now $L_gF_n\to L_gF$ in $L_p(G')$ so that $V'(L_gF_n)\to V'(L_gF)$ in $L_{p'}(G)$. Since $g*f_n$ belongs to $L_2(G)$, Corollary 6.1 applies and $L_{g*f_n}=L_gL_{f_n}=L_fF_n$. Thus $g*f_n=V'(L_{g*f_n})\to V'(L_gF)$ in $L_{p'}(G)$. Let $f=V'(F)$. Then $\|g*f-g*f_n\|_{p'}\leqq\|g\|_1\cdot\|f-f_n\|_{p'}\to 0$. For $V'(F_n)\to V'(F)$ in $L_{p'}(G)$. Thus $g*f=V'(L_gF)$.

LEMMA 7.5. *If H, K belong to $L_2(G')$, then $V'(HK)=V'(H)*V'(K)$.*

Let $h=V'(H)$ and put $k=V'(K)$. There exist h_i in $\mathfrak{D}$ such that $h_i\to h$ in $L_2(G)$. Hence $L_{h_i}\to H$ in $L_2(G')$, $L_{h_i}K\to HK$ in $L_1(G')$, and $V'(L_{h_i}K)\to V'(HK)$ in $L_\infty(G)$. By the preceding Lemma, $V'(L_{h_i}K)=h_i*k$. Now $h_i*k\to h*k$ in $L_\infty(G)$. For $\|h*k-h_i*k\|_\infty\leqq\|h-h_i\|_2\|k\|_2\to 0$, and it results that $V'(HK)=h*k$ almost everywhere.

Proof of the theorem. Let F belong to $\mathfrak{M}'$. By definition there exist F_i in $L_{p'}(G')$, $1 \leqq p_i \leqq 2$ such that $F = F_1 + \cdots + F_n$. Put $\boldsymbol{V}(F) = \boldsymbol{V}'(F_1) + \cdots + \boldsymbol{V}'(F_n)$. By Lemma 7.3, $\boldsymbol{V}(F)$ is uniquely defined and the resulting map $\boldsymbol{V}$ is obviously linear on $\mathfrak{M}'$. If F is in $L_p(G')$, then $\boldsymbol{V}(F) = \boldsymbol{V}'(F)$ and (2) is satisfied and if F is in $\mathfrak{M}'$ but not in $L_p(G')$, then $\|F\|_p = \infty$ and (2) holds trivially. Thus (1) and (2) hold and from Lemma 7.1, $\boldsymbol{V}$ is easily seen to be unique. Since $F^* = F_1^* + \cdots + F_n^*$, $\boldsymbol{V}(F^*) = \boldsymbol{V}'(F_1)^* + \cdots + \boldsymbol{V}'(F_n)^* = \boldsymbol{V}(F)^*$ and we have established (3). Let g belong to $\mathfrak{D}$. Then $L_gF = L_gF_1 + \cdots + L_gF_n$ and $\boldsymbol{V}(L_gF) = \boldsymbol{V}'(L_gF_1) + \cdots + \boldsymbol{V}'(L_gF_n)$ and setting $f_i = \boldsymbol{V}'(F_i)$ and applying Lemma 7.4, we get $\boldsymbol{V}(L_gF) = g*f_1 + \cdots + g*f_n = g*(f_1 + \cdots + f_n) = g*\boldsymbol{V}(F)$; thus if $\boldsymbol{V}(F) = 0$, $\boldsymbol{V}(L_gF) = 0$ as well. Since L_g is an element of $L_{p'}(G')$ for all p such that $1 \leqq p \leqq 2$, it follows from Holder's inequality that L_gF_i belongs to $L_1(G')$. Consequently L_gF is in $L_1(G')$ and is therefore the product of two operators in $L_2(G')$. So suppose $L_gF = HK$ with H, K in $L_2(G')$, and that $\boldsymbol{V}(F) = 0$. Putting $h = \boldsymbol{V}(H)$ and $k = \boldsymbol{V}(K)$ it results that $0 = \boldsymbol{V}(L_gF) = \boldsymbol{V}(HK) = h*k$ (by Lemma 7.5); since h, k are elements of $L_2(G)$ and $h*k = 0$ we may apply Corollary 6.1. Thus $L_{h*k} = L_hL_k = HK = 0$. As a result $L_gF = 0$ for all g in $\mathfrak{D}$. Letting g run through an approximate identity we see that $F \doteq 0$.

During the course of the proof we have shown:

Corollary 7.1. *If F belongs to $\mathfrak{M}'$ and if g is an element of $L_1(G)$ then $\boldsymbol{V}(L_gF) = g*\boldsymbol{V}(F)$.*

Corollary 7.2. *$\boldsymbol{V}\boldsymbol{U}_2 = \boldsymbol{I}_2$ (the identity on $L_2(G)$) and $\boldsymbol{U}\boldsymbol{V}_2 = \boldsymbol{I}_{2'}$ (the identity on $L_2(G')$).*

$\boldsymbol{V}\boldsymbol{U}_2 = \boldsymbol{V}_2\boldsymbol{U}_2$ where $\boldsymbol{V}_2$ is the unique continuous extension of $\boldsymbol{V}''$ to $L_2(G')$. Since $\boldsymbol{U}_2^{-1}$ is such an extension, $\boldsymbol{V}_2 = \boldsymbol{U}_2^{-1}$ and $\boldsymbol{V}_2\boldsymbol{U}_2 = \boldsymbol{U}_2^{-1}\boldsymbol{U}_2 = \boldsymbol{I}_2$. On the other hand $\boldsymbol{U}\boldsymbol{V}_2 = \boldsymbol{U}\boldsymbol{U}_2^{-1} = \boldsymbol{U}_2\boldsymbol{U}_2^{-1} = \boldsymbol{I}_{2'}$.

Corollary 7.3. *If f is in $\mathfrak{M}$ and if L_f belongs to $\mathfrak{M}'$, then $\boldsymbol{V}(L_f) = f$.*

Let g belong to $\mathfrak{D}$. Then by Corollary 6.2, $\boldsymbol{U}(g*f) = \boldsymbol{U}_2(g*f) = L_gL_f$. So $g*f = \boldsymbol{V}\boldsymbol{U}_2(g*f)$ (Corollary 7.2) $= \boldsymbol{V}(L_gL_f) = g*\boldsymbol{V}(L_f)$ (Corollary 7.1). Thus $g*f = g*\boldsymbol{V}(L_f)$ for all g in $\mathfrak{D}$. So $f = \boldsymbol{V}(L_f)$ almost everywhere.

Corollary 7.4. *If F belongs to $\mathfrak{M}'$ and if $f = \boldsymbol{V}(F)$ belongs to $\mathfrak{M}$, then $L_f = F$.*

Let g be in $\mathfrak{D}$. By Corollary 7.1, $\boldsymbol{V}(L_gF) = g*f$ and by Corollary 6.2, $\boldsymbol{U}_2\boldsymbol{V}(L_gF) = \boldsymbol{U}_2(g*f) = L_gL_f$. Thus $\boldsymbol{V}(L_gF) = \boldsymbol{V}\boldsymbol{U}_2\boldsymbol{V}(L_gF)$ (Corollary 7.2) $= \boldsymbol{V}(L_gL_f)$ and by linearity and (4) of Theorem 7, $L_gF = L_gL_f$. As a result $L_g(F - L_f) = 0$ for all g in $\mathfrak{D}$. From this it follows that $F = L_f$.

Corollary 7.5. *Suppose that f, g, and $f*g$ belong to $L_p(G)$, $L_q(G)$, and $L_r(G)$ respectively where $1 \leqq p$, $q \leqq 2$, $1/r = 1/p + 1/q - 1$ and $r > 2$. Then L_fL_g belongs to $L_{r'}(G')$, r' satisfies the inequality $1 \leqq r' < 2$, and $\boldsymbol{V}(L_fL_g) = f*g$.*

$1 \geqq 1-1/r=1/r'=1-1/p+1-1/q=1/p'+1/q'$ and as L_f, L_g belong to $L_{p'}(G')$, $L_{q'}(G')$ respectively it follows from Corollary 3.2 that $L_f L_g$ is in $L_{r'}(G')$. Since $1/r'=1-1/r \geqq 1-1/2=1/2$ we see that $1 \leqq r' < 2$ so that the inverse Fourier transform of $L_f L_g$ exists. To show that $f*g=\boldsymbol{V}(L_f L_g)$ choose $f_n g_n$ in $\mathfrak{D}$ such that $f_n \to f$ in $L_p(G)$ and $g_n \to g$ in $L_q(G)$. A simple application of Young's inequality shows that $f_n * g_n \to f*g$ in $L_r(G)$ and from the inequality given in Corollary 3.2 follows the fact that $\boldsymbol{U}(f_n * g_n) = \boldsymbol{U}(f_n)\boldsymbol{U}(g_n) \to L_f L_g$ in $L_{r'}(G)$; here we use, of course, the facts that $\boldsymbol{U}(f_n) \to L_f$ in $L_{p'}(G')$ and $\boldsymbol{U}(g_n) \to L_g$ in $L_{q'}(G')$. Hence $\boldsymbol{VU}(f_n * g_n) \to \boldsymbol{V}(L_f L_g)$ in $L_{r''}(G)$. Now $r''=r$ and by Corollary 7.2 $\boldsymbol{VU}(f_n * g_n) = f_n * g_n$. Thus $f_n * g_n \to \boldsymbol{V}(L_f L_g)$ in $L_r(G)$ and as we have already shown that $f_n * g_n \to f*g$ in $L_r(G)$ it follows that $f*g = \boldsymbol{V}(L_f L_g)$ almost everywhere.

COROLLARY 7.6. *If G is compact and if Haar measure is normalized so that the measure of G is* 1 *then for any measurable operator F on G', $\|F\|_p$ is a non-increasing function of p.*

By a standard sort of argument, which we will omit, it suffices to show that $\|F\|_1 \geqq \|F\|_\infty$. If $\|F\|_1 = \infty$, the inequality holds trivially. So suppose $\|F\|_1$ is finite and put $f=\boldsymbol{V}(F)$. Then $\|f\|_\infty \leqq \|F\|_1$ and because G has measure 1, $\|f\|_1 \leqq \|f\|_\infty$. Thus f belongs to $L_1(G)$ so that $L_f = F$ and $\|F\|_\infty \leqq \|f\|_1 \leqq \|f\|_\infty \leqq \|F\|_1$.

COROLLARY 7.7. *If G is compact every F in $\mathfrak{M}'$ is the Fourier transform of its inverse transform.*

Let $F=F_1+F_2+ \cdots +F_n$ with F_i in $L_{p_i}(G')$, $1 \leqq p_i \leqq 2$; then $\boldsymbol{V}(F) = \boldsymbol{V}(F_1) + \cdots + \boldsymbol{V}(F_n)$ and since G is compact $\boldsymbol{V}(F_i)$ belongs to $L_1(G)$. Therefore $\boldsymbol{V}(F)$ is in $L_1(G)$ and by Corollary 7.4, $\boldsymbol{UV}(F)=F$.

For compact groups, Theorem 6 and the result just established provide us with complete global analogues to both parts of the classical theorem of Hausdorff and Young; however, a more direct and much more explicit extension of the theorem can be obtained from a concrete representation of the canonical gage space as follows:

Let Λ be the collection of equivalence classes of continuous irreducible unitary representations of G which we assume to be compact with Haar measure normalized to be 1. For each λ in Λ, let ϕ_λ be a concrete representation of class λ and suppose that ϕ_λ acts on a space of dimension d_λ. The regular gage space $\Gamma = (L_2(\Lambda), \mathfrak{A}, n)$ as constructed in the example following Theorem 1 will be called a concrete representative of G'. If f is an integrable function on G the operator $F_\lambda = \int f(x)\phi_\lambda(x)dx$ is well defined and setting $F = \{F_\lambda\}$ the above terminology is justified by

THEOREM 8. *The canonical gage space G' of a compact group G is algebraically equivalent to any concrete representative Γ in such a way that when f is integrable on G, L_f corresponds to L_F.*

Since $L_q(\Lambda)$ is algebraically isomorphic to $L_q(\Gamma)$ for $1 \leqq q \leqq \infty$, Theorem 8 and the preceding results show that the Hausdorff-Young Theorem can be formulated along more traditional lines as follows:

COROLLARY 8.1. *Let G be a compact group with Haar measure* 1 *and let* Γ *be a concrete representative of the canonical gage space of G. Then $\boldsymbol{W}: f \rightarrow F$ is a bounded linear mapping from $L_1(G)$ to $L_\infty(\Lambda)$ such that for* $1 \leqq p \leqq 2$,

(a) $\|F\|_{p'} \leqq \|f\|_p$, *for all f in $L_p(G)$,*

(b) *for any F in $L_p(\Gamma)$ there exists a unique f in $L_{p'}(G)$ such that $\|f\|_{p'} \leqq \|F\|_p$ and $\boldsymbol{W}(f) = F$.*

LEMMA 8.1. *If $T = \{T_\lambda\}$ is an arbitrary element of $\mathfrak{F}$ and if L_T denotes the operation of left multiplication by T on $L_2(\Lambda)$, then $\|L_T\|_\infty \geqq \|T\|_\infty$.*

Since the ordinary trace defines a regular gage on the ring of all bounded linear operators on a complex finite dimensional space, all of our results apply to the operators T_λ. Thus for each μ there exists a sequence $F_\mu^{(n)}$ such that $\|F_\mu^{(n)}\|_2 \leqq 1$ and $\|T_\mu F_\mu^{(n)}\|_2 \rightarrow \|T_\mu\|_\infty$. Let $K_\lambda^{(n)} = d_\mu^{-1/2} F_\mu^{(n)}$ if $\lambda = \mu$ and put $K_\lambda^{(n)} = 0$ if $\lambda \neq \mu$. Let $K^{(n)} = \{K_\lambda^{(n)}\}$. Then $\|K^{(n)}\|_2 = \|F^{(n)}\|_2 \leqq 1$ and $\|L_T(K^{(n)})\|_2 = \|T_\mu F_\mu^{(n)}\|_2 \rightarrow \|T_\mu\|_\infty$ as $n \rightarrow \infty$. Hence $\|L_T\|_\infty \geqq \|T_\mu\|_\infty$ for each μ so that $\|L_T\|_\infty \geqq \|T\|_\infty$.

LEMMA 8.2. *If T belongs to $L_\infty(\Lambda)$, then L_T is bounded and $\|L_T\|_\infty = \|T\|_\infty$.*

For F in $L_2(\Lambda)$, $\|L_T(F)\|_2^2 = \sum_\lambda \|T_\lambda F_\lambda\|_2^2 d_\lambda \leqq \sum_\lambda \|T_\lambda\|_\infty^2 \|F_\lambda\|_2^2 d_\lambda \leqq \|T\|_\infty^2 \|F\|_2^2$. Thus L_T is bounded and $\|L_T\|_\infty \leqq \|T\|_\infty$; however, by the previous lemma, $\|L_T\|_\infty \geqq \|T\|_\infty$. Hence $\|L_T\|_\infty = \|T\|_\infty$.

Proof of the theorem. Let $\boldsymbol{W}_2$ be the restriction of the mapping $f \rightarrow F$ to $L_2(G)$. That $\boldsymbol{W}_2$ is a Hilbert algebra isomorphism between $L_2(G)$ and $L_2(\Lambda)$ is the essential content of the L_2 part of the Peter Weyl Theorem; we will assume this as known. Thus the left ring $\mathfrak{L}$ of $L_2(G)$ is unitarily equivalent to the left ring of $L_2(\Lambda)$, the latter ring being equal to $\boldsymbol{W}_2 \mathfrak{L} \boldsymbol{W}_2^{-1}$. If f is in $L_1(G)$ then L_f is bounded and by Theorem 6, L_f is measurable with respect to $\mathfrak{L}$. Hence L_f belongs to $\mathfrak{L}$ and we will show that $\boldsymbol{W}_2 L_f \boldsymbol{W}_2^{-1} = L_F$. Suppose K is an arbitrary element of $L_2(\Lambda)$. Then putting $k = \boldsymbol{W}_2^{-1}(K)$ it results that $\boldsymbol{W}_2 L_f \boldsymbol{W}_2^{-1}(K) = \boldsymbol{W}_2 L_f(k) = \boldsymbol{W}_2(f*k)$ and by a straight forward argument involving the Fubini Theorem $\boldsymbol{W}_2(f*k) = FK = L_F(K)$. From this follows the fact that L_F is bounded for each F in $L_2(\Lambda)$. For if $f = \boldsymbol{W}_2^{-1}(F)$, then $L_F = \boldsymbol{W}_2 L_f \boldsymbol{W}_2^{-1}$ and as $L_2(G)$ is contained in $L_1(G)$, L_f is bounded. Now if R_F denotes the operation of right multiplication by F, it is clear that $L_T R_F = R_F L_T$ for all T in $L_\infty(\Lambda)$ and all F in $L_2(\Lambda)$. As the R_F generate the commutor of the left ring of $L_2(\Lambda)$, L_T is in the left ring for each T in $L_\infty(\Lambda)$. The mapping $T \rightarrow L_T$ is easily seen to be a $*$ isomorphism of $L_\infty(\Lambda)$ into the left ring. Now let X be an arbitrary member of the left ring. Then for any T in $L_2(\Lambda)$, $XL_T = L_{X(T)}$, as L_T is bounded [4, Corollary 16.3]. For each fixed μ let $E_\lambda^\mu = I_\mu$, the identity on the μth component, if $\lambda = \mu$ and otherwise put $E_\lambda^\mu = 0$. Set $E^\mu = \{E_\lambda^\mu\}$. Using the above relation we see that $X(E^\mu F) = X(E^\mu)F$ for all F in $L_2(\Lambda)$. Since $E^\mu E^\mu = E^\mu$, $X(E^\mu) = X(E^\mu)E^\mu$ and therefore if $Y^\mu_\lambda = X(E^\mu)_\lambda$, $Y^\mu_\lambda = 0$ for $\lambda \neq \mu$. Set $Y = \{ Y^\mu_\mu \}$. Now if F is in $L_2(\Lambda)$, $F = \sum_\mu E^\mu F$ and $X(F) = \sum_\mu X(E^\mu F) = \sum_\mu X(E^\mu)F = \sum_\mu \{ Y^\mu_\lambda F_\lambda \} = \{ Y^\mu_\mu F_\mu \} = YF = L_Y(F)$. Thus $X = L_Y$, L_Y is bounded and by Lemma 8.1 $\|L_Y\|_\infty \geqq \|Y\|_\infty$. Hence the map $T \rightarrow L_T$ from $L_\infty(\Lambda)$ is a $*$ isomorphism onto the left ring of $L_2(\Lambda)$ and therefore $\mathfrak{A} = W_2 \mathfrak{L} W_2^{-1}$. Finally, the alleged gage on $\mathfrak{A}$ was just the canonical one so that the gage spaces $G' = (L_2(G), \mathfrak{L}, m)$ and $\Gamma = (L_2(\Lambda), \mathfrak{A}, n)$ are indeed equivalent.

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