WEIGHTED QUADRATIC NORMS AND ULTRA-SPHERICAL POLYNOMIALS II[1]

BY

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1. **Introduction.** The present paper continues the program announced in the preceding paper. The notations introduced in §1 of that paper are assumed in what follows. Numbers in square brackets refer to the bibliography given there.

2. **The basic identity.** Let $p(x)$ be any non-negative measurable function such that

$$q(n) = \int_{-1}^{1} [1 - W_\nu(n, x)] p(x) dx \tag{1}$$

is defined for $[n=0, 1, \cdots]$. (Note that $q(0)$ is necessarily equal to zero.) We further set $Q(x, y) = \int_{-1}^{1} C_\nu(x, y, z) p(z) dz$.

THEOREM 2a. *Let $q(n)$ and $Q(x, y)$ be defined as above. If*

1. $$\sum_0^\infty F(n)^2 \omega_\nu(n) < \infty,$$

2. $$F^\frown(x) = \sum_0^\infty F(n)\omega_\nu(n) W_\nu(n, x) \qquad (\mathrm{M}_2),$$

then

$$\sum_0^\infty F(n)^2 \omega_\nu(n) q(n) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} [F^\frown(x) - F^\frown(y)]^2 Q(x, y) d\Omega_\nu(x) d\Omega_\nu(y). \tag{2}$$

The symbol "M_2" indicates that the series defining $F^\frown(x)$ converges in the mean of order two with respect to $d\Omega_\nu(x)$. Let us first suppose that $\int_{-1}^{1} p(z)dz$ is finite. We will remove this restriction later. We expand

$$[F^\frown(x) - F^\frown(y)]^2$$

out so that the right hand side of (2) splits into three terms:

$$I_1 = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} F^\frown(x)^2 Q(x, y) d\Omega_\nu(x) d\Omega_\nu(y);$$

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$$I_2 = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} F^{\frown}(y)^2 Q(x, y)d\Omega_\nu(x)d\Omega_\nu(y);$$

$$I_3 = -\int_{-1}^{1}\int_{-1}^{1} F^{\frown}(x)F^{\frown}(y)Q(x, y)d\Omega_\nu(x)d\Omega_\nu(y).$$

We have, inserting the definition of $Q(x, y)$,

$$I_1 = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F^{\frown}(x)^2 C_\nu(x, y, z)p(z)dzd\Omega_\nu(x)d\Omega_\nu(y).$$

Since the integrand is non-negative we can invert the order of the integrations to obtain

$$I_1 = \frac{1}{2}\int_{-1}^{1} F(x)^2 d\Omega_\nu(x)\int_{-1}^{1} p(z)dz\int_{-1}^{1} C_\nu(x, y, z)d\Omega_\nu(y).$$

Using the relation

$$\int_{-1}^{1} C_\nu(x, y, z)d\Omega_\nu(y) = 1. \tag{3}$$

We find that

$$I_1 = \frac{1}{2}\left[\int_{-1}^{1} F^{\frown}(x)^2 d\Omega_\nu(x)\right]\left[\int_{-1}^{1} p(z)dz\right],$$

and thus by Parseval's equality

$$I_1 = \frac{1}{2}\left[\sum_{n=0}^{\infty} F(n)^2\omega_\nu(n)\right]\left[\int_{-1}^{1} p(z)dz\right].$$

Similarly

$$I_2 = \frac{1}{2}\left[\sum_{n=0}^{\infty} F(n)^2\omega_\nu(n)\right]\left[\int_{-1}^{1} p(z)dz\right].$$

The inequality

$$\left| F^{\frown}(x)F^{\frown}(y)\right| \leqq \frac{1}{2} F^{\frown}(x)^2 + \frac{1}{2} F^{\frown}(y)^2$$

shows that the integral

$$I_3 = -\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F^{\frown}(x)F^{\frown}(y)C_\nu(x, y, z)p(z)dzd\Omega_\nu(x)d\Omega_\nu(y)$$

is absolutely convergent if the integrals I_1 and I_2 are absolutely convergent. But we have seen that this is the case. Thus we may invert the order of the integrations in I_3 to obtain

$$I_3 = -\int_{-1}^{1} p(z)dz \int_{-1}^{1} \{F^\frown(x)\} \left\{ \int_{-1}^{1} F^\frown(y) C_\nu(x, y, z) d\Omega_\nu(y) \right\} d\Omega_\nu(x).$$

Consider the function

$$\phi(x, z) = \int_{-1}^{1} F^\frown(y) C_\nu(x, y, z) d\Omega_\nu(y).$$

By Schwarz's inequality and (3) we have

$$\begin{aligned}\phi(x, z)^2 &\leqq \left[\int_{-1}^{1} F^\frown(y)^2 C_\nu(x, y, z) d\Omega_\nu(y) \right] \left[\int_{-1}^{1} C_\nu(x, y, z) d\Omega_\nu(y) \right] \\ &\leqq \int_{-1}^{1} F^\frown(y)^2 C_\nu(x, y, z) d\Omega_\nu(y),\end{aligned}$$

and thus

$$\int_{-1}^{1} \phi(x, z)^2 d\Omega_\nu(x) \leqq \int_{-1}^{1}\int_{-1}^{1} F^\frown(y)^2 C_\nu(x, y, z) d\Omega_\nu(y) d\Omega_\nu(x).$$

Since the integrand of the integral on the right is non-negative we can invert the order of the integrations. Employing (3) again we find that

$$\int_{-1}^{1} \phi(x, z)^2 d\Omega_\nu(x) \leqq \int_{-1}^{1} F^\frown(y)^2 d\Omega_\nu(y),$$

that is as a function of x $\phi(x, z)$ is square integrable with respect to $d\Omega_\nu(x)$. Consider

$$\int_{-1}^{1} \phi(x, z) W_\nu(n, x) d\Omega_\nu(x).$$

If we replace here $\phi(x, z)$ by the integral which defines it then it is easily seen that we can invert the order of the integrations in the resulting double integral. We obtain, using (1),

$$\begin{aligned}\int_{-1}^{1} \phi(x, z) W_\nu(n, x) d\Omega_\nu(x) &= \int_{-1}^{1} F^\frown(y) d\Omega_\nu(y) \int_{-1}^{1} C_\nu(x, y, z) W_\nu(n, x) d\Omega_\nu(x) \\ &= \int_{-1}^{1} F^\frown(y) W_\nu(n, y) W_\nu(n, z) d\Omega_\nu(y) \\ &= F(n) W_\nu(n, z).\end{aligned}$$

Thus, z being fixed,

$$\int_{-1}^{1} F^{\frown}(y)C_\nu(x, y, z)d\Omega_\nu(y) = \sum_0^\infty F(n)\omega_\nu(n)W_\nu(n, z)W_\nu(n, x) \qquad (\mathrm{M}_2).$$

Since by definition

$$F^{\frown}(x) = \sum_0^\infty F(n)\omega_\nu(n)W_\nu(n, x) \qquad (\mathrm{M}_2),$$

the general Parseval equality gives

$$\int_{-1}^{1}\{F^{\frown}(x)\}\left\{\int_{-1}^{1} F^{\frown}(y)C_\nu(x, y, z)d\Omega_\nu(y)\right\}d\Omega_\nu(x) = \sum_0^\infty F(n)^2\omega_\nu(n)W_\nu(n, z),$$

and thus

$$I_3 = -\int_{-1}^{1} p(z)\left[\sum_0^\infty F(n)^2\omega_\nu(n)W_\nu(n, z)\right]dz.$$

Because $|W_\nu(z)| \leqq 1$ the series in square brackets converges uniformly. Therefore integration and summation can be interchanged and we have

$$I_3 = -\sum_0^\infty F(n)^2\omega_\nu(n)\int_{-1}^{1} W_\nu(n, z)p(z)dz.$$

Combining these results we see that (2) holds subject to the restriction $\int_{-1}^{1} p(z)dz < \infty$. If this restriction is not satisfied then we can find a sequence of non-negative integrable functions $p_k(z)$ with $p_k(z) \uparrow p(z)$. With an obvious notation

$$\sum_0^\infty F(n)^2\omega_\nu(n)q_k(n) = \int_{-1}^{1}\int_{-1}^{1}[F^{\frown}(x) - F^{\frown}(y)]^2 Q_k(x, y)d\Omega_\nu(x)d\Omega_\nu(y).$$

Since $q_k(n) \uparrow q(n)$, $Q_k(x, y) \uparrow Q(x, y)$ as $k \to \infty$ a simple limiting procedure gives us our desired result.

COROLLARY 2b. *Let $q(n)$ and $Q(x, y)$ be defined as before. If*

1. $\sum_0^\infty F(n)^2 q(n)\omega_\nu(n) < \infty, \qquad \sum_0^\infty G(n)^2 q(n)\omega_\nu(n) < \infty.$

2. $F^{\frown}(x) = \sum_0^\infty F(n)\omega_\nu(n)W_\nu(n, x), \qquad G^{\frown}(x) = \sum_0^\infty G(n)\omega_\nu(n)W_\nu(n, x),$

then

$$\sum_0^\infty F(n)G(n)q(n)\omega_\nu(n) = \int_{-1}^{1}\int_{-1}^{1}[F^{\frown}(x) - F^{\frown}(y)][G^{\frown}(x) - G^{\frown}(y)] \cdot Q(x, y)d\Omega_\nu(x)d\Omega_\nu(y).$$

To see this apply Theorem 2a to $F(n)+G(n)$ and $F(n)-G(n)$ and subtract the resulting formulas.

3. **Approximations.** Let us agree to write $A(x)\approx B(x)$ for x belonging to a certain set S if there exist finite positive constants C_1 and C_2 such that $C_1A(x)\leqq B(x)\leqq C_2A(x)$ for x in S. Similarly $A(x)<\approx B(x)$ for x in S means that $A(x)\leqq C_3B(x)$ for $x\in S$, etc.

LEMMA 3a. *If* $0<\alpha<1/2$ *and if*

1. $$p(z) = (1-z)^{-1-\alpha},$$

2. $$q(n) = \int_{-1}^{1}[1 - W_\nu(n, z)]p(z)dz,$$

then

$$q(n) \approx (n+1)^{2\alpha} \qquad (n = 1, 2, \cdots).$$

Using the formulas

$$W_\nu(n, x) = \frac{n!}{(2\nu)_n} C_n^\nu(x),$$

$$C_n^\nu(\cos\theta) = \sum_{m=0}^{n} \frac{(\nu)_m(\nu)_{n-m}}{m!(n-m)!}\cos(n-2m)\theta,$$

see [2, vol. 2, p. 175], we find that

$$\frac{(2\nu)_n}{n!}q(n) = \sum_{m=0}^{n}\frac{(\nu)_m(\nu)_{n-m}}{m!(n-m)!}\int_0^\pi [1-\cos(n-2m)\theta][1-\cos\theta]^{-1-\alpha}\sin\theta d\theta.$$

It is easy to see that

$$\int_0^\pi [1-\cos k\theta][1-\cos\theta]^{-1-\alpha}\sin\theta d\theta \approx \int_0^\pi \sin^2\left(\frac{k\theta}{2}\right)\theta^{-1-2\alpha}d\theta,$$

while

$$\int_0^\pi \sin^2\left(\frac{k\theta}{2}\right)\theta^{-1-2\alpha}d\theta = \left(\frac{k}{2}\right)^{2\alpha}\int_0^{k\pi/2}\sin^2\theta\,\theta^{-1-2\alpha}\,d\theta \approx \left(\frac{k}{2}\right)^{2\alpha} \qquad (k = 1, 2, \cdots).$$

Using the approximation

$$\frac{(\nu)_k}{k!} \approx (k+1)^{\nu-1}$$

we now obtain

$$q(n) \approx \sum_{m=0}^{n} (m+1)^{\nu-1}(n-m+1)^{\nu-1}(n+1)^{1-2\nu}\left|\frac{n}{2}-m\right|^{2\alpha},$$
$$\approx \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where Σ_1 is the sum corresponding to the range $0 \leqq m < n/3$, Σ_2 to the range $n/3 \leqq m < 2n/3$, and Σ_3 to the range $2n/3 \leqq m \leqq n$. We have

$$\Sigma_1 \approx n^{2\alpha-\nu} \sum_{0 \leqq m < n/3} (m+1)^{\nu-1} \approx (n+1)^{2\alpha},$$
$$\Sigma_2 \approx n^{-1} \sum_{n/3 \leqq m < 2n/3} \left|\frac{n}{2}-m\right|^{2\alpha} \approx (n+1)^{2\alpha},$$
$$\Sigma_3 \approx n^{2\alpha-\nu} \sum_{2n/3 \leqq m \leqq n} (n-m+1)^{\nu-1} \approx (n+1)^{2\alpha}.$$

Combining these estimates we obtain our desired result.

LEMMA 3b. *If* $0<\alpha<1/2$ *and if*

1. $$p(z) = (1-z)^{-1-\alpha},$$

2. $$Q(x, y) = \int_{-1}^{1} C_\nu(x, y, z)p(z)dz,$$

then

$$Q(\cos\theta, \cos\phi) \approx [1-\cos(\theta-\phi)]^{-\alpha-1/2}[1-\cos(\theta+\phi)]^{-\nu}.$$

Let us set

$$u = xy + (1-x^2)^{1/2}\cdot(1-y^2)^{1/2},$$
$$v = xy - (1-x^2)^{1/2}\cdot(1-y^2)^{1/2}.$$

Since $1-x^2-y^2-z^2+2xyz=(u-z)(z-v)$ we have

$$[(1-x^2)(1-y^2)]^{\nu-1/2}Q(x, y) = 2^{1-2\nu}\Gamma(2\nu)\Gamma(\nu)^{-2}\int_v^u [(u-z)(z-v)]^{\nu-1} \cdot (1-z)^{-1/2-\nu-\alpha}(1+z)^{-\nu+1/2}dz.$$

If we put $s=(z-v)/(u-v)$ then we find that

$$\int_v^u [u-z)(z-v)]^{\nu-1}(1-z)^{-1/2-\nu-\alpha}(1+z)^{-\nu+1/2}dz$$
$$= (u-v)^{\nu-\alpha-3/2}\int_0^1 s^{\nu-1}(1-s)^{\nu-1}\left[\frac{1-v}{u-v}-s\right]^{-\nu-\alpha-1/2} \cdot [(u-v)s+1+v]^{-\nu+1/2}ds$$
$$\approx (u-v)^{\nu-\alpha-3/2}\int_0^1 s^{\nu-1}(1-s)^{\nu-1}\left[\frac{1-v}{u-v}-s\right]^{-\nu-\alpha-1/2}ds.$$

Thus our problem is reduced to that of estimating the integral

$$I(c) = \int_0^1 s^{\nu-1}(1-s)^{\nu-1}(c-s)^{-\nu-\alpha-1/2}ds$$

where $c>1$.

Suppose first that $c \geqq 3/2$, then

$$I(c) \approx c^{-\nu-\alpha-1/2}\int_0^1 s^{\nu-1}(1-s)^{\nu-1}ds \approx c^{-\nu-\alpha-1/2}.$$

On the other hand if $1<c<3/2$ we have $I(c)=I_1(c)+I_2(c)$ where

$$I_1(c) = \int_0^{2-c} s^{\nu-1}(1-s)^{\nu-1}(c-s)^{-\nu-\alpha-1/2}ds,$$

$$I_2(c) = \int_{2-c}^1 s^{\nu-1}(1-s)^{\nu-1}(c-s)^{-\nu-\alpha-1/2}ds.$$

It is easily seen that

$$I_1(c) \approx \int_0^{2-c} s^{\nu-1}(1-s)^{-\alpha-3/2}ds$$
$$\approx (c-1)^{-\alpha-1/2},$$

and that

$$I_2(c) \approx (c-1)^{-\nu-\alpha-1/2}\int_{2-c}^1 s^{\nu-1}(1-s)^{\nu-1}ds$$
$$\approx (c-1)^{-\alpha-1/2}.$$

Thus we have (in either case)

$$I(c) \approx (c-1)^{-\alpha-1/2}c^{-\nu} \qquad (1<c<\infty).$$

Combining our results we have

$$[(1-x^2)(1-y^2)]^{\nu-1/2}Q(x,y) \approx (u-v)^{2\nu-1}(1-v)^{-\nu}(1-u)^{-\alpha-1/2}.$$

If we set $x=\cos\theta$, $y=\cos\phi$ then $u=\cos(\theta-\phi)$, $v=\cos(\theta+\phi)$, $u-v=2\sin\theta\sin\phi$, and we find that

$$Q(\cos\theta,\cos\phi) \approx [1-\cos(\theta-\phi)]^{-\alpha-1/2}[1-\cos(\theta+\phi)]^{-\nu}.$$

Let us recall that §1 that

$$\mathfrak{N}_\alpha^\nu[F(n)] = \left\{\sum_0^\infty F(n)^2\omega_\nu(n)(n+1)^{2\alpha}\right\}^{1/2} \qquad \left(-\frac{1}{2}<\alpha<\frac{1}{2}\right),$$

and also that $\mathfrak{N}_\alpha^\nu$ denotes the space of functions $F(n)$ for which $\mathfrak{N}_\alpha^\nu[F]$ is finite. Using Theorem 2a and Lemmas 2a and 2b we have the following result.

THEOREM 3c. *If* $0<\alpha<1/2$ *and if*

1. $$F(n) \in \mathfrak{N}_\alpha^\nu,$$

2. $$F^\frown(x) = \sum_0^\infty F(n)\omega_\nu(n)W_\nu(n, x),$$

then

$$\mathfrak{N}_\alpha^\nu[F]^2 - \omega_\nu(0)^2F(0)^2 \approx \int_0^\pi\int_0^\pi [F^\frown(\cos\theta) - F^\frown(\cos\phi)]^2|\theta-\phi|^{-2\alpha-1}$$
$$[1-\cos(\theta+\phi)]^{-\nu}\sin^{2\nu}\theta\sin^{2\nu}\phi\, d\theta d\phi.$$

4. **Multipliers.** Let $t(x)$ be a bounded measurable function on $[-1, 1]$, and consider the transformation t defined by the formula

$$tF\cdot(n) = \int_{-1}^1\left[\sum_0^\infty F(m)\omega_\nu(m)W_\nu(m, x)\right]W_\nu(n, x)t(x)d\Omega_\nu(x). \tag{1}$$

In this section we shall obtain sufficient conditions to insure that t be a bounded linear transformation of $\mathfrak{N}_\alpha^\nu$ into itself. Actually it will be convenient to change variables. We replace x by $\cos\theta$ and we set

$$F^*(\theta) = \sum_0^\infty F(m)\omega_\nu(m)W_\nu(m, \cos\theta), \tag{2}$$

$$\tau(\theta) = t(\cos\theta), \tag{3}$$

and instead of (1) we write

$$\tau F\cdot(n) = \int_0^\pi F^*(\theta)\tau(\theta)W_\nu(n, \cos\theta)\sin^{2\nu}\theta d\theta. \tag{4}$$

Let us rewrite the equation defining $F^*(\theta)$ in the form

$$\sin^\nu\theta F^*(\theta) = \sum_0^\infty [F(m)\omega_\nu(m)^{1/2}][\omega_\nu(m)^{1/2}W_\nu(m, \cos\theta)\sin^\nu\theta].$$

The functions $\omega_\nu(n)^{1/2}W_\nu(n, \cos\theta)\sin^\nu\theta$ are orthonormal on the interval $[0, \pi]$ and they are uniformly bounded there. See [12, p. 166]. A general theorem [5] on uniformly bounded orthonormal systems yields, as a special case, the following result

LEMMA 4a. *If*

1. $F(n) \in \mathfrak{N}_\alpha^\nu \qquad (0 \leqq \alpha < 1/2)$,
2. $F^*(\theta)$ *is defined by* (2),

then for any $\theta_0 \in [0, \pi]$

$$\int_0^\pi F^*(\theta)^2 \sin^{2\nu} \theta \left| \theta - \theta_0 \right|^{-2\alpha} d\theta \leqq A'(\alpha, \nu) \mathfrak{N}_\alpha^\nu [F]^2.$$

LEMMA 4b. *If*

1. $K_\rho(\theta)$ *is* 1 *for* $0 \leqq \theta \leqq \rho$ *and* 0 *for* $\rho < \theta \leqq \pi$,
2. $F \in \mathfrak{N}_\alpha^\nu \qquad 0 \leqq \alpha < 1/2$,
3. $K_\rho F$ *is defined as in* (4),

then

$$\mathfrak{N}_\alpha^\nu [K_\rho F] \leqq A(\nu, \alpha) \mathfrak{N}_\alpha^\nu (F).$$

Our lemma is evidently true if $\alpha = 0$, so that we may assume $0 < \alpha < 1/2$. Since $F \in \mathfrak{N}_\alpha^\nu$ we have *a fortiori* $F \in \mathfrak{N}_0^\nu$; thus the series defining $F^*(\theta)$ converges in the mean of order 2 (with respect to the measure $\sin^{2\nu} \theta$) and the formula (4) is meaningful. By Theorem 3c we see that if $F_\rho = K_\rho F$ and if

$$I = \sum_1^\infty [F_\rho(n)]^2 \omega_\nu(n)(n+1)^{2\alpha}$$

then

$$I \approx \int_0^\pi \int_0^\pi [F^*(\theta) K_\rho(\theta) - F^*(\phi) K_\rho(\phi)]^2 K(\theta, \phi) \sin^{2\nu} \theta \sin^{2\nu} \phi d\theta d\phi$$

where

$$K(\theta, \phi) = \left| \theta - \phi \right|^{-2\alpha - 1} [1 - \cos(\theta + \phi)]^{-\nu}.$$

Using the specific form of K_ρ we find that

$$I \approx I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^\rho \int_0^\rho [F^*(\theta) - F^*(\phi)]^2 K(\theta, \phi) \sin^{2\nu} \theta \sin^{2\nu} \phi d\theta d\phi,$$

$$I_2 = \int_0^\rho F^*(\phi)^2 \sin^{2\nu} \phi d\phi \int_\rho^\pi K(\theta, \phi) \sin^{2\nu} \theta d\theta,$$

$$I_3 = \int_0^\rho F^*(\theta)^2 \sin^{2\nu} \theta d\theta \int_\rho^\pi K(\theta, \phi) \sin^{2\nu} \phi d\phi.$$

By Theorem 3c we have $I_1 < \approx \mathfrak{N}_\alpha^\nu[F]^2$. A simple estimation shows that

$$I_2 < \approx \int_0^\rho F^*(\phi)^2 \,|\, \phi - \rho \,|^{-2\alpha} \sin^{2\nu} \phi d\phi$$

and thus by Lemma 4a $I_2 < \approx \mathfrak{N}_\alpha^\nu[F]^2$. Similarly $I_3 < \approx \mathfrak{N}_\alpha^\nu[F]^2$. Combining these we have $I < \approx \mathfrak{N}_\alpha^\nu[F]^2$. We have

$$F_\rho(0) = \int_0^\rho F^*(\theta) \sin^{2\nu} \theta d\theta,$$

$$|\, F_\rho(0)\,|^2 \leqq A \int_0^\pi F^*(\theta)^2 \sin^{2\nu} \theta d\theta = A \sum_0^\infty F(n)^2 \omega_\nu(n) \leqq A \mathfrak{N}_\alpha^\nu[F]^2.$$

Finally

$$\mathfrak{N}_\alpha^\nu[F_\rho]^2 = I + |\, F_\rho(0)\,|^2 \leqq A \mathfrak{N}_\alpha^\nu[F]^2,$$

as desired.

Let

$$t_\mu = \frac{\pi}{2} \{1 + (\operatorname{sgn} \mu) 2^{-|\mu|+1}\} \qquad (\mu = \pm 1, \pm 2, \cdots)$$

and let S_μ be the interval $[t_{\mu+1}, t_\mu]$ if $\mu > 0$ and the interval $[t_\mu, t_{\mu-1}]$ if $\mu < 0$. Let b_μ be the mid point of S_μ and r_μ the length of S_μ. Finally let σ_μ be the interval $[b_\mu - r_\mu, b_\mu + r_\mu]$ if $|\mu| > 1$ and the intersection of $[b_\mu - r_\mu, b_\mu + r_\mu]$ with $[0, \pi]$ if $|\mu| = 1$. We set

$$\rho_\mu(\theta) = \begin{cases} [1 - r_\mu^{-2}(\theta - b_\mu)^2] & \theta \in \sigma_\mu, \\ 0 & \theta \notin \sigma_\mu. \end{cases}$$

LEMMA 4c. *If*

1. $F(n) \in \mathfrak{N}_\alpha^\nu \qquad (0 \leqq \alpha < 1/2),$
2. $F_\mu(n) = \rho_\mu F \cdot (n),$

then

$$\sum_{-\infty}^{\infty}{}' \mathfrak{N}_\alpha^\nu[F_\mu]^2 \leqq A(\alpha, \nu) \mathfrak{N}_\alpha^\nu[F]^2.$$

Here the "$'$" indicates that there is no term corresponding to $\mu = 0$. By Theorem 3c if

$$I_\mu = \sum_{n=1}^\infty F_\mu(n)^2 \omega_\nu(n)(n+1)^{2\alpha}$$

then

$$I_\mu = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{\sigma_\mu}\int_{\sigma_\mu} [F^*(\theta)\rho_\mu(\theta) - F^*(\phi)\rho_\mu(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi,$$

$$I_2 = \int_{\sigma_\mu} [F^*(\phi)\rho_\mu(\phi)]^2 \sin^{2\nu}\phi d\phi \int_{\sigma_\mu'} K(\theta, \phi) \sin^{2\nu}\theta d\theta,$$

$$I_3 = \int_{\sigma_\mu} [F^*(\theta)\rho_\mu(\theta)]^2 \sin^{2\nu}\theta \int_{\sigma_\mu'} K(\theta, \phi) \sin^{2\nu}\phi d\phi.$$

Here σ_μ' is the complement of σ_μ in $[0, \pi]$. It is easily verified that

$$\int_{\sigma_\mu} K(\theta, \phi) \sin^{2\nu}\theta d\theta \leqq A\{ |\phi - b_\mu - r_\mu|^{-2\alpha} + |\phi - b_\mu + r_\mu|^{-2\alpha}\}.$$

(Throughout we shall use A for any constant depending only upon α and ν and not necessarily the same at each occurrence.)

A simple estimation shows that if $\phi \in \sigma_\mu$ then

$$\rho_\mu(\phi)^2 |\phi - b_\mu - r_\mu|^{-2\alpha} \leqq A\left|\phi - \frac{\pi}{2}\right|^{-2\alpha},$$

$$\rho_\mu(\phi)^2 |\phi - b_\mu + r_\mu|^{-2\alpha} \leqq A\left|\phi - \frac{\pi}{2}\right|^{-2\alpha}.$$

Thus

$$I_2 \leqq A\int_{\sigma_\mu} [F^*(\phi)]^2 \left|\phi - \frac{\pi}{2}\right|^{-2\alpha} \sin^{2\nu}\phi d\phi,$$

and similarly

$$I_3 \leqq A\int_{\sigma_\mu} [F^*(\theta)]^2 \left|\theta - \frac{\pi}{2}\right|^{-2\alpha} \sin^{2\nu}\theta d\theta.$$

Since

$$F^*(\theta)\rho_\mu(\theta) - F^*(\phi)\rho_\mu(\phi) = [F^*(\theta) - F^*(\phi)]\rho_\mu(\theta) + F^*(\phi)[\rho_\mu(\theta) - \rho_\mu(\phi)]$$

we have

$$[F^*(\theta)\rho_\mu(\theta) - F^*(\phi)\rho_\mu(\phi)]^2 \leqq 2[F^*(\theta) - F^*(\phi)]^2\rho_\mu(\theta)^2 + 2F^*(\phi)^2[\rho_\mu(\theta) - \rho_\mu(\phi)]^2.$$

Inserting this inequality in the integral defining I_1 we find that $I_1 \leqq 2I_1' + 2I_1''$ where

$$I_1' = \int_{\sigma_\mu}\int_{\sigma_\mu} [F^*(\phi) - F^*(\theta)]^2 \rho_\mu(\theta)^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi,$$

$$I_2' = \int_{\sigma_\mu}\int_{\sigma_\mu} F^*(\phi)^2 [\rho_\mu(\theta) - \rho_\mu(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi.$$

Now $0 \leqq \rho_\mu(\theta) \leqq 1$ for $\theta \in \sigma_\mu$ and thus

$$I_1' \leqq \int_{\sigma_\mu}\int_{\sigma_\mu} [F^*(\theta) - F^*(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi.$$

Using $|\rho_\mu(\theta) - \rho_\mu(\phi)| \leqq A|\theta - \phi| r_\mu^{-1}$ we find that

$$\int_{\sigma_\mu} [\rho_\mu(\theta) - \rho_\mu(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta d\theta \leqq A\left|\phi - \frac{\pi}{2}\right|^{-2\alpha}$$

and hence that

$$I_2' \leqq A \int_{\sigma_\mu} [F^*(\phi)]^2 \left|\phi - \frac{\pi}{2}\right|^{-2\alpha} \sin^{2\nu}\phi d\phi.$$

We now have

$$I_\mu \leqq A \int_{\sigma_\mu}\int_{\sigma_\mu} [F^*(\theta) - F^*(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi$$
$$+ A \int_{\sigma_\mu} F^*(\theta)^2 \sin^{2\nu}\theta \left|\theta - \frac{\pi}{2}\right|^{-2\alpha} d\theta.$$

Also

$$F_\mu(0) = \int_{\sigma_\mu} \rho_\mu(\theta) F^*(\theta) \sin^{2\nu}\theta d\theta$$

and thus

$$|F_\mu(0)|^2 \leqq A \int_{\sigma_\mu} F^*(\theta)^2 \sin^{2\nu}\theta \left|\theta - \frac{\pi}{2}\right|^{-2\alpha} d\theta.$$

Finally

$$\mathfrak{N}_\alpha^\nu [F_\mu]^2 = I_\mu + [F_\mu(0)]^2,$$

$$\mathfrak{N}_\alpha^\nu [F]^2 \leqq A \int_{\sigma_\mu}\int_{\sigma_\mu} [F^*(\theta) - F^*(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi$$
$$+ A \int_{\sigma} F^*(\theta)^2 \sin^{2\nu}\theta \left|\theta - \frac{\pi}{2}\right|^{-2\alpha} d\theta.$$

Summing over μ and noting that no point θ in $[0, \pi]$ belongs to more than three σ_μ's we see that

$$\sum_{-\infty}^{\infty}{}' \mathfrak{N}_\alpha^\nu[F_\mu]^2 \leqq A \int_0^\pi \int_0^\pi [F^*(\theta) - F^*(\phi)]^2 K(\theta, \phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi$$
$$+ A \int_0^\pi \int_0^\pi F^*(\theta)^2 \sin^{2\nu}\theta \left|\theta - \frac{\pi}{2}\right|^{-2\alpha} d\theta$$

and from this using Theorem 3c and Lemma 4a it follows that

$$\sum_{-\infty}^{\infty}{}' \mathfrak{N}_\alpha^\nu[F_\mu]^2 \leqq A \mathfrak{N}_\alpha^\nu[F]^2, \qquad \text{q.e.d.}$$

LEMMA 4d. *If*

1. $F(n) \in \mathfrak{N}_\alpha^\nu \quad (0 \leqq \alpha < 1/2)$,
2. $F^*(\theta)$ *is defined by* (2),
3. $\phi_\mu \in S_\mu \quad (\mu = \pm 1, \pm 2, \cdots)$,

then

$$\sum_{-\infty}^{\infty}{}' \int_{S_\mu} F^*(\theta)^2 \sin^{2\nu}\theta \,|\, \phi_\mu - \theta \,|^{-2\alpha} \leqq A(\nu, \alpha) \mathfrak{N}_\alpha^\nu[F]^2.$$

Since $\rho_\mu(\theta) \geqq 3/4$ for $\theta \in S_\mu$ we have

$$\int_{S_\mu} F^*(\theta)^2 \sin^{2\nu}\theta \,|\, \theta - \phi_\mu \,|^{-2\alpha} d\theta \leqq A \int_{\sigma_\mu} [\rho_\mu(\theta) F^*(\theta)]^2 \sin^{2\nu}\theta \,|\, \theta - \phi_\mu \,|^{-2\alpha} d\theta,$$
$$\leqq A \mathfrak{N}_\alpha^\nu[F_\mu]^2,$$

the second inequality following from Lemma 4a. Summing over μ and using Lemma 4c we obtain our desired result.

DEFINITION. $\tau = \tau(\theta)$ $(0 \leqq \theta \leqq \pi)$ is said to belong to class $M(C)$ if:

1. $\tau(\theta) \leqq C \qquad 0 \leqq \theta \leqq \pi;$
2. $\int_{S_\mu} |\, d\tau(\theta) \,| \leqq C \qquad \mu = \pm 1, \pm 2, \cdots.$

THEOREM 5c. *If*

1. $F(n) \in \mathfrak{N}_\alpha^\nu \qquad -1/2 < \alpha < 1/2.$
2. $\tau \in M(C)$,

then

$$\mathfrak{N}_\alpha^\nu[\tau F] \leqq A(\nu, \alpha)\mathfrak{N}_\alpha^\nu[F].$$

Here τF is defined by (4). Our theorem is evidently true if $\alpha = 0$. Suppose $0 < \alpha < 1/2$. If $G = \tau F$ then

$$G(0) = \int_0^\pi F^*(\theta)\tau(\theta) \sin^{2\nu} \theta d\theta,$$

$$[G(0)]^2 \leqq AC^2 \int_0^\pi F^*(\theta)^2 \sin^{2\nu} \theta d\theta = AC^2 \sum_0^\infty F(n)^2 \omega_\nu(n) \leqq AC^2 \mathfrak{N}_\alpha^\nu[F]^2.$$

Thus it is sufficient to show that

$$\sum_1^\infty G(n)^2 \omega_\nu(n)(n+1)^{2\alpha} \leqq AC^2 \mathfrak{N}_\alpha^\nu[F]^2.$$

Let $\delta_\mu^*(\theta)$ be $\tau(\theta)F^*(\theta)$ if $\theta \in S_\mu$ and zero otherwise. We define

$$\delta_\mu(n) = \int_0^\pi \delta_\mu^*(\theta) W_\nu(n, \cos\theta) \sin^{2\nu} \theta d\theta.$$

Since $G(n) = \sum_{-\infty}^{\prime\infty} \delta_\mu(n)$ it is enough to show that if

$$G_M(n) = \sum_1^M \delta_\mu(n), \qquad G_{-M} = \sum_{-M}^{-1} \delta_\mu(n)$$

then

$$\sum_{n=1}^\infty G_{\pm M}(n)^2 \omega_\nu(n)(n+1)^{2\alpha} \leqq AC^2 \mathfrak{N}_\alpha^\nu[F]^2, \tag{5}$$

where A depends only upon ν and α and not upon M. Let $p(z)$, $q(n)$, and $Q(x, y)$ be defined as in §2 and §3. Then

$$\sum_1^\infty G_M(n)^2 \omega_\nu(n)(n+1)^{2\alpha} \approx \sum_1^\infty G_M(n)^2 \omega_\nu(n) q(n) \approx \sum_{\mu=1}^M \sum_{n=1}^\infty \delta_\mu(n)^2 \omega_\nu(n) q(n)$$
$$+ \sum_{\lambda,\mu=1;\lambda\neq\mu}^M \sum_{n=1}^\infty \delta_\mu(n)\delta_\lambda(n)\omega_\nu(n)q(n).$$

By Corollary 2b

$$I_{\mu,\lambda} = \sum_{n=1}^\infty \delta_\mu(n)\delta_\lambda(n)\omega_\nu(n)q(n)$$
$$= \frac{1}{2}\int_0^\pi \int_0^\pi [\delta_\mu^*(\theta) - \delta_\mu^*(\phi)][\delta_\lambda^*(\theta) - \delta_\lambda^*(\phi)] Q(\cos\theta, \cos\phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi$$
$$= -\int_{S_\mu}\int_{S_\lambda} \delta_\mu^*(\theta)\delta_\lambda^*(\phi) Q(\cos\theta, \cos\phi) \sin^{2\nu}\theta \sin^{2\nu}\phi d\theta d\phi,$$

and from this it follows that

$$(6)\quad \begin{aligned} |I_{\mu,\lambda}| \leqq A \int_{S_\mu} \delta_\mu^*(\theta)^2 \sin^{2\nu}\theta d\theta \int_{S_\lambda} K(\theta,\phi)\sin^{2\nu}\phi d\phi \\ + A\int_{S_\lambda} \delta_\lambda^*(\phi)^2 \sin^{2\nu}\phi d\phi \int_{S_\mu} K(\theta,\phi)\sin^{2\nu}\theta d\theta. \end{aligned}$$

If S_μ' is the complement of S_μ in $[0, \pi]$ then

$$\begin{aligned} &\sum_{\lambda=1;\lambda\neq\mu}^{\infty} \int_{S_\mu} \delta_\mu^*(\theta)^2 \sin^{2\nu}\theta d\theta \int_{S_\lambda} K(\theta,\phi)\sin^{2\nu}\phi d\phi \\ &\leqq \int_{S_\mu} \delta_\mu^*(\theta)^2 \sin^{2\nu}\theta d\theta \int_{S_{\mu'}} K(\theta,\phi)\sin^{2\nu}\phi d\phi, \\ &\leqq A\int_{S_\mu} \delta_\mu^*(\theta)^2 \sin^{2\nu}\theta\left\{|\theta - t_\mu|^{-2\alpha} + |\theta - t_{\mu+1}|^{-2\alpha}\right\} d\theta \\ &\leqq AC^2 \int_{S_\mu} F^*(\theta)^2 \sin^{2\nu}\theta\left\{|\theta - t_\mu|^{-2\alpha} + |\theta - t_{\mu+1}|^{-2\alpha}\right\} d\theta. \end{aligned}$$

The last inequality coming from the fact that $|\delta_\mu^*(\theta)| \leqq C|F^*(\theta)|$. Applying Lemma 4d we have

$$\sum_{\mu=1}^{\infty} \sum_{\lambda=1;\lambda\neq\mu}^{\infty} \int_{S_\mu} \delta^*(\theta)^2 \sin^{2\nu}\theta d\theta \int_{S_\lambda} K(\theta,\phi)\sin^{2\nu}\phi d\phi \leqq AC^2 \mathfrak{N}_\alpha^\nu[F]^2.$$

Since the second term in (6) differs from the first only in that μ and λ are interchanged we have proved that

$$\sum_{\lambda,\mu=1;\mu\neq\lambda}^{\infty} |I_{\mu,\lambda}| \leqq AC^2 \mathfrak{N}_\alpha^\nu[F]^2.$$

Next let

$$I_\mu = \sum_{n=1}^{\infty} \delta_\mu(n)^2 \omega_\nu(n) q(n).$$

Let F_μ be defined as in Lemma 4c, and set

$$F_\mu(\eta, n) = \int_0^\eta F_\mu^*(\theta) W_\nu(n, \cos\theta)\sin^{2\nu}\theta d\theta.$$

By Lemma 4b

$$(7)\quad \mathfrak{N}_\alpha^\nu[F_\mu(\eta, n)] \leqq A\mathfrak{N}_\alpha^\nu[F_\mu].$$

If $u(\eta) = \tau(\eta)/\rho_\mu(\eta)$ then since $F_\mu^*(\theta) = \rho_\mu(\theta)F(\theta)$ we have

$$\delta_\mu(n) = \int_{S_\mu} u(\eta) dF_\mu(\eta, n).$$

Integrating by parts we obtain

$$\delta_\mu(n) = -\int_{S_\mu} F(\eta, n) du(\eta) + [u(\eta) F_\mu(\eta, n)]_{t_{\mu+1}}^{t_\mu}.$$

Using (7) we find that

$$I_\mu \leqq A \mathfrak{N}_\alpha^\nu [F]^2 \left\{ \int_{S_\mu} |du(\eta)| + |u(t_\mu)| + |u(t_{\mu+1})| \right\}^2.$$

It is easily verified that

$$\int_{S_\mu} |du(\eta)| + |u(t_\mu)| + |u(t_{\mu+1})| \leqq AC^2$$

and hence

$$I_\mu \leqq AC^2 \mathfrak{N}_\alpha^\nu [F_\mu]^2.$$

It follows using Lemma 4d that

$$\sum_{\mu=1}^{M} I_\mu \leqq AC^2 \mathfrak{N}_\alpha^\nu [F]^2.$$

We have thus proved the first of the inequalities (5). The second is established in exactly the same way, and our theorem is proved for $0 < \alpha < 1/2$.

If $-1/2 < \alpha < 0$ then $F^*(\theta)$ is not necessarily defined and the formula (4) may be without meaning. However it is meaningful if $F(n) \in \mathfrak{N}_0^\nu \cap \mathfrak{N}_\alpha^\nu$. A familiar duality argument shows that for such an $F \mathfrak{N}_\alpha^\nu[\tau F] \leqq A(\nu, \alpha) \mathfrak{N}_\alpha^\nu[F]$. Since $\mathfrak{N}_0^\nu \cap \mathfrak{N}_\alpha^\nu$ is dense in $\mathfrak{N}_\alpha^\nu$ t has a unique extension (as a bounded linear transformation) to all of $\mathfrak{N}_\alpha^\nu$.

An application of Theorem 5e to the theory of "fractional differences" is stated without proof in (6). The missing demonstration can however easily be supplied using (4) as a model.

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