THE FINITE CONVOLUTION TRANSFORM

BY

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1. **Introduction.** There exists a large literature concerning convolution transforms on the whole real line, but the corresponding problem for semi-infinite intervals has received little attention. The problem is that of determining a function ϕ from the relation

$$f(x) = \int_0^x \phi(x - t)K(t)dt, \qquad 0 \leqq x < \infty \tag{1}$$

given f and the kernel K.

If ϕ and K have suitable behavior at infinity the classical solution by Laplace transforms is available. This method of solution, however, determines ϕ from a knowledge of $f(x)$ for all x in $0 \leqq x < \infty$, whereas the form of (1) suggests that the determination of $\phi(x)$ in an interval $0 \leqq x < a$ should use the values of $f(x)$ only in this interval. This suggestion is confirmed by a theorem of Titchmarsh [1, p. 327] concerning the equation

$$0 = \int_0^x \phi(x - t)K(t)dt, \qquad 0 \leqq x < a, \tag{2}$$

where a is a fixed positive number. It is a consequence of his theorem that *if ϕ and K are locally integrable and $K(x) \not\equiv 0$ in a neighborhood $0 \leqq x < \eta$ of the origin, then the only solution of* (2) *is $\phi \equiv 0$.*

The main object of this paper is to give a solution of the equation in (1) for an interval $0 \leqq x < a$. Clearly this solves the problem for the whole interval $0 \leqq x < \infty$. There is no added difficulty in solving the more general equation

$$f(x) = \int_0^x \phi(x - t)dk(t), \qquad 0 \leqq x < a, \tag{3}$$

and this is what we shall do. It is supposed that $k(t)$ is of bounded variation in each interval $0 \leqq x \leqq a_1$, $0 < a_1 < a$, and that ϕ is Borel measurable in $0 \leqq x < a$.

2. **Assumptions.** In solving (3) the following assumptions will be made about k and ϕ.

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A1. k has positive variation near 0;
2. $k(0)=0$;
3. k is real-valued.
B1. $\phi \in C^2$ $(0 \leqq x < a)$;
2. $\phi(0)=\phi'(0)=0$.

Assumption A1 is necessary since otherwise (3) does not determine ϕ. A2 can be achieved by replacing $k(t)$ with $k(t)-k(0)$ in (3). If k is not real-valued to begin with, (3) can be replaced by a new equation of the same form for which the kernel *is* real-valued. The procedure is this: Define

$$k_1(x) = \int_0^x k(x-t)d\bar{k}(t)$$

and

$$f_1(x) = \int_0^x f(x-t)d\bar{k}(t),$$

where $\bar{k}$ is the complex-conjugate of k. Then

$$f_1(x) = \int_0^x \phi(x-t)dk_1(t), \qquad 0 \leqq x < a,$$

and k_1 is real because of A2.

Assumptions B1 and B2 are satisfied by taking the precautionary measure of integrating (3) from 0 to x three times before starting.

3. **A study of the kernel** k. Because of A1 we can choose a number b, $0<b<a/2$ so that

$$B = \int_0^{2b} dk(u) \neq 0. \tag{4}$$

An important step in solving (3) is the determination of $\phi(x)$ in $0 \leqq x < b$, and thus depends on a preliminary study of the Laplace transform

$$\hat{k}(s) \equiv \hat{k}_b(s) = \int_0^{2b} e^{-su}dk(u). \tag{5}$$

$\hat{k}$ is an entire function. Because of (4) it has the form

$$\hat{k}(s) = Be^{-c's}\prod_z \left(1-\frac{s}{z}\right)e^{s/z}, \tag{6}$$

where c' is a constant and the product is taken over the roots z of $\hat{k}$. Since k is real the roots of $\hat{k}$ appear in conjugate-complex pairs unless they are real. Let α denote the real zeros and β the complex zeros with positive imaginary part.

$\hat{k}(s)$ is bounded in the right half-plane and $e^{2bs}\hat{k}(s)$ in the left. Consequently the two series

$$\sum_{\alpha} \frac{1}{|\alpha|} \quad \textit{and} \quad \sum_{\beta} \frac{\cos(\arg \beta)}{|\beta|} \tag{7}$$

converge, [2, p. 131]. (6) may therefore be written

$$\hat{k}(s) = Be^{-cs} \prod_{\alpha} \left(1 - \frac{s}{\alpha}\right) \prod_{\beta} \left(1 - \frac{s}{\beta}\right)\left(1 - \frac{s}{\bar{\beta}}\right). \tag{8}$$

An argument of Titchmarsh [1, p. 323], shows that because of A1 and (5) c is real and satisfies

$$0 \leqq c \leqq b. \tag{9}$$

It will be necessary to arrange the product (8) in another order, the justification coming from the convergence of the two series (7). Define this function of a complex variable w:

$$S(w) = \begin{cases} (2[(Iw)^2 - (Rw)^2])^{1/2}, & |Iw| \geqq |Rw|, \\ -1, & |I(w)| < |Rw|. \end{cases}$$

The roots of $\hat{k}$ are decomposed into classes A_n, $n \equiv 0, 1, 2, \cdots$ according to the rule

$$z \in A_n \textit{ if } n - 1 \leqq S(z) < n.$$

The following lemma is trivial.

LEMMA 1. *If t is real then $|1 - it/z| \geqq 1$ for real zeros z. Also*

$$\left|\left(1 - \frac{it}{z}\right)\left(1 - \frac{it}{\bar{z}}\right)\right| \geqq 1$$

if and only if $|t| \geqq S(z)$.

(8) may now be rearranged as

$$k(s) = Be^{-cs} \prod_{j=0}^{\infty} \prod_{z \in A_j}{}' \left(1 - \frac{s}{z}\right)\left(1 - \frac{s}{\bar{z}}\right).$$

An empty product is interpreted as unity and the prime ′ in the inner product means that only a single factor is written when z is real. (This can occur only for $j = 0$.) For a fixed j the order of interior factors does not matter and we regard them as fixed once and for all, say by absolute value. For a fixed j the zeros z belonging to A_j will be denoted generally by $z_1, z_2, \cdots$, thus avoiding the use of a double subscript. The last formula is conveniently broken up into

$$g_j(s) = \prod_{z\in A_j}\left(1-\frac{s}{z}\right)\left(1-\frac{s}{\bar{z}}\right), \tag{10}$$

$$\hat{k}(s) = Be^{-cs}\prod_{j=0}^{\infty} g_j(s).$$

LEMMA 2. *If* $|t| \geqq j$ *then* $|g_k(it)| \geqq 1$, $k=0, 1, \cdots, j$. *If* $|t| \leqq j$ *then* $|g_k(it)| \leqq 1$, $k=j+1, j+2, \cdots$.

Proof. If $|t| \geqq j$ then for all $k \leqq j$ we have $|t| \geqq j \geqq k$ so that $|t| \geqq S(z)$ for t in A_k. By Lemma 1 each factor in the product composing g_k is at least 1 in absolute value.

For the second part suppose $|t| \leqq j$ and $k \geqq j+1$. Then $|t| \leqq j \leqq k-1$ so that $|t| \leqq S(z)$ for $z \in A_k$. Now apply Lemma 1 again.

Using (10) and Lemma 2 it is not difficult to see that

$$\left|\frac{\hat{k}(it)}{\prod_0^k g_j(it)}\right| \leqq \max\left[\,|\hat{k}(it)|,\ |B|\,\right]. \tag{11}$$

In fact by Lemma 2 the left side for t fixed will (as a function of k) first decrease then increase. It is consequently less than or equal to the larger of the values for $k=0$ and $k=\infty$.

4. **The main lemma.** With each zero z of k we associate a function $h_z(x)$ on $0 \leqq x < \infty$ as follows

$$h_z(x) = \begin{cases} |z|^2 \int_0^x e^{z(x-t)}e^{\bar{z}t}dt, & z \text{ imaginary}, \\ -ze^{zx}, & z \text{ real}. \end{cases}$$

These have the respective Laplace transforms

$$\frac{1}{1-s/z}\,\frac{1}{1-s/\bar{z}}$$

and

$$\frac{1}{1-s/z},$$

valid for $Rs > Rz$.

We use l^*m to denote the convolution

$$\int_0^x l(t)m(x-t)dt = \int_0^x m(t)l(x-t)dt$$

of two functions l and m, and $\prod^*$ to denote the convolution product of several functions. Wherever we use such a product it will be obviously associative.

Now define by induction) (3)

$$(12)\qquad \begin{aligned} H_0(x) &= \lim_{m\to\infty} f(x)^* \prod_{i=1;z_i\in A_0}^{m}{}^* h_{z_i}(x), \\ H_k(x) &= \lim_{m\to\infty} H_{k-1}(x)^* \prod_{i=1;z_i\in A_k}^{m}{}^* h_{z_i}(x), \end{aligned}$$

whenever the defining expressions have meaning. Our object in this section is to prove the

MAIN LEMMA. *The functions* $H_k(x)$ *exist for* $0 \leqq x \leqq 2b$ *and*

$$(13)\qquad \lim_{k\to\infty} H_k(x) = B\phi(x-c), \qquad c \leqq x \leqq 2b,$$

where c and B are defined by (8).

We begin by considering the new integral equation

$$(13a)\qquad f_1(x) = \int_0^x \phi_1(x-t)dk_1(t), \qquad 0 \leqq x < \infty,$$

where

$$\phi_1(x) = \begin{cases} \phi(x), & 0 \leqq x \leqq 2b, \\ \epsilon C^2, & 0 \leqq x \leqq 3b, \\ 0 & x \geqq 3b; \end{cases}$$

and

$$k_1(x) = \begin{cases} k(x), & 0 \leqq x \leqq 2b, \\ k(2b), & x \geqq 2b. \end{cases}$$

By comparison with (3)

$$(14)\qquad f_1(x) = \begin{cases} f(x), & 0 \leqq x \leqq 2b, \\ 0, & x \geqq 5b. \end{cases}$$

Taking Laplace transforms on both sides of the new integral equation we have

$$(15)\qquad \hat{f}_1(s) = \hat{\phi}_1(s)\hat{k}(s),$$

where $\hat{k}(s)$ is defined by (5) and $\hat{f}_1$, $\widehat{\phi_1}$ by

$$\int_0^{5b} e^{-su} f_1(u)du \quad \textit{and} \quad \int_0^{2b} e^{-su}\phi_1(u)du$$

(3) In (12) and similar products we take $\prod^*$ to have no effect on the other factor if the corresponding A_j is empty.

respectively. Laplace transforms in general will be denoted by the roof symbol $\widehat{\ }$.

In view of assumptions B1 and B2 we know that

$$\widehat{\phi}_1(it) = O(1/t^2), \qquad |t| \to \infty. \tag{16}$$

Since $\hat{k}(it)$ is bounded (15) yields

$$\hat{f}_1(it) = O(1/t^2), \qquad |t| \to \infty. \tag{17}$$

The Main Lemma will be a consequence of the next one. First *define the functions $L_k(x)$ just as the functions $H_k(x)$ are defined in* (12), *but replacing f by f_1*. Then by (14)

$$L_k(x) = H_k(x), \qquad 0 \leqq x \leqq 2b. \tag{18}$$

Secondly define

$$l_0(x) = \lim_{m\to\infty} \int_0^x \left(\prod_{i=1;z_i\in A_0}^{m}{}^{*} h_{z_i}(x-t) \right) dk_1(t),$$

$$l_k(x) = \lim_{m\to\infty} l_{k-1}(x) * \prod_{i=1;z_i\in A_k}^{m}{}^{*} h_{z_i}(x).$$

LEMMA 3. *The limits defining $l_k(x)$ exist for $k=0, 1, \cdots$ boundedly in $0 \leqq x < \infty$; the $l_k(x)$ vanish for $x \geqq 2b$ and their Laplace transforms are*

$$\hat{l}_k(s) = \hat{k}(s) \Big/ \prod_{j=0}^{k} g_j(s), \tag{19}$$

where g_j is defined as in (10). *Moreover the functions $L_k(x)$ exist for $k=0, 1, \cdots$, $0 \leqq x < \infty$ and equal the absolutely convergent integrals*

$$L_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \widehat{\phi}_1(it) \hat{l}_k(it) dt. \tag{20}$$

This will be proved in the next section. Before doing so we show how it is used in proving the Main Lemma. By combining (18), (19) and (20) we have

$$H_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \widehat{\phi}_1(it) \frac{\hat{k}(it)}{\prod_{j=0}^{k} g_j(it)} dt, \qquad 0 \leqq x \leqq 2b.$$

By virtue of (11) and (16) we may let $k \to \infty$ under the integral. (10) tells us then that for $x > c$

$$\lim_{k\to\infty} H_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \widehat{\phi}_1(it) B e^{-cit} dt = B\phi_1(x-c).$$

Since ϕ_1 and ϕ agree on $0 \leqq x \leqq 2b$, this establishes (13).

5. **Proof of Lemma 3.** The assertion about $l_k(x)$ will be proved by induction on k. We start with $k=0$. Define for $m \geqq 1$

$$l_0^{(m)}(x) = \int_0^x \left(\prod_{i=1;z_i \in A_0}^{m}{}^{*} h_{z_i}(x-t) \right) dk_1(t).$$

Taking Laplace transforms we obtain

$$\hat{l}_0^{(m)}(s) = \hat{k}_1(s) \left\{ \prod_{i=1;z_i \in A_0}^{m}{}' \left(1 - \frac{s}{z_i}\right)\left(1 - \frac{s}{\bar{z}_i}\right)\right\}^{-1}. \tag{21}$$

By the complex inversion formula for Laplace transforms

$$l_0^{(m)}(x) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{sx}\hat{k}(s)\{ \cdots \}^{-1} ds$$

if $d > Rz_1, Rz_2, \cdots, Rz_m$. By Cauchy's theorem we may move the line of integration to $d=0$ since the integrand is entire and sufficiently small at $\pm i\infty$. Hence

$$l_0^{(m)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx}\hat{k}(it) \left\{ \prod_{j=1}^{m}{}' \left(1 - \frac{it}{z_j}\right)\left(1 - \frac{it}{\bar{z}_j}\right)\right\}^{-1} dt.$$

By Lemma 1 $\lim_{m\to\infty} l_0^{(m)}(x)$ exists boundedly as $m \to \infty$.

To prove that $l_0(x)$ vanishes outside $(0, 2b)$ it is enough to prove it for each $l_0^{(m)}(x)$. We prove it by induction noting that

$$l_0^{(1)}(x) = \int_0^x h_{z_1}(x-t)dk_1(t),$$

$$l_0^{(m)}(x) = l_0^{(m-1)}(x) * h_{z_m}(x).$$

Consider first $l_0^{(1)}(x)$. If z_1 is real we have

$$l_0^{(1)}(x) = z_1 \int_0^x e^{(x-t)z_1} dk_1(t),$$

$$= z_1 e^{xz_1} \int_0^x e^{-tz_1} dk_1(t).$$

Now if $x \geqq 2b$ the last integral is

$$\int_0^{2b} e^{-tz_1} dk(t)$$

since $k_1(x)$ and $k(x)$ agree on $(0, 2b)$. But this vanishes since it is $k(z_1)$ and z_1 is a zero of $k(s)$. If z_1 is not real we note that

$$h_{z_1}(x) = (-z_1 e^{z_1 x})*(-\bar{z}_1 e^{\bar{z}_1 x})$$

so that a similar argument applies. Remember that $\bar{z}_i$ is also a zero of $\hat{k}(s)$.

Now suppose, to continue this induction that $l_0^{(m-1)}(x)$ vanishes for $x \geqq 2b$. We have

$$l_0^{(m)}(x) = \int_0^x h_{z_m}(x-t) l_0^{(m-1)}(t) dt.$$

If $x \geqq 2b$

$$l_0^{(m)}(x) = \int_0^{2b} h_{z_m}(x-t) l_0^{(m-1)}(t) dt.$$

The argument used for $m=1$ still holds provided z_m and $\bar{z}_m$ are zeros of $\hat{l}_0^{(m-1)}(s)$. Since they are zeros of $k(s)$, this follows from (21) with m replaced by $m-1$.

To complete the case $k=0$ we must establish (19), i.e. that

$$\hat{l}_0(s) = \hat{k}(s)/g_0(s).$$

We know that

$$\hat{l}_0^{(m)}(s) = \int_0^{2b} e^{-su} l_0^{(m)}(u) du,$$

since $l_0^{(m)}$ vanishes outside $(0, 2b)$. By bounded convergence

$$\lim_{m\to\infty} \hat{l}_0^{(m)}(s) \int_0^{2b} e^{-su} l_0(u) du = \hat{l}_0(s).$$

Now compare (21).

To continue the induction on k *suppose the assertions of the lemma hold for* $k-1$. Then defining

$$l_k^{(m)}(x) = l_{k-1}(x)* \prod_{i=1;z_i\in A_k}^{m} h_{z_i}(x) \tag{22}$$

we know that

$$\lim_{m\to\infty} l_{k-1}^{(m)}(x) = l_{k-1}(x) \text{ \textit{boundedly},}$$

that

$$\hat{l}_{k-1}(s) = \int_0^{2b} e^{-su} l_{k-1}(u) du, \tag{23}$$

and that

$$\hat{l}_{k-1}(s) = \frac{\hat{k}(s)}{\prod_{j=0}^{k-1} g_i(s)}. \tag{24}$$

According to (22) we have

$$\hat{l}_k^{(m)}(s) = \hat{l}_{k-1}(s)\left\{ \prod_{i=1;z_i\in A_k}^{m}{}' \left(1 - \frac{s}{z_i}\right)\left(1 - \frac{s}{\bar{z}_i}\right)\right\}^{-1}.$$

We now complete the proof as for $k=0$ with the following changes. At the stage where Cauchy's theorem is used to move the line of investigation use the boundedness of $\hat{l}_{k-1}(s)$ in $0 \leqq Rs \leqq d$. This follows from (23). To prove the bounded existence of $\lim_{m\to\infty} l_k^{(m)}(x)$ use Lemma 2 in place of Lemma 1. To prove the vanishing of $l_k^{(m)}(x)$ outside of $(0, 2b)$ we employ an induction as before, noting that by (24) the elements of A_k are zeros of $\hat{l}_{k-1}(s)$. The details are left to the reader.

It remains to prove the assertion of Lemma 3 concerning $L_k(x)$. By (13a) and the associative property of convolution it follows from the definition of $L_k(x)$ (see the sentence preceding (18)) that

$$L_0(x) = \lim_{m\to\infty} \phi_1 * l_0^{(m)}(x),$$

so that by Lemma 3

$$L_0(x) = \phi_1 * l_0(x).$$

By induction

$$L_k(x) = \phi_1 * l_k(x), \tag{25}$$

for suppose

$$L_{k-1}(x) = \phi_1 * l_{k-1}(x).$$

Then

$$\begin{aligned} L_k(x) &= \lim_{m\to\infty} L_{k-1}(x) * \prod_{i=1;z_i\in A_k}^{m} h_{z_i}(x) \\ &= \lim_{m\to\infty} \phi_1(x) * l_{k-1}(x) * \prod_{i=1;z_i\in A_k}^{m} h_{z_i}(x) \\ &= \lim_{m\to\infty} \phi_1(z) * l_k^{(m)}(x) \\ &= \phi_1(x) * l_k(x), \end{aligned}$$

by Lemma 3 again. This completes the inductive proof of (25). By (25)

$$\hat{L}_k(s) = \widehat{\phi}_1(s)\hat{l}_k(s),$$

$$L_k(x) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{sx}\widehat{\phi}_1(s)\hat{l}_k(s)ds$$

for $d>0$ and now (20) follows since the line of integration can be moved once again to $d=0$.

6. Recapitulation. In view of the mass of details let us restate where matters stand. We begin with (3) as given.

THEOREM 1. *Let b be a number such that $0<b<a/2$ and such that* (4) *holds. Define $\hat{k}$ by* (5) *and c as in* (8). *Arrange the roots of $\hat{k}$ into the classes A_n described in* §3. *Define $h_z(x)$ as in* §4 *and $H_k(x)$ as in* (12). *Then* (3) *has the solution*

$$\lim_{k\to\infty} \frac{1}{B} H_k(x+c) = \phi(x), \qquad 0 \leqq x \leqq b. \tag{26}$$

(26) is a restatement of (13) and (9). The solution is valid in the (possibly) larger interval $0\leqq x\leqq 2b-c$, but this fact will not be useful for our ultimate goal which is a solution valid in $0\leqq x<a$. This is described in the following section.

Note that (26) uses the values of $f(x)$ in at most the interval $0\leqq x\leqq 2b$. This is due to the definition of $H_0(x)$ in (12).

7. The solution completed. It is now our object to show how the integral equation in (3) can be solved in any interval $0\leqq x\leqq a-\eta$, $0<\eta<a$, by a finite number of repetitions of the procedure described in Theorem 1. Choose b so small that (4) holds, and that two of the successive numbers $b, 2b, 3b, \cdots$ fall into the interval $a-\eta\leqq x<a$, say

$$a - \eta \leqq Nb < (N+1)b < a. \tag{27}$$

If $N=1$ Theorem 1 accomplishes the solution so suppose $N>1$.

We shall show how to use the procedure of Theorem 1 to find $\phi(x)$ in an interval $0\leqq x\leqq (n+1)b$, given the solution in $0\leqq x\leqq nb$, $1\leqq n\leqq N-1$. Since ϕ is given in $0\leqq x\leqq b$ by Theorem 1, an N-fold repetition of the procedure gives the solution in $0\leqq x\leqq Nb$ and this includes the prescribed interval $0\leqq x\leqq a-\eta$.

Define

$$f_n(x) = f(x) + \int_0^{nb} \phi(t)dk(x-t).$$

(3) takes the form

$$f_n(x) = -\int_{nb}^{x} \phi(t)dk(x-t),$$

valid if $nb \leqq x < a$. Replace x by $x+nb$. Then

$$f_n(x + nb) = -\int_{nb}^{x+nb} \phi(t)dk(x + nb - t),$$

valid if $0 \leqq x < a - nb$. Changing variables we find that

$$f_n(x + nb) = \int_0^x \phi(s + nb - t)dk(t)$$

valid in $0 \leqq x < a - nb$, hence, by (27) in $0 \leqq x \leqq 2b$. This final integral equation is of the same type as the original, so that the method of Theorem 1 applied with $f_n(x+nb)$ in place of $f(x)$ yields $\phi(x+nb)$ in $0 \leqq x \leqq b$, and hence $\phi(x)$ in $nb \leqq x \leqq (n+1)b$. Since $\phi(x)$ is known in $0 \leqq x \leqq nb$, this completes the program.

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