

SOME THEOREMS ON L^p FOURIER SERIES

BY

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1. The almost everywhere convergence of lacunary sequences of partial sums has long been known for L^2 Fourier series [8, p. 251]. Somewhat later the result was established for class L^p by Littlewood and Paley [4] and for class H by Zygmund, [6] and [7]. In the papers of Zygmund, he also extended to class H an unpublished result of Paley concerning L^p series. This stated that for f in L^p , $p > 1$, and for almost every x , the positive integers can be divided into complementary sequences, $\{m_\nu\}$ and $\{n_\nu\}$, in general depending on x , such that $s_{m_\nu}(x; f)$ converges to $f(x)$ and such that $\sum 1/n_\nu < \infty$. Here $s_m(x; f)$ denotes as usual the m th partial sum of the Fourier series of f at x . The sequence $\{m_\nu\}$ depends then on both the function f and the point x . Results intermediate between these have recently appeared [1] in which the sequence of indices of the partial sums for which convergence takes place are more dense in a certain sense than lacunary sequences, less dense than those described immediately above; and they depend on the function f but not on the point x . In particular, for each f in L^2 , there is a sequence $\{m_\nu\}$ of upper density one such that $s_{m_\nu}(x; f)$ converges to f almost everywhere.

In this paper we shall generalize our initial results on L^p series, which were largely based on the Hausdorff-Young theorem. Much stronger results will be obtained by the use of powerful theorems of Littlewood and Paley. In the next section we give our preliminary lemmas; in the third section our two main theorems; and in the final section additional results which are mainly corollaries of our main results.

2. Our theorems depend to a large extent on a certain result of Littlewood and Paley [4, II], which we now describe. Let $\{m_k\}$ and $\{n_k\}$ be two sequences of positive integers satisfying $1 < \alpha \leq m_{k+1}/m_k \leq \beta$, $1 < \alpha \leq n_{k+1}/n_k \leq \beta$ for fixed α, β . Let f belong to L^p , $p > 1$, and let $\sum_{m=-\infty}^{+\infty} c_m e^{imx}$ be its Fourier series. We may assume throughout that $c_0 = 0$. For $k > 0$, let $\Delta_k(x) = \sum_{\nu=n_{k-1}+1}^{n_k} c_\nu e^{i\nu x}$; $\Delta_{-k}(x) = \sum_{\nu=-m_{k-1}-1}^{-m_k} c_\nu e^{i\nu x}$ if it is understood that $m_0 = n_0 = 0$. The theorem states that

$$(1) \quad A_{p,\alpha,\beta} \int_0^{2\pi} |f(x)|^p dx \leq \int_0^{2\pi} \left(\sum_{k=-\infty}^{+\infty} |\Delta_k(x)|^2 \right)^{p/2} dx \leq B_{p,\alpha,\beta} \int_0^{2\pi} |f(x)|^p dx$$

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where $A_{p,\alpha,\beta}, B_{p,\alpha,\beta}$ are constants depending only on their subscripts.

In general it will be sufficient to consider only those f such that $c_n = 0$ for $n \leq 0$; and we do this now. Letting $n_k = \lambda^k$ for λ an integer exceeding 1, we have that the constants A and B in (1) depend on p and λ . We shall be interested in blocks of consecutive $\Delta_k(x)$'s. Thus, for $K_2 > K_1$ by (1)

$$(2) \int_0^{2\pi} \left(\sum_{k=K_1+1}^{K_2} |\Delta_k(x)|^2 \right)^{p/2} dx \leq B_{\lambda,p} \int_0^{2\pi} |s_{\lambda^{K_2}}(x;f) - s_{\lambda^{K_1}}(x;f)|^p dx.$$

The constant $B_{\lambda,p}$ may be large, but by the well known fact of the mean convergence of L^p Fourier series, the right side of (2) may be made arbitrarily small for fixed λ by taking K_1 and K_2 sufficiently large. We let

$$\varepsilon_k = \int_0^{2\pi} |\Delta_k(x)|^p dx, \quad \delta = \int_0^{2\pi} \left(\sum_{k=K}^{2K-1} |\Delta_k(x)|^2 \right)^{p/2} dx.$$

For a real number y , let $\langle y \rangle$ denote the greatest integer not exceeding y . The principle involved in our first lemma will be useful in all our proofs.

LEMMA 1. For $\beta > 0$, among the K numbers ε_k , no more than $\langle K/\beta \rangle$ of them exceed $\beta\delta K^{-p/2}$.

The proof follows directly from Hölder's inequality which gives

$$\sum_{k=K}^{2K-1} |\Delta_k(x)|^p \leq K^{1-p/2} \left(\sum_{k=K}^{2K-1} |\Delta_k(x)|^2 \right)^{p/2}.$$

Integration of this inequality gives

$$\sum_{k=K}^{2K-1} \varepsilon_k \leq K^{1-p/2} \delta.$$

If N of the numbers ε_k exceed $\beta\delta K^{-p/2}$, then from the above $N\beta\delta K^{-p/2} \leq K^{1-p/2} \delta$; or $N \leq K/\beta$ as required.

We shall also need another result of Littlewood and Paley [4, III].

LEMMA 2. If f belongs to L^2 , then

$$\int_0^{2\pi} \sup_n \left\{ \frac{|s_n(x;f)|^2}{\log(n+2)} \right\} dx \leq A_2 \int_0^{2\pi} |f(x)|^2 dx.$$

The original proof covers the L^p case, $1 < p \leq 2$, and is complicated. A relatively simple proof covers the L^2 case and will be found in [2]. The L analogue of Lemma 2 is false [7] although it is well known, for example, that almost everywhere $s_n(x;f) = o(\log n)$ [8, p. 32]. However we shall require the somewhat different result of the following lemma, which involves the function

$$s_n^*(x; f) = \sup_{m \leq n} |s_m(x; f)|.$$

LEMMA 3. Given $M > 0$ and f in L , let E_M be the set of x values in $[0, 2\pi]$ for which $s_n^*(x; f) > M \log(n+2)$. Then E_M has measure not exceeding $AM^{-1} \int_0^{2\pi} f(x) dx$, where A is a constant.

Let $F = \int_0^{2\pi} |f(t)| dt$. If $F/M > 2\pi(n+2)$, then the condition on $|E_M|$, the measure of E_M , is automatically satisfied with $A \geq 1$ since $|E_M| \leq 2\pi$. If, on the other hand, $2\pi(n+2)F < M$, then E_M has measure 0 since

$$|s_m(x; f)| \leq \frac{1}{\pi} \int_0^{2\pi} |f(t)| |D_m(x-t)| dt \leq (n+1) \frac{F}{\pi}, \quad m \leq n$$

where $D_m(x)$ is the Dirichlet kernel. Hence we may assume that

$$(2\pi(n+2))^{-1} \leq FM^{-1} \leq 2\pi(n+2).$$

Let Q be the least integer exceeding both MF^{-1} and $4(n+2)FM^{-1}$ so that $1 \leq Q \leq 8\pi(n+2)^2 + 1$. Let I_ν be the interval $2\pi\nu Q^{-2} \leq x \leq 2\pi(\nu+1)Q^{-2}$, $\nu = 0, 1, \dots, Q^2 - 1$. Then, except possibly for Q values of ν ,

$$\int_{I_\nu} |f(t)| dt \leq FQ^{-1}.$$

Let G_1 be the exceptional set of I_ν 's. Then, $|G_1| \leq 2\pi Q^{-1}$. Let G_2 be the set of intervals of length $2\pi Q^{-3}$ situated symmetrically about the points $2\pi\nu Q^{-2}$, $\nu = 0, 1, \dots, Q^2 - 1$. Then $|G_2| \leq 2\pi Q^{-1}$. If x belongs to the complement of $G_1 \cup G_2$ and also belongs to I_ν , then

$$\int_{0 \leq |x-t| \leq \pi/Q^3} |f(t)| dt \leq \int_{I_\nu} |f(t)| dt \leq FQ^{-1}.$$

For $m \leq n$,

$$\begin{aligned} |s_m(x; f)| &\leq \frac{(n+1/2)}{\pi} \int_{0 \leq |x-t| \leq \pi/Q^3} |f(t)| dt + \frac{1}{2} \int_{\pi/Q^3 \leq |x-t| \leq \pi} \frac{|f(t)|}{|x-t|} dt \\ &= T_1(x) + T_2(x). \end{aligned}$$

Since $\int_0^{2\pi} T_2(x) dx \leq 2F \log Q^3$, then $2T_2(x) \leq M \log(n+2)$ for x outside a set G_3 of measure not exceeding $12F \log Q / M \log(n+2)$. For x outside $G_1 \cup G_2$

$$T_1(x) \leq \frac{n+1/2}{\pi} \frac{F}{Q} \leq \frac{M}{4} < \frac{M}{2} \log(n+2).$$

Thus for x in the complement of $G_1 \cup G_2 \cup G_3$, $s_n^*(x; f) \leq M \log(n+2)$. The measure of this set does not exceed

$$\frac{4\pi}{Q} + \frac{12F \log Q}{M \log(n+2)} \leq \frac{4\pi F}{M} + \frac{12F}{M \log(n+2)} \log [8\pi(n+2)^2 + 1] \leq \frac{CF}{M}$$

for some constant C .

Lemma 3 leads in an easy way to a weak analogue of Lemma 2; thus if f belongs to L ,

$$\int_0^{2\pi} [s_n^*(x; f)]^r dx \leq C_r [\log(n + 2)]^r \left[\int_0^{2\pi} |f(x)| dx \right]^r, \quad 0 < r < 1.$$

To prove this we may assume that $s_n^*(x; f) \geq F \log(n + 2)$. Let G_m be the set of x values for which $s_n^*(x; f) \geq Fm \log(n + 2)$. By Lemma 3, $|G_m| \leq Am^{-1}$. Then our result follows from the sequence of inequalities

$$\begin{aligned} \int_0^{2\pi} [s_n^*(x; f)]^r dx &\leq [\log(n + 2)]^r F^r \sum_{m=1}^{\infty} (m + 1)^r [|G_m| - |G_{m+1}|] \\ &\leq [2 \log(n + 2)]^r F^r \sum_{m=1}^{\infty} m^{r-1} |G_m| \leq A [2 \log(n + 2)]^r F^r \sum_{m=1}^{\infty} m^{r-2}. \end{aligned}$$

3. Our first theorem deals with the density function for a sequence of distinct positive integers: given $\{m_\nu\}$, let $\sigma(n)$ be the number of terms of the sequence not exceeding n . For the L^2 case we were able to prove that $s_{m_\nu}(x; f)$ converged to f almost everywhere for some sequence of upper density one: i.e. such that $\limsup \sigma(n)/n = 1$ [1, p. 396]. Here the corresponding result for L^p , $1 < p < 2$, is not the same, but it merges, so to speak, with the L^2 result. We shall say that a sequence $\{m_\nu\}$ satisfies condition C_p if

$$(C_p) \quad \limsup_{n \rightarrow \infty} \frac{(\log n)^{(2-p)/2(p-1)} \sigma(n)}{n} \geq 1.$$

THEOREM 1. *If f belongs to L^p , $1 < p \leq 2$, there is a sequence $\{m_\nu\}$ satisfying (C_p) such that $s_{m_\nu}(x; f)$ converges to f almost everywhere.*

We assume first that $c_n = 0$ for $n \leq 0$. Let $\{\lambda_r\}$ and $\{k_r\}$ be sequences of positive integers, in general large; and let $n_k = \lambda_r^k$, $k = k_r, k_r + 1, \dots, 2k_r$. Let

$$\int_0^{2\pi} \left(\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 \right)^{p/2} dx = \delta_r, \quad \Delta_k(x) = \sum_{n=n_{k+1}}^{n_{k+1}} c_n e^{inx}.$$

As noted in the previous section, δ_r depends not only on f but also on λ_r and k_r ; but it can be made as small as we please, given λ_r , by proper choice of k_r . Let E_r be the set of x values in $[0, 2\pi]$ for which

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 > (k_r)^{1/(p-1)}.$$

From the preceding, $|E_r|$, the measure of E_r , does not exceed $\delta_r k_r^{-q/2}$ where q is the conjugate of p , i.e. $p^{-1} + q^{-1} = 1$. Let $\gamma = (2 - p)/2(p - 1)$. On E_r' , the complement of E_r ,

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 \leq k_r^\gamma \left(\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 \right)^{p/2}.$$

Integrating this inequality over E'_r , we obtain

$$\sum_{k=k_r}^{2k_r-1} \int_{E'_r} |\Delta_k(x)|^2 dx \leq k_r^\gamma \delta_r.$$

By the same reasoning as that used in the proof of Lemma 1, no more than $\langle k_r/3 \rangle$ of the numbers $\int_{E'_r} |\Delta_k(x)|^2 dx$, $k = k_r, \dots, 2k_r - 1$, exceed $3k_r^{\gamma-1} \delta_r$. By Lemma 1 itself, no more than $\langle k_r/3 \rangle$ of the numbers $\int_0^{2\pi} |\Delta_k(x)|^p dx$, $k = k_r, \dots, 2k_r - 1$, exceed $3k_r^{-p/2} \delta_r$. Hence, for at least one k , say $k(r)$ satisfying $k_r \leq k(r) \leq 2k_r - 1$, the following simultaneous inequalities hold.

$$(3) \quad \int_{E'_r} |\Delta_{k(r)}(x)|^2 dx \leq 3k_r^{\gamma-1} \delta_r, \quad \int_0^{2\pi} |\Delta_{k(r)}(x)|^p dx \leq 3k_r^{-p/2} \delta_r.$$

Let \mathfrak{X} be the characteristic function of the set E_r . To investigate the partial sums of the Fourier series of $\Delta_{k(r)}$, we consider separately the partial sums of the Fourier series of the two functions $\mathfrak{X}\Delta_{k(r)}$ and $(1 - \mathfrak{X})\Delta_{k(r)}$. By Holder's inequality and (3) above

$$\int_0^{2\pi} |\mathfrak{X}(x)\Delta_{k(r)}(x)| dx \leq \left(\int_0^{2\pi} |\Delta_{k(r)}(x)|^p dx \right)^{1/p} |E_r|^{1/q} \leq 3^{1/p} \delta_r k_r^{-1}.$$

By Lemma 3, there is a set F_r of measure not more than $A\delta_r^{1/2} \log \lambda_r$ for some constant A such that for x in F'_r

$$(4) \quad \sup_{n \leq \lambda^{2k_r}} |s_n(x; \mathfrak{X}\Delta_{k(r)})| \leq \delta_r^{1/2}.$$

By (3), the square of the L^2 norm of the function $(1 - \mathfrak{X})\Delta_{k(r)}$ does not exceed $3k_r^{\gamma-1} \delta_r$. We apply to this function the methods of [1]. Let its Fourier series be $\sum_{n=-\infty}^{+\infty} d_n e^{inx}$; let $L_r = \langle (n_{k(r)+1} - n_{k(r)})/k_r^\gamma \rangle$, $J_r = \langle k_r^\gamma \rangle - 1$; and let

$$\mathcal{E}_\mu(x) = \sum_{|n|=n_{k(r)+1+\mu L_r}^{n_{k(r)}+(\mu+1)L_r}} d_n e^{inx}, \quad \mu = 0, 1, \dots, J_r.$$

Since

$$\sum_{\mu=0}^{J_r} \int_0^{2\pi} |\mathcal{E}_\mu(x)|^2 dx \leq \int_0^{2\pi} |(1 - \mathfrak{X}(x))\Delta_{k(r)}(x)|^2 dx \leq 3k_r^{\gamma-1} \delta_r$$

then for at least one μ , say $\mu(r)$, $0 \leq \mu(r) < \langle k_r^\gamma \rangle$,

$$(5) \quad \int_0^{2\pi} |\mathcal{E}_{\mu(r)}(x)|^2 dx \leq Ck_r^{-1} \delta_r$$

for some constant C . Combining this result with Lemma 2, we obtain

$$(6) \quad \int_0^{2\pi} \sup_{n \leq \lambda_r^{2k_r}} |s_n(x; \mathcal{E}_{\mu(r)})|^2 dx \leq 2CA_2 \delta_r (\log \lambda_r).$$

Let $N_r = n_{k(r)} + \mu(r)L_r$. Let the sequence $\{m_r\}$ take on the values m such that $N_r \leq m \leq N_r + L_r$, $r = 1, 2, \dots$. For any such m there is an r such that

$$(7) \quad s_m(x; f) = s_{N_r}(x; f) + \{s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)})\}.$$

Now we write

$$(8) \quad s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)}) = \{s_m(x; \mathfrak{X}\Delta_{k(r)}) - s_{N_r}(x; \mathfrak{X}\Delta_{k(r)})\} + \{s_m(x; (1 - \mathfrak{X})\Delta_{k(r)}) - s_{N_r}(x; (1 - \mathfrak{X})\Delta_{k(r)})\}.$$

By (4), the first bracketed term on the right of (8) does not exceed in absolute value $2\delta_r^{1/2}$ outside a set of measure not more than $A (\log \lambda_r) \delta_r^{1/2}$. The second bracketed term on the right of (8) is $s_m(x; \mathcal{E}_{\mu(r)})$, which, by virtue of (6), does not exceed in absolute value $\delta_r^{1/4}$ outside a set of measure not more than $2CA_2 (\log \lambda_r) \delta_r^{1/2}$. Thus

$$(9) \quad \sup_{N_r \leq m \leq N_r + L_r} |s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)})| \leq 2\delta_r^{1/2} + \delta_r^{1/4},$$

for x in a set G'_r where $|G_r| \leq B (\log \lambda_r) \delta_r^{1/2}$ for some constant B . From our previous comments about the smallness of δ_r , it follows that choices of $\{\lambda_r\}$ and $\{k_r\}$ can be made, even if λ_r is allowed to increase to ∞ slowly enough, so that $\sum_{r=1}^{\infty} |G_r| < \infty$, i.e. so that almost every x belongs to all sets G'_r for all sufficiently large r . The sequence $\{N_r\}$ can be made lacunary so that $s_{N_r}(x; f)$ converges to f almost everywhere. From (7) and (9) we deduce that $s_{m_r}(x; f)$ converges to f almost everywhere.

Since $\lambda_r^{k_r} \leq N_r + L_r \leq \lambda_r^{k(r)+1}$, we have for the sequence $\{m_r\}$

$$\begin{aligned} (\log(N_r + L_r))^\gamma \frac{\sigma(N_r + L_r)}{N_r + L_r} &\geq k_r^\gamma (\log \lambda_r)^\gamma \frac{L_r}{N_r + L_r} \\ &\geq (\log \lambda_r)^\gamma \left(1 - \frac{1}{\lambda_r}\right) - \frac{k_r^\gamma (\log \lambda_r)^\gamma}{\lambda_r^{k(r)+1}}. \end{aligned}$$

The second term on the right goes to 0, and the first term can be made larger than 1 so that (C_p) is satisfied. The limit can, in fact, be made infinite if $\gamma > 0$, i.e. if $1 < p < 2$; but then the analogy with the L^2 theorem is lost.

It remains only to get rid of the restriction that $c_n = 0$ for $n \leq 0$. Thus we may write $f(x) = f_1(x) + f_2(-x)$ where f_i belongs to L^p and where the Fourier coefficients of negative index for the two functions $f_1(x)$ and $f_2(x)$ are 0. We may proceed as before to find an integer $k(r)$ such that inequalities analogous to (3) hold simultaneously for both of these functions; and then a single

integer $\mu(\tau)$ such that inequalities analogous to (5) hold for both. Hence a single sequence $\{m_\nu\}$ satisfying (C_p) is constructed so that $s_{m_\nu}(x; f_i)$ converges almost everywhere to f_i , $i = 1, 2$. A generalization of this technique leads to the following generalization of Theorem 1: *given a sequence $\{f_n\}$ of functions in L^p , $1 < p \leq 2$, there is a sequence of integers satisfying (C_p) such that almost everywhere $s_{m_\nu}(x; f_n)$ converges to f_n for every n .* The same principle applies to all our theorems, and we do not mention it again.

With respect to a lacunary sequence $\{n_k\}$ we shall say that the sequence $\{m_\nu\}$ of positive integers satisfies the condition (c_γ) if in every block (n_k, n_{k+1}) there is a block of terms from $\{m_\nu\}$ of length at least

$$(c_\gamma) \quad \left\langle \frac{n_{k+1} - n_k}{(\log n_{k+1})^\gamma} \right\rangle.$$

The next theorem is stated in terms of the (c_γ) condition, and it merges with the corresponding L^2 theorem except for minor adjustments [1, p. 392].

THEOREM 2. *Let f belong to L^p , $1 < p < 2$; let $\gamma > 3q/2 - 2$ where q is the conjugate of p ; and let $\{n_k\}$ be a lacunary sequence. Then there is a sequence $\{m_\nu\}$ satisfying (c_γ) with respect to $\{n_k\}$ such that $s_{m_\nu}(x; f)$ converges to f almost everywhere.*

In order to make use of the Littlewood-Paley result, we must have a lacunary sequence $\{N_j\}$ such that N_{j+1}/N_j is bounded above. There is no difficulty in imbedding the sequence $\{n_k\}$ in a sequence $\{N_j\}$ satisfying the condition $1 < \alpha \leq N_{j+1}/N_j \leq \beta$. This can be done, for example, by adjoining to $\{n_k\}$ the appropriate terms of a geometric progression. If the theorem is proved for the sequence $\{N_j\}$, and if $n_{k+1} = N_{j+1}$, then

$$\left\langle \frac{N_{j+1} - N_j}{(\log N_{j+1})^\gamma} \right\rangle \geq \left\langle \frac{n_{k+1}(1 - \alpha^{-1})}{(\log n_{k+1})^\gamma} \right\rangle \geq \frac{1 - \alpha^{-1}}{2} \left\langle \frac{n_{k+1} - n_k}{(\log n_{k+1})^\gamma} \right\rangle$$

for n_k big enough. Hence, apart from the constant factor $(1 - \alpha^{-1})/2$, the sequence $\{m_\nu\}$ satisfies (c_γ) with respect to $\{n_k\}$. As our proof will show, compensation can be made for this factor. Thus we begin by assuming that $1 < \alpha \leq n_{k+1}/n_k \leq \beta$. In fact we shall assume that $n_k = 3^k$ and, as before, that $c_n = 0$ for $n \leq 0$. The adjustments in the proof for the general case are quite minor. $\Delta_k(x)$ will have the same meaning as previously. By Hölder's inequality

$$\sum_{k=1}^{\infty} k^{-(\gamma-q+1)(p-1)} |\Delta_k(x)|^p \leq \left(\sum_{k=1}^{\infty} |\Delta_k(x)|^2 \right)^{p/2} \left(\sum_{k=1}^{\infty} k^{-\tau} \right)^{1-p/2}$$

where $\tau = 2(\gamma - q + 1)(p - 1)/(2 - p)$. The second factor of the right side is a convergent series since $\tau > 1$. Integrating this inequality gives

$$\sum_{k=1}^{\infty} \delta_k = \sum_{k=1}^{\infty} k^{-(\gamma-q+1)(p-1)} \int_0^{2\pi} |\Delta_k(x)|^p dx < \infty$$

by (1). Let E_k be the set of x values for which $|\Delta_k(x)| > k^\gamma$. On E_k , $|\Delta_k(x)| < k^{-\gamma(p-1)} |\Delta_k(x)|^p$. Let $\mathfrak{X}_k(x)$ be the characteristic function of E_k . Then

$$\int_0^{2\pi} |\mathfrak{X}_k(x) \Delta_k(x)| dx \leq k^{-\gamma(p-1)} \int_0^{2\pi} |\Delta_k(x)|^p dx = k^{-1} \delta_k.$$

Hence, by Lemma 3, if $F_{k,M}$ is the set where $\sup_{n \leq 3^{k+1}} |s_n(x; \Delta_k \mathfrak{X}_k)| > M$, then

$$(10) \quad |F_{k,M}| \leq \frac{A \delta_k}{M}$$

for some constant A . For E'_k , the complement of E_k , we have as before

$$(11) \quad \int_{E'_k} |\Delta_k(x)|^2 dx \leq k^{\gamma(2-p)} \int_0^{2\pi} |\Delta_k(x)|^p dx = k^{\gamma-1} \delta_k.$$

Let d_n be the n th Fourier coefficient of $\Delta_k(1 - \mathfrak{X}_k)$, let $L_k = \langle 2(3)^k / k^\gamma \rangle$, and let

$$\varepsilon_j(x) = \sum_{|n|=3^{k+1}+jL_k}^{3^{k+(j+1)L_k}} d_n e^{inx}, \quad j = 0, 1, \dots, \langle k^\gamma \rangle - 1 = J.$$

Now by (11)

$$\sum_{j=0}^J \int_0^{2\pi} |\varepsilon_j(x)|^2 dx \leq \int_0^{2\pi} |\Delta_k(x)|^2 (1 - \mathfrak{X}_k(x)) dx \leq k^{\gamma-1} \delta_k.$$

For at least one j in the given range,

$$\int_0^{2\pi} |\varepsilon_j(x)|^2 dx \leq 2k^{-1} \delta_k$$

since there are $J+1 = \langle k^\gamma \rangle$ numbers $\int_0^{2\pi} |\varepsilon_j(x)|^2 dx$. Denote a suitable j by $j(k)$. Lemma 2 implies

$$(12) \quad \int_0^{2\pi} \sup_{n \leq 3^{k+1}} |s_n(x; \varepsilon_{j(k)})|^2 dx \leq 2A_2 \frac{k+1}{k} (\log 3) \delta_k.$$

Now we let $\{m_\nu\}$ take on the values m such that $N_k = 3^k + j(k)L_k \leq m \leq 3^k + [j(k)+1]L_k$, $k = 1, 2, \dots$. The sequence $\{m_\nu\}$ satisfies (c_γ) . For any such m , there is a k such that

$$(13) \quad \begin{aligned} s_m(x; f) &= s_{N_k}(x; f) + \{s_m(x; \Delta_k) - s_{N_k}(x; \Delta_k)\} \\ &= s_{N_k}(x; f) + \{s_m(x; \Delta_k \mathfrak{X}_k) - s_{N_k}(x; \Delta_k \mathfrak{X}_k)\} + s_m(x; \varepsilon_{j(k)}). \end{aligned}$$

Let $\{\mu_k\}$ be a sequence increasing to ∞ slowly enough so that $\sum_{k=1}^{\infty} \mu_k^2 \delta_k < \infty$. From (10) and (12) it follows that

$$\sup_{N_k \leq m \leq N_k + L_k} \left| \{s_m(x; \Delta_k \mathfrak{X}_k) - s_{N_k}(x; \Delta_k \mathfrak{X}_k)\} + s_m(x; \mathfrak{E}_{j(k)}) \right| \leq \frac{1}{\mu_k}$$

outside a set G_k of measure not exceeding $8A_2[(k+1)/k](\log 3)\delta_k\mu_k^2 + 4A\delta_k\mu_k$. Thus $\sum_{k=1}^{\infty} |G_k| < \infty$ so that almost everywhere the above is true for all sufficiently large k . The sequences $\{N_k\}$, k odd and k even, are separately lacunary so that $s_{N_k}(x; f)$ converges to f almost everywhere. From (13) we deduce that $s_{m_\nu}(x; f)$ converges to f almost everywhere.

4. The tools used in the proofs of Theorems 1 and 2 are also available for Walsh series, at least for the L^2 case [5], but not for general orthonormal series. In fact the construction used in the proof of a theorem of Menchoff [3, p. 167] can be modified so as to show the following: *there is an orthonormal system $\{\phi_n\}$ and an L^2 function f such that for any sequence $\{m_\nu\}$ of upper density one, the sequence of partial sums of index m_ν of the $\{\phi_n\}$ expansion of f diverges almost everywhere.*

Our theorems do have analogues for the case of Fourier integrals. We give a sample in the following theorem. Let

$$S_\omega(x; f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \frac{\sin \omega(x - t)}{x - t} dt$$

which is defined for all x and ω if f belongs to $L(-\infty, \infty)$.

THEOREM 3. *If f belongs to $L(-\infty, \infty)$, and if for some p , $1 < p \leq 2$, f belongs to L^p over any finite interval, then there is a sequence $\{m_\nu\}$ of positive integers satisfying (C_p) such that $S_{m_\nu}(x; f)$ converges to f almost everywhere.*

For $n=0, \pm 1, \pm 2, \dots$, we define $f_n(x) = f(x + 2\pi n)$, $0 \leq x < 2\pi$, and by periodicity elsewhere. Since each f_n belongs to $L^p(0, 2\pi)$, there is according to our remarks following the proof of Theorem 1 a sequence $\{m_\nu\}$ satisfying (C_p) such that almost everywhere $s_{m_\nu}(x; f_n)$ converges to f_n for each n . Our theorem then follows from the equiconvergence principle [8, p. 306] which implies that $S_{m_\nu}(x; f) - s_{m_\nu}(x; f_n)$ converges to 0 for $2\pi n < x < 2\pi(n+1)$.

The techniques of the proof of Theorem 1 can also be used to find a sequence $\{m_\nu\}$ satisfying (C_p) , corresponding to a function f satisfying a certain continuity condition, such that the order of convergence of $s_{m_\nu}(x; f)$ to f reflects this continuity. Let

$$\omega_p(\delta; f) = \sup_{0 < h \leq \delta} \left[\int_0^{2\pi} |f(x+h) - f(x)|^p dx \right]^{1/p}.$$

THEOREM 4. *If f belongs to L^p , $1 < p \leq 2$, and if $\omega_p(\delta; f) = o[(\log 1/\delta)^{-\alpha}]$, $\alpha \geq 0$, then there is a sequence $\{m_\nu\}$ satisfying (C_p) such that almost everywhere $s_{m_\nu}(x; f) - f(x) = o[(\log m_\nu)^{-\alpha}]$.*

Let $\sigma_n(x; f)$ be the n th Cèsaro mean of the Fourier series of f at x . We have [8, p. 85]

$$\int_{-\pi}^{\pi} |\sigma_n(x; f) - f(x)|^p dx \leq \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) dt \int_{-\pi}^{\pi} |f(x+t) - f(x)|^p dx$$

where K_n is the n th Fejer kernel. The integration along the t axis can be split into two parts: $|t| \leq n^{-1/2}$ and $n^{-1/2} < |t| \leq \pi$. If use is made of the inequality $K_n(t) \leq \pi^2(n+1)^{-1}t^{-2}$ for the second part, it follows that

$$\int_{-\pi}^{\pi} |\sigma_n(x; f) - f(x)|^p dx = o[(\log n)^{-\alpha p}].$$

Then [8, p. 153] if we let $f(x) = f_n(x) + \sigma_n(x; f)$

$$\mathfrak{M}_p[f - s_n] = \left[\int_0^{2\pi} |f(x) - s_n(x; f)|^p dx \right]^{1/p} \leq A_p \mathfrak{M}_p[f_{n-1}]$$

for some constant A_p . Now we use the same notation as in the proof of Theorem 1 except for the following minor change: $\Delta_k(x) = \sum_{|n|=k+1}^{n_{k+1}} c_n e^{inx}$. We may write $\delta_r = k_r^{-\alpha p} \delta_{r,1}$ where $\delta_{r,1} = o(1)$ as r goes to ∞ , from what has been shown about the mean convergence of $s_n(x; f)$. E_r is now defined as the set of x values for which

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 > k_r^{1/(p-1)-2\alpha}.$$

In the same way as before, $k(r)$ is found such that

$$\int_{E'_r} |\Delta_{k(r)}(x)|^2 dx \leq 3k_r^{\gamma-2\alpha-1} \delta_{r,1}, \quad \int_0^{2\pi} |\Delta_{k(r)}(x)|^p dx \leq 3k_r^{-p/2-\alpha p} \delta_{r,1}.$$

Thus there is a set F_r with measure not exceeding $A \delta_{r,1}^{1/2} (\log \lambda_r)^{\alpha+1}$ such that for x in F'_r

$$\sup_{n \leq \lambda^{2k_r}} |s_n(x; \mathfrak{A} \Delta_{k(r)})| \leq \delta_{r,1}^{1/2} k_r^{-\alpha} (2 \log \lambda_r)^{-\alpha}$$

L_r and \mathcal{E}_μ have their previous meaning. Analogous to (6) is the following:

$$\int_0^{2\pi} \sup_{n \leq \lambda^{2k_r}} |s_n(x; \mathcal{E}_\mu(r))|^2 dx \leq C A_2 (\log \lambda_r) \delta_{r,1} k_r^{-2\alpha}.$$

That is, $\sup_{n \leq \lambda^{2k_r}} |s_n(x; \mathcal{E}_\mu(r))| \leq \delta_r^{1/2} (2k_r \log \lambda_r)^{-\alpha}$ outside a set of small measure. Since

$$\int_0^{2\pi} |s_{N_r}(x; f) - f(x)|^p dx = o(k_r^{-\alpha p})$$

we have, for some small δ , $|s_{N_r}(x; f) - f(x)| \leq \delta (2k_r \log \lambda_r)^{-\alpha}$ outside a set of small measure. Now the proof can be completed by our original method.

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