ON A DECOMPOSITION THEOREM OF FEDERER(1)

BY

EARL J. MICKLE

1. **Introduction.** 1.1. We shall be concerned in this paper with a fundamental decomposition theorem of Federer [2] (numbers in square brackets refer to the bibliography at the end of this paper). In the next few sections we shall give some definitions leading to the statement of this result of Federer and the statement of the results of the present paper. Points in Euclidean three space R^3 will be denoted by x and the distance between two points $x, y \in R^3$ will be denoted by $|x-y|$. For $x \in R^3$ and $r>0$, $K(x, r)$ denotes the open sphere and $C(x, r)$ denotes the closed sphere with center at x and radius r.

1.2. The 2-sphere $|x|=1$ will be denoted by U and points on U will be denoted by P. For sets $E \subset U$, $\sigma(E)$ will denote the Hausdorff 2-dimensional measure on E. For $P \in U$, $R^1(P)$ is the line through the origin and P, $R^2(P)$ is the plane through the origin with normal $R^1(P)$ and for $i=1, 2$, $T_P^i: R^3 \Rightarrow R^i(P)$ is the orthogonal projection of R^3 onto $R^i(P)$. For $x \in R^3$, $P \in U$, $R^1(P, x)$ is the line passing through x and parallel to $R^1(P)$. For a set $E \subset R^2(P)$, $L_2(E)$ is the Lebesgue planar exterior measure of E.

1.3. Let Λ be a Carathéodory outer measure (written for brevity C.o.m.) in R^3 (see, for example, Saks [6, p. 43]). Λ is called Borel regular if every set is contained in a Borel set of equal Λ measure. Λ will be said to satisfy the projection inequality if for every set $E \subset R^3$ and every $P \in U$, $\Lambda(E) \geqq L_2 T_P^2(E)$. Λ will be said to satisfy a weak projection inequality if for every set $E \subset R^3$ with $\Lambda(E) < \infty$ there is a set $Z \subset U$ with $\sigma(Z)=0$ such that

$$\Lambda(E^*) \geqq L_2 T_P^2(E^*) \quad \text{for } E^* \subset E, P \in U - Z.$$

In §2 we show that there is a smallest Borel regular C.o.m. μ in R^3 which satisfies a weak projection inequality and hence, if for $E \subset R^3$, $\mu(E)=0$ then $L_2 T_P^2(E)=0$ for σ a.e. $P \in U$. In §6 (see Theorem 6.11) we show that if $\mu(B) < \infty$ for a Borel set B then $\mu(B)$ is equal to the integral geometric Favard 2-dimensional measure of B.

1.4. A set $E \subset R^3$ is called rectifiable if there is a Lipschitz transformation from a bounded set in a plane onto E, countably rectifiable if it is the union of a countable number of rectifiable sets and Λ unrectifiable if for every rectifiable set $E_0 \subset E$, $\Lambda(E_0)=0$ where Λ is a C.o.m. in R^3.

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1.5. Let Λ be a Borel regular C.o.m. in R^3 that satisfies the projection inequality. For a Λ measurable set A with $\Lambda(A)<\infty$ Federer [2, Theorem 9.6] has shown that A can be decomposed into three Λ measurable sets A_1, A_2, A_3 for which $A=A_1\cup A_2\cup A_3$, $A_1\cap A_2=A_2\cap A_3=A_3\cap A_1=\emptyset$ and each of these sets satisfies certain properties. In this paper we shall be concerned with the following of these properties.

(i) A_1 is countably rectifiable.

(ii) $\mu(A_2)=0$ (see 1.3).

(iii) $\limsup_{r\to 0}\Lambda[K(x, r)\cap A]/r^2=0$ or ∞ for $x\in A_3$. In this paper we shall show (see Theorem 6.10) that if Λ is a Borel regular C.o.m. in R^3 that satisfies a weak projection inequality and if A is a Λ measurable set with $\Lambda(A)<\infty$ then A can be decomposed into two Λ measurable sets A_1^*, A_2^* for which $A=A_1^*\cup A_2^*$, $A_1^*\cap A_2^*=\emptyset$ and A_1^*, A_2^* satisfy the following properties.

(i*) A_1^* is countably rectifiable.

(ii*) $\mu(A_2^*)=0$.

1.6. The results of the present paper would follow as a corollary to the results of Federer and the following statement. If $A_0\subset R^3$ is a μ measurable set and $\limsup_{r\to 0}\mu[K(x, r)\cap A_0]/r^2=0$ for $x\in A_0$ then $\mu(A_0)=0$. Actually this statement is true and follows from the results of the present paper. However, the writer was unable to prove this statement directly and to prove (i*) and (ii*) had to essentially repeat the proof of Federer by modifying certain intermediate results.

While we have limited ourselves in this paper to a discussion of 2-dimensional measures in 3-space the reader will have no difficulty in seeing that the results are valid for the corresponding $(n-1)$-dimensional measures in n-space.

2. **The weak projection inequality.** 2.1. Let $\mathfrak{B}$ be the family of Borel sets in R^3. For $B\in\mathfrak{B}$, $E\subset U$, let $\nu_E(B)=\text{l.u.b. } L_2T_P^2(B)$ for $P\in U-E$ and for $B\in\mathfrak{B}$ set

$$\mu_E(B) = \text{l.u.b.} \sum_i \nu_E(B_i) \text{ for}$$

$$B = \bigcup_i B_i,\ B_i\cap B_j = \emptyset \text{ for } i\neq j,\ B_i\in\mathfrak{B} \text{ for } i = 1, 2, \cdots.$$

It is easily verified that $\mu_E(B)$ is a measure over $\mathfrak{B}$. Let $\mathfrak{Z}$ be the family of sets $Z\subset U$ for which $\sigma(Z)=0$. For $B\in\mathfrak{B}$ we set

$$\mu(B) = \text{gr.l.b. } \mu_Z(B) \qquad \text{for } Z\in\mathfrak{Z}.$$

In the next few sections we shall be concerned with some properties of this set function μ.

2.2. Lemma. *For $B\in\mathfrak{B}$ there is a $Z\in\mathfrak{Z}$ such that $\mu(B)=\mu_Z(B)$.*

Proof. If $\mu(B)=\infty$ then $\mu(B)=\mu_Z(B)$ for every $Z\in\mathfrak{Z}$. If $\mu(B)<\infty$ then for each integer i there is a $Z_i\in\mathfrak{Z}$ such that $\mu_{Z_i}(B)<\mu(B)+1/i$. Hence for $Z=Z_1\cup Z_2\cup\cdots$, $Z\in\mathfrak{Z}$,

$$\mu(B) \leqq \mu_Z(B) \leqq \mu_{Z_i}(B) < \mu(B) + 1/i, \qquad i = 1, 2, \cdots.$$

Thus $\mu(B)=\mu_Z(B)$, $Z\in\mathfrak{Z}$.

2.3. LEMMA. *$\mu(B)$ is a measure on $\mathfrak{B}$.*

Proof. Let $B_0=B_1\cup B_2\cup\cdots$, $B_i\in\mathfrak{B}$, $B_i\cap B_j=\emptyset$ for $i\neq j$, $i,j=1,2,\cdots$. By Lemma 2.2, for $i=0,1,2,\cdots$, there is a $Z_i\in\mathfrak{Z}$ such that $\mu(B_i)=\mu_{Z_i}(B_i)$. Hence, for $Z=Z_0\cup Z_1\cup\cdots$, $Z\in\mathfrak{Z}$,

$$\mu(B_i) = \mu_Z(B_i), \qquad i = 0, 1, 2, \cdots.$$

Since μ_Z is a measure on $\mathfrak{B}$, $\mu(B_0)=\mu(B_1)+\mu(B_2)+\cdots$, and hence μ is a measure over $\mathfrak{B}$.

2.4. LEMMA. *For $B\in\mathfrak{B}$ with $\mu(B)<\infty$ let $Z\in\mathfrak{Z}$ be such that $\mu(B)=\mu_Z(B)$. If $B^*\subset B$, $B^*\in\mathfrak{B}$, then*

$$\mu(B^*) \geqq L_2\overset{2}{T}_P(B^*) \qquad \textit{for } P \in U - Z. \tag{1}$$

Proof. There are sets Z', $Z''\in\mathfrak{Z}$ such that

$$\mu(B^*) = \mu_{Z'}(B^*) \qquad \mu(B - B^*) = \mu_{Z''}(B - B^*).$$

Set $Z_0=Z\cup Z'\cup Z''$. Then $Z_0\in\mathfrak{Z}$,

$$\begin{aligned} \mu(B) &= \mu_Z(B) = \mu_Z(B^*) + \mu_Z(B - B^*) \geqq \mu_{Z_0}(B^*) + \mu_{Z_0}(B - B^*) \\ &= \mu(B^*) + \mu(B - B^*) = \mu(B). \end{aligned} \tag{2}$$

Since the equality sign must hold in (2) we have

$$[\mu_Z(B^*) - \mu(B^*)] + [\mu_Z(B - B^*) - \mu(B - B^*)] = 0. \tag{3}$$

Since both the quantities in square brackets in (3) are non-negative, it follows that each is zero. Hence $\mu(B^*)=\mu_Z(B^*)$ and (1) follows from this fact and the definition of μ_Z.

2.5. THEOREM. *For $E\subset R^3$*

$$\mu(E) = gr.l.b.\ \mu(B) \qquad \textit{for } E \subset B \in \mathfrak{B},$$

is an extension of the measure μ over $\mathfrak{B}$ to a Borel regular C.o.m. in R^3 that satisfies a weak projection inequality.

Proof. The fact that this extended μ is a Borel regular C.o.m. in R^3 follows readily. The fact that μ satisfies a weak projection inequality follows from Lemma 2.4.

2.6. THEOREM. *If Λ is a Borel regular C.o.m. in R^3 which satisfies a weak projection inequality then $\mu(E) \leqq \Lambda(E)$ for $E \subset R^3$.*

Proof. Since Λ and μ are both Borel regular it is sufficient to prove this inequality for the case where E is a Borel set B. Let $Z' \in \mathcal{Z}$ be such that $\mu(B) = u_{Z'}(B)$ and let $Z'' \in \mathcal{Z}$ be such that for $B^* \subset B$, $B^* \in \mathcal{B}$, $\Lambda(B^*) \geqq L_2 T_P^2(B^*)$ for $P \in U - Z''$. Set $Z = Z' \cup Z''$. For $B = B_1 \cup B_2 \cup \cdots$, $B_i \in \mathcal{B}$, $B_i \cap B_j = \emptyset$ for $i \neq j$,

$$\sum_i \nu_Z(B_i) \leqq \sum_i \Lambda(B_i) = \Lambda(B).$$

Thus $\mu(B) = \mu_Z(B) \leqq \Lambda(B)$.

2.7. For $P \in U$ let x_P designate a point in $R^2(P)$. For $E \subset R^3$ we denote by $N(x_P, T_P^2, E)$ the number (possibly $+\infty$) of points $x \in E$ for which $x_P = T_P^2(x)$. If $B \in \mathcal{B}$ then it is well known that $N(x_P, T_P^2, B)$ is an L_2 measurable function of x_P in $R^2(P)$.

LEMMA. *If $B \in \mathcal{B}$ and $Z \in \mathcal{Z}$ is such that $\mu(B) = \mu_Z(B)$ then*

$$\mu(B) \geqq \iint N(x_P, T_P^2, B)\, dL_2 \qquad \text{for } P \in U - Z. \tag{1}$$

Proof. For each positive integer i, R^3 is the union of a countable number of mutually disjoint Borel sets $B_i^1, B_i^2, \cdots$ each of which is of diameter less than $1/i$. For $P \in U - Z$ let $f_i^j(x_P)$ be the characteristic function of $T_P^2(B \cap B_i^j)$, $j = 1, 2, \cdots$, and set $f_i(x_P) = f_i^1(x_P) + f_i^2(x_P) + \cdots$. Then

$$\begin{aligned} \iint f_i(x_P)\, dL_2 &= \sum_j \iint f_i^j(x_P)\, dL_2 = \sum_j L_2 T_P^2(B \cap B_i^j) \\ &\leqq \sum_j \mu(B \cap B_i^j) = \mu(B). \end{aligned} \tag{2}$$

Since $f_i(x_P) \to N(x_P, T_P^2, B)$ for $i \to \infty$, (1) follows by the lemma of Fatou from (2).

2.8. THEOREM. *If $E_0 \subset R^3$ is a rectifiable set then $\mu(E_0) = H^2(E_0)$ where H^2 is the Hausdorff 2-dimensional measure in R^3.*

Proof. Since H^2 satisfies the projection inequality (and hence a weak projection inequality), by Theorem 2.6

$$\mu(E) \leqq H^2(E) \qquad \text{for } E \subset R^3. \tag{1}$$

Since E_0 is a rectifiable set there is a Lipschitz transformation from a bounded set in a plane onto E_0. Thus there is a Lipschitz transformation T from a bounded set in the unit square Q: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, in the uv-plane onto E_0. By Mickle [3] this transformation can be extended to a Lipschitz transformation from Q into R^3. Since $\mu[T(Q)] \leqq H^2[T(Q)] < \infty$ and since μ and

H^2 are both Borel regular, to prove that $\mu(E_0)=H^2(E_0)$ it is sufficient to prove that

$$(2) \qquad \mu(B) = H^2(B), \qquad B = T(Q).$$

For $P\in U$ the Jacobian $J_P(u, v)$ of the transformation T_P^2T exists L_2 a.e. in Q, is a Borel measurable function and is summable on Q. Let $J(u, v)$ be a vector whose components are $J_P(u,v)$ for $P=(1,0,0), P=(0,1,0), P=(0,0,1)$. There is a Borel set $B^*\subset Q$ for which the following conditions are satisfied.

(i) T is one-to-one on B^*.

(ii) $J(u, v)$ exists everywhere and $0<|J(u, v)|<\infty$ on B^*.

(iii) $\mu[B-T(B^*)]=H^2[B-T(B^*)]=0$.

Let us now consider $P\in U$ as a vector in R^3. Then $J_P(u, v)=P\cdot J(u, v)$. Let $Z\in \mathcal{Z}$ be such that $\mu(B)=\mu_Z(B)$ and let $P_1, P_2, \cdots$, be a dense set in $U-Z$ (and hence dense in U). For $i=1, 2, \cdots$, the sets

$$B_i^* = \{(u, v) \mid (u, v) \in B^*, 0 \leqq |J(u, v)| - |J_{P_i}(u, v)| < \epsilon\}$$

are Borel sets and $B^*=B_1^*\cup B_2^*\cup \cdots$. Thus the sets

$$B_1 = B_1^*, \qquad B_i = B_i^* - (B_1^* \cup \cdots \cup B_{i-1}^*), \qquad i = 2, 3 \cdots,$$

are mutually disjoint Borel sets and $B^*=B_1\cup B_2\cup \cdots$. By Federer [1, Theorem 4.5; 2, Theorem 5.9] and Lemma 2.7

$$\begin{aligned} H^2(B) &= H^2[T(B^*)] = \iint_{B^*} |J(u, v)|\, dL_2 = \sum_i \iint_{B_i} |J(u, v)|\, dL_2 \\ &\leqq \sum_i \iint_{B_i} |J_{P_i}(u, v)|\, dL_2 + \epsilon = \sum_i \iint N[x_{P_i}, T_{P_i}^2, T(B_i)]dL_2 + \epsilon \\ &\leqq \sum_i \mu[T(B_i)] + \epsilon = \mu(B) + \epsilon. \end{aligned}$$

Since $\epsilon>0$ is arbitrary, $H^2(B)\leqq\mu(B)$ and hence, by (1), (2) holds.

3. **Outer measures.** 3.1. Let M be a metric space with metric $d(x, y)$. For $x\in M$, $0<r<\infty$ we set $c(x, r)=\{y\mid y\in M, d(x, y)\leqq r\}$. Throughout this section $\sigma(E)$ will be a Borel regular C.o.m. in M which satisfies the following condition. There is a λ, $1<\lambda<\infty$, such that for any closed sphere $c(x, r)$, $\sigma[c(x, 5r)]<\lambda\sigma[c(x, r)]$. This condition implies that $0<\sigma[c(x, r)]<\infty$ for $x\in M$, $0<r<\infty$. In our application of the results of this section M will be the unit sphere U and σ will be the Hausdorff 2-dimensional measure on U.

3.2. A set function $\psi(E)$ defined for all sets $E\subset M$ will be called an outer measure if it satisfies all the conditions for being a C.o.m. except the condition that it be additive on any pair of sets that are a positive distance apart. It follows readily (see, for example, Mickle and Radó [4]) that

$$\limsup_{r\to 0} \psi[c(x, r)]/\sigma[c(x, r)]$$

is a Borel measurable function of x in M.

3.3. LEMMA. *Let ψ be an outer measure in M. If E is a σ measurable set for which*

$$\psi(E) = 0 \tag{1}$$

$$0 < \limsup_{r\to 0} \psi[c(x, r)]/\sigma[c(x, r)] < \infty \qquad \textit{for } x \in E, \tag{2}$$

then $\sigma(E)=0$.

Proof. By (1), for $x\in M$,

$$\limsup_{r\to 0} \psi[c(x, r)]/\sigma[c(x, r)] = \limsup_{r\to 0} \psi[c(x, r) \cap (M - E)]/\sigma[c(x, r)]. \tag{3}$$

By a simple modification of a result of Mickle and Radó [4],

$$\limsup_{r\to 0} \psi[c(x, r) \cap (M - E)]/\sigma[c(x, r)] = 0 \text{ or } \infty \qquad \text{for } \sigma \text{ a.e. } x \in E. \tag{4}$$

From (2), (3) and (4) it follows that $\sigma(E)=0$.

3.4. THEOREM. *Let ψ_j be a sequence of outer measures in M and let Z_j be a sequence of σ measurable sets in M for which the following conditions are satisfied.*

$$\psi_{j+1}(E) \leqq \psi_j(E) \qquad \textit{for } E \subset M, j = 1, 2, \cdots. \tag{1}$$

$$\psi_j(M - Z_j) = 0 \qquad \textit{for } j = 1, 2, \cdots. \tag{2}$$

$$Z_{j+1} \subset Z_j \qquad \textit{for } j = 1, 2, \cdots. \tag{3}$$

Set

$$W_j = \left\{x \,\middle|\, x \in M, \limsup_{r\to 0} \psi_j[c(x, r)]/\sigma[c(x, r)] = \infty\right\},$$

$$X_j = \left\{x \,\middle|\, x \in M, \limsup_{r\to 0} \psi_j[c(x, r)]/\sigma[c(x, r)] = 0\right\},$$

then, for

$$Z = \bigcap_{j=1}^{\infty} Z_j, \qquad W = \bigcap_{j=1}^{\infty} W_j, \qquad X = \bigcup_{j=1}^{\infty} X_j,$$

we have that

$$\sigma[M - (W \cup X \cup Z)] = 0. \tag{4}$$

Proof. By the remark in 3.2 X and W_j are σ measurable sets. Thus $M-(Z_j\cup W_j\cup X)$ is a σ measurable set, $\psi_j[M-(Z_j\cup W_j\cup X)]=0$ and $0<\limsup_{r\to 0} \psi_j[c(x, r)]/\sigma[c(x, r)]<\infty$ for $x\in M-(Z_j\cup W_j\cup X)$. By Lemma 3.3

$$\sigma[M-(Z_j \cup W_j \cup X)] = 0, j = 1, 2, \cdots. \tag{5}$$

By (1) and (3) W_j and Z_j are monotone decreasing sequences of sets and hence

$$M-(Z \cup W \cup X) = \bigcup_{j=1}^{\infty} [M-(Z_j \cup W_j \cup X)]. \tag{6}$$

(4) then follows from (5) and (6).

4. **Lemmas on measurability.** 4.1. For $P \in U$, $x \in R^3$, $0<\eta<1$ we set

$$\begin{aligned}\diamond(P, \eta, x) &= \{y \mid y \in R^3, \mid y - x \mid < (1+\eta^2)^{1/2} \mid T^1_P(y) - T^1_P(x) \mid \} \\ &= \{y \mid y \in R^3, \mid T^2_P(y) - T^2_P(x) \mid < \eta \mid T^1_P(y) - T^1_P(x) \mid \} \\ &= \{y \mid y \in R^3, y \neq x, y \in \bigcup R^1(P^*, x) \text{ for } \mid P^* - P \mid \\ &\qquad\qquad < 2 \sin 2^{-1} \operatorname{arc} \tan \eta\}.\end{aligned}$$

For a C.o.m. Λ in R^3, $A \subset R^3$, $P \in U$, $0<\eta<1$, $0<r<\infty$, $x \in R^3$ we set

$$\nabla(\Lambda, A, P, \eta, r, x) = \Lambda[A \cap \diamond(P, \eta, x) \cap K(x, r)]/\pi r^2 \eta^2,$$

and for $j=1, 2, \cdots$, we set

$$\nabla_j(\Lambda, A, P, \eta, x) = \underset{0<r\leqq 1/j}{\text{l.u.b.}} \nabla(\Lambda, A, P, \eta, r, x),$$

$$\nabla_j^*(\Lambda, A, P, x) = \limsup_{\eta \to 0} \nabla_j(\Lambda, A, P, \eta, x).$$

4.2. Lemma. *$\nabla(\Lambda, A, P, \eta, x)$ is a lower semi-continuous function of (P, η, r, x).*

Proof. For (P_0, η_0, r_0, x_0) let λ be any number less than $\Lambda[A \cap \diamond(P_0, \eta_0, x_0) \cap K(x_0, r_0)]$. Then there is a closed set F such that

$$F \subset \diamond(P_0, \eta_0, x_0) \cap K(x_0, r_0), \qquad \Lambda(A \cap F) > \lambda.$$

For (P, η, r, x) sufficiently close to (P_0, η_0, r_0, x_0), $F \subset \diamond(P, \eta, x) \cap K(x, r)$ and

$$\Lambda[A \cap \diamond(P, \eta, x) \cap K(x, r)] \geqq \Lambda(A \cap F) > \lambda.$$

Thus $\Lambda[A \cap \diamond(P, \eta, x) \cap K(x, r)]$ is a lower semi-continuous function of (P, η, r, x) and hence so is $\nabla(\Lambda, A, P, \eta, r, x)$.

4.3. Lemma. *$\nabla_j(\Lambda, A, P, \eta, x)$ is a lower semi-continuous function of (P, η, x).*

Proof. Follows from Lemma 4.2.

4.4. Lemma. *$\nabla_j^*(\Lambda, A, P, x)$ is a Borel measurable function of (P, x) in the cartesian product space $U \times R^3$.*

Proof. Follows from Lemma 4.3.

4.5. LEMMA. *If $A \subset R^3$ is a Borel set then in the cartesian product space $U \times A$,*

$$W_j(\Lambda, A) = \{(P, x) \mid (P, x) \in U \times A, \nabla_j^*(\Lambda, A, P, x) = \infty\},$$
$$X_j(\Lambda, A) = \{(P, x) \mid (P, x) \in U \times A, \nabla_j^*(\Lambda, A, P, x) = 0\},$$
$$W(\Lambda, A) = \bigcap_{j=1}^{\infty} W_j(\Lambda, A), \quad X(\Lambda, A) = \bigcup_{j=1}^{\infty} X_j(\Lambda, A),$$

are Borel sets.

Proof. Follows from Lemma 4.4 and the fact that A is a Borel set.

4.6. LEMMA. *If $A \subset R^3$ is a closed set then in the cartesian product space $U \times A$*

$$V(A) = \{(P, x) \mid (P, x) \in U \times A, x \text{ is a cluster point of } R^1(P, x) \cap A\}$$

is a Borel set.

Proof. For $i, j = 1, 2, \cdots,$

$$V_{i,j} = \{(P, x) \mid (P, x) \in U \times A, A \cap [C(x, 1/i) - K(x, 1/(i+j))] \cap R^1(P, x) \neq \emptyset\}$$

is a closed set in $U \times A$ and since

$$V(A) = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} V_{i,j}$$

it follows that $V(A)$ is a Borel set in $U \times A$.

5. **Λ unrectifiable sets.**

5.1. THEOREM. *If $A \subset R^3$, $P \in U$ and for each $x \in A$ there are numbers $0 < \eta < 1$, $0 < r < \infty$ such that $A \cap \diamond(P, \eta, x) \cap K(x, r) = \emptyset$ then A is countably rectifiable.*

Proof. See Federer [2, Theorem 4.3].

5.2. LEMMA. *Let Λ be a C.o.m. in R^3 and for a positive integer j let A be a set in R^3 of diameter less than $1/j$. If, for $P \in U$, $0 < \delta < 1$, $0 < \gamma < \infty$,*

$$(1) \qquad \nabla_j(\Lambda, A, P, \eta, x) < \gamma \qquad \textit{for } x \in A, 0 < \eta < \delta,$$

and

$$(2) \quad B = \{x \mid x \in A, A \cap \diamond(P, \eta, x) \cap K(x, r) \neq \emptyset \textit{ for } r > 0, 0 < \eta < 1\},$$

then for $a \in B$, $a'' = T_P^2(a)$,

$$(3) \qquad \Lambda\{B \cap (T_P^2)^{-1}[K(a'', r)]\} < 10^5 \gamma \pi r^2 \qquad \textit{for } 0 < r < \delta/12j.$$

Proof. The proof given here is with certain necessary modifications the proof given in Federer [2, Theorem 4.5]. Since the diameter of A is less than $1/j$, (1) implies that

$$\nabla(\Lambda, A, P, \eta, r, x) < \gamma \qquad \text{for } x \in A, 0 < \eta < \delta, r > 0. \tag{4}$$

Set $x' = T_P^1(x)$, $x'' = T_P^2(x)$ for $x \in R^3$. For $a \in B$, $0 < r < \delta/12j$ set $\eta = 12rj < \delta$, $\epsilon = \eta/12 = rj$. Set

$$= EB \cap (T_P^2)^{-1}[K(a'', r)].$$

For $x \in E$ set

$$h(x) = \text{l.u.b. } | y' - x' | \qquad \text{for } y \in A \cap \diamond(P, \epsilon, x) \cap (T_P^2)^{-1} [K(a'', r)].$$

By (2) and the fact that the diameter of A is less than $1/j$,

$$0 < h(x) \leqq 1/j.$$

For each $x \in E$ we associate a point $\bar{x} \in A \cap \diamond(P, \epsilon, x) \cap (T_P^2)^{-1}[K(a'', r)]$ such that

$$12 | \bar{x}' - x' | > 11h(x). \tag{5}$$

Since $\bar{x} \in \diamond(P, \epsilon, x)$ (see 4.1)

$$| \bar{x}'' - x'' | < \epsilon | \bar{x}' - x' |. \tag{6}$$

For $x \in E$ we set

$$Q(x) = \{ y \mid y \in E, | y'' - x'' | < 5\epsilon h(x) \}.$$

Part 1. For $x \in E$, $Q(x) \subset \diamond(P, \eta, x) \cup \diamond(P, \eta, \bar{x})$.

Proof. Assume that there is a point $y \in Q(x)$ for which $| y'' - x'' | < 5\epsilon h(x)$, $| y'' - x'' | \geqq \eta | y' - x' |$, $| y'' - \bar{x}'' | \geqq \eta | y' - \bar{x}' |$. From these inequalities, (5), (6) and the fact that $12\epsilon = \eta$ we obtain

$$\begin{aligned} 0 < 11\epsilon h(x) &< 12\epsilon | \bar{x}' - x' | \\ &\leqq \eta | \bar{x}' - y' | + \eta | y' - x' | \\ &\leqq | y'' - \bar{x}'' | + | y'' - x'' | \\ &\leqq | x'' - \bar{x}'' | + 2 | y'' - x'' | \\ &< \epsilon | x' - \bar{x}' | + 10\epsilon h(x) \leqq 11\epsilon h(x) < \infty. \end{aligned}$$

This is a contradiction. Hence Part 1 holds.

Part 2. If $x \in E$ then $K[x'', \epsilon h(x)] \subset K(a'', 2r)$.

Proof. If $| y'' - x'' | < \epsilon h(x)$ we have (since $| x'' - a'' | < r$)

$$| y'' - a'' | \leqq | y'' - x'' | + | x'' - a'' | < \epsilon h(x) + r \leqq r + r = 2r.$$

Part 3. If $y \in Q(x)$ then $| y' - x' | < 5h(x)$.

Proof. Assume that there is a $y \in Q(x)$ with $| y' - x' | \geqq 5h(x)$. Then

$$| y'' - x'' | < 5\epsilon h(x) \leqq \epsilon | y' - x' |.$$

Thus $y \in A \cap \diamond(P, \epsilon, x) \cap (T_P^2)^{-1}[K(a'', r)]$ and hence

$$1/j \geqq h(x) \geqq |y' - x'| \geqq 5h(x) > 0.$$

This is a contradiction and Part 3 holds.

PART 4. If $x \in E$ then $\Lambda[Q(x)] \leqq 2 \cdot 84^2 \gamma\pi[\epsilon h(x)]^2$.

Proof. By Part 3, $y \in Q(x)$ implies that

$$|y - x| \leqq |y' - x'| + |y'' - x''| < 5h(x) + 5\epsilon h(x) = 5(\epsilon + 1)h(x) < 6h(x),$$

$$\begin{aligned} |y - \bar{x}| &\leqq |y - x| + |x - \bar{x}| \leqq 5(\epsilon + 1)h(x) + |x' - \bar{x}'| + |x'' - \bar{x}''| \\ &< 5(\epsilon + 1)h(x) + (\epsilon + 1)|x' - \bar{x}'| \\ &< (\epsilon + 1)6h(x) < (13/12)6h(x) \\ &< 7h(x). \end{aligned}$$

Hence

$$Q(x) \subset K[x, 7h(x)] \cap K[\bar{x}, 7h(x)]. \tag{7}$$

From (7) and Part 1,

$$Q(x) \subset \{A \cap \diamond(P, \eta, x) \cap K[x, 7h(x)]\} \cup \{A \cap \diamond(P, \eta, \bar{x}) \cap K[\bar{x}, 7h(x)]\}.$$

Thus by (4) and the fact that $\eta = 12\epsilon$,

$$\Lambda[Q(x)] \leqq 2\gamma\pi[7h(x)]^2\eta^2 = 2 \cdot 84^2\gamma\pi[\epsilon h(x)]^2.$$

PART 5. $\Lambda(E) \leqq 10^5 \gamma\pi r^2$.

Proof. Since $T_P^2(E) \subset \bigcup K[x'', \epsilon h(x)]$ for $x \in E$, by a covering theorem of Morse [5] there is a sequence of points $x_1, x_2, \cdots$, in E such that

$$T_P^2(E) \subset \bigcup_i K[x_i'', 5\epsilon h(x_i)], \qquad K[x_i'', \epsilon h(x_i)] \cap K[x_k'', \epsilon h(x_k)] = \emptyset \quad \text{for } i \neq k. \tag{8}$$

Now $x \in E$ implies that there is an x_i such that $|x'' - x_i''| < 5^\epsilon h(x_i)$ and hence that $x \in Q(x_i)$. Thus $E \subset \bigcup_i Q(x_i)$. From Parts 2 and 4 and (8) we have that

$$\Lambda(E) \leqq \sum_i \Lambda[Q(x_i)] \leqq 2 \cdot 84^2\gamma\pi \sum_i [\epsilon h(x_i)]^2 \leqq 2 \cdot 84^2\gamma\pi 4r^2 < 10^5\gamma\pi r^2.$$

Thus (3) holds.

5.3. THEOREM. *Let Λ be a C.o.m. in R^3, let A be a Λ unrectifiable set in R^3, let $P \in U$ and set* (*see* 4.5)

$$X^*(\Lambda, A, P) = \{x \mid x \in A, (P, x) \in X(\Lambda, A)\}.$$

Then $\Lambda[X^*(\Lambda, A, P)] = 0$.

Proof. Set

$$X_j^*(\Lambda, A, P) = \{x \mid x \in A, (P, x) \in X_j(\Lambda, A)\}, \qquad j = 1, 2, \cdots.$$

Since $X^*(\Lambda, A, P) = X_1^*(\Lambda, A, P) \cup X_2^*(\Lambda, A, P) \cup \cdots$, to show that $\Lambda[X^*(\Lambda, A, P)] = 0$ it is sufficient to show that $\Lambda[X_j^*(\Lambda, A, P)] = 0$ for

$j=1, 2, \cdots$. We have that

$$\nabla_j^*(\Lambda, X_j^*, P, x) = 0 \qquad \text{for } x \in X_j^*.$$

Assume that $\Lambda(X_j^*) > 0$. Using Lemma 4.3 there is a set $A_j^* \subset X_j^*$ and a sequence of positive numbers $\delta_1, \delta_2, \cdots$, such that

$$\Lambda(A_j^*) > 0, \qquad \nabla_j^*(\Lambda, A_j, P, \eta, x) < 1/i \qquad \text{for } 0 < \eta < \delta_i, x \in A_j^*.$$

Since $\Lambda(A_j^*) > 0$, A_j^* contains a subset A_j of diameter less than $1/j$ for which

$$\Lambda(A_j) > 0, \qquad \nabla_j(\Lambda, A_j, P, \eta, x) < 1/i \qquad \text{for } 0 < \eta < \delta_i, x \in A_j.$$

Let

$$B_j = \{x \mid x \in A_j, A_j \cap \Diamond(P, \eta, x) \cap K(x, r) \neq \emptyset \text{ for } r > 0, 0 < \eta < 1\}.$$

Since A_j is Λ unrectifiable, by Theorem 5.1, $\Lambda(A_j - B_j) = 0$. Hence $\Lambda(B_j) = \Lambda(A_j) > 0$. Thus there is a point $a \in B_j$ such that

$$\limsup_{r \to 0} \Lambda\{B_j \cap (T_P^2)^{-1}[K(a'', r)]\}/\pi r^2 > \lambda > 0, \ a'' = T_P^2(a). \tag{1}$$

Choose i such that $1/i < \lambda/10^5$. By Lemma 5.2

$$\Lambda\{B_j \cap (T_P^2)^{-1}[K(a'', r)]\} < \lambda \pi r^2 \qquad \text{for } 0 < r < \delta_i/12j. \tag{2}$$

Since (2) contradicts (1) it follows that $\Lambda(X_j^*) = 0$.

6. **The decomposition theorem.** 6.1. Throughout this section $\sigma(E)$ will denote the Hausdorff 2-dimensional measure of a set $E \subset U$ and for $P \in U$, $0 < r < \infty$,

$$c(P, r) = \{P^* \mid P^* \in U, \ |P^* - P| \leqq r\},$$
$$c^0(P, r) = \{P^* \mid P^* \in U, \ |P^* - P| < r\}.$$

As noted in Federer [2, §6.2], there is a λ, $1 < \lambda < \infty$, such that $\sigma[c(P, 5r)] < \lambda \sigma[c(P, r)]$ for $P \in U$, $0 < r < \infty$.

6.2. For the sets defined in 4.5 and 4.6 we set

$$W_j^*(\Lambda, A, x) = \{P \mid P \in U, (P, x) \in W_j(\Lambda, A)\},$$
$$W_j^*(\Lambda, A, P) = \{x \mid x \in A, (P, x) \in W_j(\Lambda, A)\}.$$

$W^*(\Lambda, A, x)$, $W^*(\Lambda, A, P)$, $X_j^*(\Lambda, A, x)$, $X_j^*(\Lambda, A, P)$, $X^*(\Lambda, A, x)$, $X^*(\Lambda, A, P)$, $V^*(A, x)$ and $V^*(A, P)$ are defined in the same way with respect to $W(\Lambda, A)$, $X_j(\Lambda, A)$, $X(\Lambda, A)$, $V(A)$ respectively.

6.3. For a C.o.m. Λ in R^3, $A \subset R^3$, $x \in R^3$, $E \subset U$, $j = 1, 2, \cdots$, we set

$$\psi_j(\Lambda, A, E, x) = \underset{0 < r \leqq 1/j}{\text{l.u.b.}} \Lambda[(A - x) \cap K(x, r) \cap \{\bigcup R^1(P, x) \text{ for } P \in E\}]/r^2.$$

The proof of the following lemma is easily verified.

LEMMA. *For* $j=1, 2, \cdots, \psi_j(\Lambda, A, E, x)$ *is an outer measure in* U *and for* $\epsilon=2 \sin 2^{-1} \arc \tan \eta$, $0<\eta<1$,

$$(1) \qquad \psi_j[\Lambda, A, c^0(P, \epsilon), x]/\pi\eta^2 = \nabla_j(\Lambda, A, P, \eta, x).$$

6.4. LEMMA. *For the sets defined in* 6.2

$$W_j^*(\Lambda, A, x) = \left\{P \,\middle|\, P \in U, \limsup_{\epsilon\to 0} \psi_j[\Lambda, A, c^0(P, \epsilon), x]/\sigma[c^0(P, \epsilon)] = \infty \right\},$$

$$X_j^*(\Lambda, A, x) = \left\{P \,\middle|\, P \in U, \limsup_{\epsilon\to 0} \psi_j[\Lambda, A, c^0(P, \epsilon), x]/\sigma[c^0(P, \epsilon)] = 0 \right\}.$$

Proof. These relations follow from (1) of 6.3 and the fact that for $\epsilon=2 \sin 2^{-1} \arc \tan \eta$,

$$\lim_{\eta\to 0} \{\sigma[c^0(P, \epsilon)]/\pi\epsilon^2\}(\epsilon/\eta)^2 = 1.$$

6.5. LEMMA. *If* Λ *is a C.o.m. in* R^3, A *is a closed set in* R^3, $x\in R^3$ *and for* $j=1, 2, \cdots$, $Z_j=\{P|P\in U, (A-x)\cap K(x, 1/j)\cap R^1(P, x)\neq\emptyset\}$, *then*

(i) Z_j *is a Borel set for* $j=1, 2, \cdots$,
(ii) $\psi_j(U-Z_j)=0$ *for* $j=1, 2, \cdots$,
(iii) $Z_{j+1}\subset Z_j$ *for* $j=1, 2, \cdots$,
(iv) $V^*(A, x)=\bigcap_{j=1}^\infty Z_j$.

Proof. (ii), (iii) and (iv) are easily verified. To prove (i), for $m=2, 3, \cdots$, set

$$Z_j^m = \{P \mid P \in U, A \cap [C(x, m/j(m+1)) - K(x, 1/jm)] \cap R^1(P, x) \neq \emptyset\}.$$

Z_j^m is a closed set and $Z_j=Z_j^2\cup Z_j^3\cup \cdots$ is a Borel set.

6.6. LEMMA. *Let* Λ *be a C.o.m. in* R^3 *and let* A *be a closed set in* R^3. *Then*

$$(1) \qquad \sigma\{U - [V^*(\Lambda, A, x) \cup W^*(\Lambda, A, x) \cup X^*(\Lambda, A, x)]\} = 0 \qquad \textit{for } x \in A.$$

Proof. The outer measures $\psi_j(\Lambda, A, E, x)$ and the sets Z_j defined in 6.5 satisfy the conditions of Theorem 3.4. Since $\sigma[c(P, r)-c^0(P, r)]=0$ for $P\in U$, $0<r<1$, either closed or open spheres can be used in finding upper densities. Hence, by Lemma 6.4, (iv) of Lemma 6.5 and Theorem 3.4, (1) holds.

6.7. LEMMA. *If* Λ *is a C.o.m. in* R^3, A *is a set in* R^3 *with* $\Lambda(A)<\infty$, $P\in U$, *then* $L_2T_P^2[W^*(\Lambda, A, P)]=0$.

Proof. For $E\subset R^2(P)$ let $\Lambda_P(E)=\Lambda[A\cap(T_P^2)^{-1}(E)]$. Then Λ_P is a finite-valued C.o.m. in $R^2(P)$. Hence, for $x_P\in R^2(P)$,

$$(1) \qquad \limsup_{t\to 0} \Lambda_P[K(x_P, t)]/\pi t^2 < \infty \qquad \text{for } L_2 \text{ a.e. } x_P \in R^2(P).$$

If $x \in W^*(\Lambda, A, P)$, $x_P = T_P^2(x)$ then, for $0<\eta<1$, $0<r<\infty$,

$$\diamond(P, \eta, x) \cap K(x, r) \subset \{y \mid y \in R^3, T_P^2(y) \in K(x_P, \eta r)\}$$

and there is a sequence $(\eta_j, r_j) \to (0, 0)$ for $j \to \infty$ such that

$$\lim_{j\to\infty} \Lambda_P[K(x_P, \eta_j r_j)]/\pi\eta_j^2 r_j^2 = \lim_{j\to\infty} \nabla(\Lambda, A, P, \eta_j, r_j, x) = \infty.$$

It thus follows from (1) that $L_2 T_P^2[W^*(\Lambda, A, P)] = 0$.

6.8. LEMMA. *If Λ is a Borel regular C.o.m. in R^3 that satisfies a weak projection inequality, $A \subset R^3$ is a Borel set with $\Lambda(A) < \infty$, $Z \in \mathcal{Z}$ is such that $\mu(A) = \mu_Z(A)$ then*

$$L_2 T_P^2[V^*(A, P)] = 0 \qquad \text{for } P \in U - Z. \tag{1}$$

Proof. By Theorem 2.6, $\mu(A) < \infty$. By Lemma 2.7, for $P \in U - Z$, $N(x_P, T_P^2, A)$ is an L_2 summable function and hence $N(x_P, T_P^2, A) < \infty$ for L_2 a.e. $x_P \in R^2(P)$. Since $x \in V^*(A, P)$ implies that $N[T_P^2(x), T_P^2, A] = \infty$, (1) follows.

6.9. THEOREM. *If Λ is a Borel regular C.o.m. in R^3 which satisfies a weak projection inequality and A is a Λ unrectifiable closed set in R^3 with $\Lambda(A) < \infty$ then $\mu(A) = 0$.*

Proof. By Lemma 6.6

$$\sigma\{U - [V^*(A, x) \cup W^*(\Lambda, A, x) \cup X^*(\Lambda, A, x)]\} = 0 \qquad \text{for } x \in A. \tag{1}$$

By Lemmas 4.5 and 4.6, $V(A)$, $W(\Lambda, A)$, $X(\Lambda, A)$ are Borel sets in the cartesian product space $U \times A$ with product measure $\sigma \times \Lambda$. By applying Fubini's theorem to the characteristic function of $U \times A - [V(A) \cup W(\Lambda, A) \cup X(\Lambda, A)]$ it follows from (1) that

$$\Lambda\{A - [V^*(A, P) \cup W^*(\Lambda, A, P) \cup X^*(\Lambda, A, P)]\} = 0 \quad \text{for } \sigma \text{ a.e. } P \in U. \tag{2}$$

Let $Z \in \mathcal{Z}$ be such that (2) holds for $P \in U - Z$ and $\mu(A) = \mu_Z(A)$. Set

$$A_0 = A - [V^*(A, P) \cup W^*(\Lambda, A, P) \cup X^*(\Lambda, A, P)]. \tag{3}$$

By (2), Lemma 6.8, Lemma 6.7, Theorem 5.3 we have respectively

$$L_2 T_P^2(A_0) \leqq \mu(A_0) \leqq \Lambda(A_0) = 0 \qquad \text{for } P \in U - Z, \tag{4}$$

$$L_2 T_P^2[V^*(A, P)] = 0 \qquad \text{for } P \in U - Z, \tag{5}$$

$$L_2 T_P^2[W^*(\Lambda, A, P)] = 0 \qquad \text{for } P \in U, \tag{6}$$

$$L_2 T_P^2[X^*(\Lambda, A, P)] \leqq \mu[X^*(\Lambda A, P)] \leqq \Lambda[X^*(\Lambda, A, P)], = 0, P \in U. \tag{7}$$

From (3), (4), (5), (6), and (7) it follows that $L_2 T_P^2(A) = 0$ for $P \in U - Z$ and hence $\mu(A)=0$.

6.10. THEOREM. *Let Λ be a Borel regular C.o.m. in R^3 that satisfies a weak projection inequality and let A be a Λ measurable set in R^3 with $\Lambda(A)<\infty$. Then A can be decomposed into two Λ measurable sets A_1, A_2 with $A=A_1\cup A_2$, $A_1\cap A_2=\varnothing$, A_1 is countably rectifiable and $\mu(A_2)=0$.*

Proof. Let $\gamma=\text{l.u.b. }\Lambda(A_0)$, $A_0\subset A$, A_0 rectifiable. For each positive integer i there is a rectifiable set $B_i\subset A$ such that $\Lambda(B_i)>\gamma-1/i$. Then there is a Lipschitz transformation from a bounded set E_i in a plane π onto B_i and by Mickle [3] this transformation can be extended to the whole plane π with the same Lipschitz constant. If F_i is a bounded closed set in π containing E_i then the intersection of A and the image of F_i under this extended transformation is a Λ measurable and rectifiable subset of A containing B_i. Hence we may assume that B_i is Λ measurable for each i. Set $A_1=B_1\cup B_2\cup \cdots$. Then A_1 is countably rectifiable and Λ measurable. The set $A_2=A-A_1$ is Λ measurable and Λ unrectifiable, $A=A_1\cup A_2$, $A_1\cap A_2=\varnothing$. Since A_2 is the union of a countable number of closed (and necessarily Λ unrectifiable) sets and a set of Λ measure zero (and hence of μ measure zero), it follows from Theorem 6.9 that $\mu(A_2)=0$.

6.11. THEOREM. *If $A\subset R^3$ and $\mu(A)<\infty$ then $\mu(A)=I^2(A)$ where I^2 is the Borel regular integralgeometric Favard 2-dimensional measure in R^3.*

Proof. Since μ and I^2 are Borel regular we can assume that A is a Borel set. By Theorem 6.10 A can be decomposed into two Borel sets A_1, A_2 such that $A=A_1\cup A_2$, $A_1\cap A_2=\varnothing$, A_1 is countably rectifiable and $\mu(A_2)=0$. Then $I^2(A_2)=\mu(A_2)=0$ and by Theorem 2.8 and Federer [2, Theorem 5.14], $\mu(A_1)=H^2(A_1)=I^2(A_1)$. Hence $\mu(A)=I^2(A)$.

The problem as to whether $\mu(A)=I^2(A)$ for all sets $A\subset R^3$ is unsolved and is equivalent to the problem of showing that Theorem 6.10 is true if the assumption that Λ satisfies a weak projection inequality is replaced by $\Lambda=I^2$.

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THE OHIO STATE UNIVERSITY,
COLUMBUS, OHIO