ON CANONICAL CONFORMAL MAPS OF MULTIPLY CONNECTED REGIONS

BY

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The first-named author has recently proved the following theorem [3]:

**Theorem 1.** Let \( D \) be a region of the extended \( z \)-plane whose boundary consists of mutually disjoint Jordan curves \( B_1, B_2, \ldots, B_\mu; \ C_1, C_2, \ldots, C_\nu \), \( \mu \neq 0 \). There exists a conformal map of \( D \) onto a region \( \Delta \) of the extended \( Z \)-plane one to one and continuous in the closures of the two regions, where \( \Delta \) is defined by

\[
1 < | T(Z) | < e^{1/\tau}, \quad T(Z) = \frac{A(Z - a_1)^{M_1} \cdots (Z - a_\mu)^{M_\mu}}{(Z - b_1)^{N_1} \cdots (Z - b_\nu)^{N_\nu}},
\]

with

\[
M_i, N_j, \tau > 0, \quad \sum M_i = \sum n_j = 1.
\]

The locus \( | T(Z) | = 1 \) consists of \( \mu \) mutually disjoint Jordan curves \( B_\mu^* \), respective images of the \( B_i \), which separate \( \Delta \) from the \( a_i \); the locus \( | T(Z) | = e^{1/\tau} \) consists of \( \nu \) mutually disjoint Jordan curves \( C_\nu^* \), respective images of the \( C_j \), which separate \( \Delta \) from the \( b_j \). The number \( \tau \) is a conformal invariant of \( D \) and represents the period of the conjugate function of the harmonic measure of the set \( \cup_{i=1}^\mu B_i \) with respect to \( D \) around the set of curves \( \cup_{j=1}^\nu C_j \).

The region \( \Delta \) represents a new canonical region for the conformal mapping of multiply connected domains; it has the property that the harmonic measure of the union of the curves \( C_\nu^* \) with respect to \( \Delta \) is extendable harmonically to the entire \( Z \)-plane, except for one point in each of the components of the complement of \( \Delta \).

The object of the present paper is to demonstrate, using closely related methods, that the new theorem extends to the case of domains in which the sets \( \cup_{i=1}^\mu B_i \) and \( \cup_{j=1}^\nu C_j \) are made up of Jordan curves that are not necessarily disjoint. This result will later prove useful in the study of approximation by rational and by bounded analytic functions.

We begin with some necessary definitions:

A **Jordan configuration** is a finite collection of Jordan curves, no subset of which forms a closed cycle of Jordan curves; that is, every connected set of constituent curves is disconnected by the removal of any two points from any one of the curves.

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A multiple point of a Jordan configuration is a point of the configuration which lies on more than one of the constituent curves. A multiple point is described uniquely by specifying the curves on which it lies, for if two different multiple points lie on each curve of a set of curves, the curves involved form a cycle, contrary to hypothesis.

Theorem 2. Let $D$ be a region of the extended $z$-plane bounded by a Jordan configuration consisting of Jordan curves $B_1, B_2, \ldots, B_\mu, C_1, C_2, \ldots, C_\nu$, with $\mu, \nu \neq 0$, and $U_{i=1}^\mu B_i$ disjoint from $U_{j=1}^\nu C_j$. There exists a conformal map of $D$ onto a region $\Delta$ of the extended $Z$-plane, one to one and continuous in the closures of the two regions, where $\Delta$ is defined by

$$1 < \left| T(Z) \right| < e^{1/\tau}, \quad T(Z) = \frac{A(Z-a_1)^{M_1} \cdots (Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1} \cdots (Z-b_\nu)^{N_\nu}}, \text{ with } M_i, N_j, \tau > 0,$$

(2)

and $\sum M_i = \sum N_j = 1$.

The locus $\left| T(Z) \right| = 1$ is a Jordan configuration composed of $\mu$ Jordan curves $B_i^*$, respective images of the $B_i$, which separate $\Delta$ from the $a_i$; the locus $\left| T(Z) \right| = e^{1/\tau}$ is a Jordan configuration composed of $\nu$ Jordan curves $C_j^*$, respective images of the $C_j$, which separate $\Delta$ from the $b_j$. The number $\tau$ is a conformal invariant of $D$ and represents the period of the conjugate function of the harmonic measure of the set $U_{i=1}^\mu B_i$ with respect to $D$ around the set of curves $U_{j=1}^\nu C_j$.

If a point $a_i$ or $b_j$ is at infinity the corresponding factor in (2) is to be omitted.

We may assume that $D$ is bounded and that the curves $B_i$ and $C_j$ are analytic in the neighborhood of every point which is not a multiple point of the configuration, since that property may always be obtained by a preliminary conformal transformation of $D$.

Let $D_k$ be a sequence of domains in the $z$-plane which satisfy:

1. $D_{k+1} \subset D_k$; $D \subset D_k$ for all $k$;
2. $D_k$ is bounded by analytic curves $B_i^k$, $C_j^k$, separated from $D$ by $B_i$, $C_j$ respectively, $i = 1, \ldots, \mu$, $j = 1, \ldots, \nu$;
3. $\lim_{k \to \infty} D_k = D$.

Let $u_k(z)$ be the function harmonic in $D_k$ which takes the values 0, 1 on $U_{i=1}^\mu B_i^k$ and on $U_{j=1}^\nu C_j^k$ respectively, and let $u(z)$ be the function harmonic in $D$ which takes the values 0, 1 on $U_{i=1}^\mu B_i$ and on $U_{j=1}^\nu C_j$ respectively.

The domains $D_k$ are bounded by mutually disjoint Jordan curves and thus, by Theorem 1, for every $D_k$ there exists a canonical conformal map $Z = f_k(z)$, taking $D_k$ onto a domain $\Delta_k$ of the $Z$-plane, defined by $1 < \left| T_k(Z) \right| < e^{1/\tau_k}$, where

$$T_k(Z) = \frac{A_k(Z-a_{1k})^{M_{1k}} \cdots (Z-a_{\mu k})^{M_{\mu k}}}{(Z-b_{1k})^{N_{1k}} \cdots (Z-b_{\nu k})^{N_{\nu k}}} \text{, with } M_{i,k}, N_{j,k}, \tau_k > 0,$$

(3)

$$\sum M_{i,k} = \sum N_{j,k} = 1.$$
and $a_{ik}, b_{jk}$ are separated from $\Delta_k$ by $B^*_i, C^*_j$, the images under $f_k(z)$ of $B^*_i, C^*_j$ respectively.

The boundary of $D$ (thus of every $D_k$) consists of at least three Jordan curves, or else $D$ falls into the category, already studied, of a domain with disjoint boundary curves. Following [3], select $B_1, B_2, C_1$ on different boundary components of $D$, and choose, by a suitable linear transformation of the $Z$-plane, $a_{1k} = 0, a_{2k} = 1, b_{1k} = \infty$ in $T_k(Z)$. As $k \to \infty$ there exists a partial sequence of $k$ such that all the numbers $A_k, a_{ik}, M_{ik}, b_{jk}, N_{jk}, \tau_k$ approach limits (finite or infinite) $A, a, M, b, N, \tau$; we will consider only this partial sequence hereforth. The functions $f_k(z)$ admit in $D_k$, thus also in $D$, the exceptional values $0, 1, \infty$ and consequently form a normal family in $D$. We may therefore restrict ourselves further to a subsequence of values $k$ for which the functions $f_k(z)$ approach a limit $f(z)$, uniformly on every compact in $D$.

**Proposition 1.** $f(z)$ is not identically constant, hence defines a univalent map of $D$.

Continuing with the method of [3] we choose an analytic Jordan curve $\gamma$ in $D$ which separates $U^*_i \cup B_i$ from $U^*_j \cup C_j$. The image of $\gamma$ under $f_k(z)$ surrounds both $Z = 0$ and $Z = 1$, thus

$$\Delta_\gamma[\arg f_k(z)] = \Delta_\gamma[\arg (f_k(z) - 1)] = 2\pi,$$

which contradicts $f_k(z) \to \infty$ uniformly on $\gamma$. By use of an auxiliary linear transformation, $f_k(z) \to \infty$ is similarly seen to be impossible.

The above proof requires modification if $D$ is doubly connected. In this case we choose a point $\alpha$ in $D$, and require $a_{1k} = 0, b_{1k} = \infty, f_k(\alpha) = 1$. The functions $f_k(z)$ admit in $D_k - \alpha$ and in $D - \alpha$ the exceptional values $0, 1, \infty$. On a circumference with center $\alpha$ whose closed interior lies in $D$ we have $\Delta \arg [f_k(z) - 1] = 2\pi$, so the proof can be completed as before. —For this modification the writers are indebted to Mr. Vincent Williams. A similar modification may be used in [3] to treat the case that $D$ is doubly connected, and indeed was used [3, p. 142] in the proof of a limiting case of Theorem 1, as in the proof of Theorem 3 below.

**Proposition 2.** $0 < \tau < \infty$; $\tau$ represents the period around the set of curves, $U^*_i \cup B_i$ of the conjugate function of the harmonic measure $u(z)$.

The function $\tau_k \log |T_k[f_k(z)]|$ is defined and harmonic on $D_k$ and takes the values $0, 1$ on $U^*_i \cup B^*_i$ and $U^*_j \cup C^*_j$ respectively; it thus coincides everywhere in $D_k$ with $u_k(z)$. Hence, denoting by $n$ the interior normal,

$$\frac{1}{2\pi} \int_{U^*_i \cup B^*_i} \frac{\partial u_k}{\partial n} \, ds = \frac{\tau_k}{2\pi} \int_{U^*_i \cup B^*_i} \frac{\partial}{\partial n} \log |T_k[f_k(z)]| \, ds$$

$$= \frac{\tau_k}{2\pi} \int_{U^*_i \cup B^*_i} \frac{\partial}{\partial n} \log |T_k(Z)| \, ds = \tau_k.$$
The function $u_k(z) - u_{k-1}(z)$ is harmonic everywhere in $D_k$ and has values $<0, >0$ on $U_{j-1}^k B_j^k$ and $U_{j+1}^k C_j^k$ respectively. Let $v_k(z)$ be the function harmonic in $D_k$ which takes the value 0 on $U_{j-1}^k B_j^k$ and which coincides with $u_k(z) - u_{k-1}(z)$ on $U_{j+1}^k C_j^k$. Then, letting $w_k(z) = v_k(z) - [u_k(z) - u_{k-1}(z)]$, we have $w_k(z)$ harmonic in $D_k$, $w_k(z) = 0$ on $U_{j+1}^k C_j^k$, and $w_k(z) > 0$ on $U_{j-1}^k B_j^k$. The function $w_k(z)$ may be extended harmonically beyond the curves $U_{j+1}^k C_j^k$; consequently $\partial w_k(z)/\partial n$ is defined on $U_{j+1}^k C_j^k$ and, since $w_k(z) > 0$ throughout $D_k$, is positive there. Thus

\[
(5) \quad \int_{U_{j+1}^k C_j^k} \frac{\partial w_k}{\partial n} ds > 0.
\]

The function $w_k(z)$ may be extended harmonically across the curves $U_{j+1}^k B_j^k$ since $v_k(z)$, $u_k(z)$, and $u_{k-1}(z)$ may all be. Hence $\partial w_k(z)/\partial n$ exists on $U_{j+1}^k B_j^k$, and since

\[
\int_{U_{j-1}^k B_j^k + U_{j+1}^k C_j^k} \frac{\partial w_k}{\partial n} ds = 0
\]

we have by (5)

\[
\int_{U_{j-1}^k B_j^k} \frac{\partial w_k}{\partial n} ds < 0,
\]

that is

\[
(6) \quad \int_{U_{j-1}^k B_j^k} \frac{\partial}{\partial n} [u_k(z) - u_{k-1}(z)] ds > \int_{U_{j-1}^k B_j^k} \frac{\partial v_k(z)}{\partial n} ds.
\]

But $v_k(z) = 0$ on $U_{j+1}^k B_j^k$ and $>0$ in $D_k$, thus $\partial v_k/\partial n > 0$ on $U_{j+1}^k B_j^k$, which implies, by (4), $\tau_k > \tau_{k-1}$, and in particular $\tau_k > \tau_1 > 0$.

Next let $\gamma_0$ be an analytic Jordan curve in $D$ homotopic to $U_{j+1}^k B_j^k$. By virtue of (4) and of the harmonicity of $u_k(z)$ in $D_k$ we have

\[
\tau_k = \frac{1}{2\pi} \int_{\gamma_0} \frac{\partial u_k}{\partial n} ds.
\]

By a theorem of Lebesgue [2], $\lim_{k \to \infty} u_k(z) = u(z)$, uniformly on every closed subset of $D$, thus also in some closed neighborhood of $\gamma_0$, whence

\[
\tau = \lim_{k \to \infty} \tau_k = \frac{1}{2\pi} \int_{\gamma_0} \frac{\partial u}{\partial n} ds < \infty.
\]

Thus $0 < \tau < \infty$, and $\tau$ represents the period around the set of curves $U_{j+1}^k B_j^k$ of the conjugate of the harmonic measure $u(z)$.

**Proposition 3.** $M_r, N_s > 0$.

From the form (2) of $T_k(Z)$, and as in Proposition 2,
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\[ M_{rk} = \frac{1}{2\pi} \int_{|z| = r} \frac{\partial \log |T_k(z)|}{\partial n} \, ds = \frac{1}{2\pi \tau_k} \int_{|z| = r} \frac{\partial u_k(z)}{\partial n} \, ds. \]

Consider the function \( v^k_r(z) \) harmonic in the region bounded by \( B^k_r, U_v^* B^k_r, \) and \( U_{j-1} C^k_j \), which takes the values 0 on \( B^k_r \) and on \( U_v^* B^k_r \), and 1 on \( U_{j-1} C^k_j \). We have \( v^k_r(z) > 0 \) on \( B^k_r \), so that \( v^k_r(z) - u_k(z) > 0 \) on \( B^k_r \), and on \( U_{j-1} C^k_j \), whence

(7) \[ \frac{\partial}{\partial n} [v^k_r(z) - u_k(z)] > 0, \quad z \text{ in } U_v^* B^k_r \text{ and in } U_{j-1} C^k_j, \text{ with } n \text{ the interior normal}. \]

Since \( v^k_r - u_k \) is harmonic in \( D_k \) and in a neighborhood of the boundary of \( D_k \),

\[ \int_{\partial D_k} \frac{\partial}{\partial n} [v^k_r - u_k] \, ds = 0, \]

whence by (7)

\[ \int_{\partial D_k} \frac{\partial}{\partial n} [v^k_r - u_k] \, ds < 0. \]

Thus

(8) \[ \int_{\partial D_k} \frac{\partial}{\partial n} u_k \, ds > \int_{\partial D_k} \frac{\partial}{\partial n} v^k_r \, ds = \int_{\partial D_k} \frac{\partial}{\partial n} v_r \, ds. \]

Next let \( w_r(z) \) be the function harmonic in the region bounded by \( B^k_r, U_v B^k_r, \) and by \( U_{j-1} C^k_j \), which takes the values 0 on \( B^k_r \) and on \( U_v B^k_r \), and 1 on \( U_{j-1} C^k_j \). Then the function \( v^k_r(z) - w_r(z) \) is zero on \( B^k_r \), >0 on \( U_v B^k_r \) and on \( U_{j-1} C^k_j \), so that \( \partial [v^k_r - w_r] / \partial n > 0 \) on \( B^k_r \), whence

(9) \[ \int_{\partial B^k_r} \frac{\partial}{\partial n} v^k_r \, ds > \int_{\partial B^k_r} \frac{\partial}{\partial n} w_r \, ds. \]

Combining (8) and (9) we obtain

\[ M_{rk} = \frac{1}{2\pi \tau_k} \int_{\partial B^k_r} \frac{\partial u_k}{\partial n} \, ds > \frac{1}{2\pi \tau_k} \int_{\partial B^k_r} \frac{\partial w_r}{\partial n} \, ds. \]

The normal derivative \( \partial w_r / \partial n > 0 \) on \( B^k_r \), and \( 1/\tau_k > 1/\tau > 0 \), thus

(10) \[ \frac{1}{\tau} \int_{\partial B^k_r} \frac{\partial w_r(z)}{\partial n} \, ds > 0. \]

The identical argument applied to \( N_{rk} \) yields \( N_r > 0 \).

PROPOSITION 4. \( a_i \neq b_j, i = 1, \cdots, \mu; j = 1, \cdots, v \).
We may find in \( D \) an analytic Jordan curve \( \gamma \) which separates \( \bigcup_{i=1}^{r} B_i \) from \( \bigcup_{j=1}^{s} C_j \), that is one for which
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f_k'(z)dz}{f_k(z) - a_{ik}} = 1,
\]
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f_k'(z)dz}{f_k(z) - b_{jk}} = 0.
\]
Letting \( k \to \infty \) in the above integrals we obtain
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)dz}{f(z) - a_i} = 1
\]
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)dz}{f(z) - b_j} = 0,
\]
which implies that \( a_i \neq b_j \).

If an analytic Jordan curve \( \gamma \) in \( D \) separates \( B_r \) and \( B_s \), it follows as in the proof of Proposition 4 that the image of \( \gamma \) under the transformation \( Z = f(z) \) separates the images of \( B_r \) and \( B_s \). Otherwise expressed, if \( \gamma \) separates an annular neighborhood of \( B_r \) in \( D \) from such a neighborhood of \( B_s \), then the image of \( \gamma \) separates the images in the \( Z \)-plane of the corresponding neighborhoods. We use here the following generalization (Carathéodory, [1, §120]) of Hurwitz's Theorem: if \( f_k(z) \) is schlicht, \( f_k(z) \to f(z) \neq \text{const} \), uniformly in a region \( D_0 \), and if \( D_0^* \) is a closed subregion of \( D_0 \), then all points of \( w = f(D_0^*) \) lie in \( w = f_k(D_0) \) for \( k \) sufficiently large. Similarly for \( B_r \) and \( C_s \).

**Proposition 5.** \( a_i \neq a_s, b_r \neq b_s \) if \( r \neq s \).

By a preliminary linear transformation of the \( Z \)-plane let all the \( a_i, b_j \) lie in a finite part of the \( Z \)-plane. If \( a_r = a_s, r \neq s \), then given \( \epsilon > 0 \) we may find \( k_0 \) so that \( |a_{rk} - a_{sk}| < \epsilon \) for all \( k > k_0 \). Any two points \( a_{rk}, a_{sk} \) are separated from each other by a curve of the locus \( |T_k(Z)| = 1 \); thus there will exist points \( Z_k \) with \( |Z_k - a_{rk}| < \epsilon \) and \( |T_k(Z_k)| = 1 \). However, because the points \( a_i \) are disjoint from the \( b_j \) (Proposition 4) the quantities \( 1/|Z - b_{jk}|^N_{jk} \) are all bounded independently of \( k \) if only \( k \) is sufficiently large and \( Z \) is in a sufficiently small neighborhood of \( a_{rk} \). The \( |Z - a_{ik}|^M_{ik} \) are also bounded uniformly in \( k \) under the same conditions, and since by the proof of Proposition 3 we have \( M_{rk} > \alpha > 0 \), it follows that \( |Z_k - a_{rk}| < \epsilon \) implies \( |T_k(Z_k)| < G\epsilon^{M_{rk}} < G\epsilon^\alpha < 1 \) for \( \epsilon \) sufficiently small, contradicting the requirement \( |T_k(Z_k)| = 1 \). By the identical argument \( b_r \neq b_s \) if \( r \neq s \), and Proposition 5 is established. Let
\[
T(Z) = \frac{A(Z - a_1)^{M_1} \cdots (Z - a_\mu)^{M_\mu}}{(Z - b_1)^{N_1} \cdots (Z - b_\nu)^{N_\nu}};
\]
we have \( M_i, N_j > 0, \sum M_i = \sum N_j = 1 \). On any closed set in the \( Z \)-plane which contains none of the points \( b_j, j = 1, \cdots, \nu \), the functions \( T_k(Z) \) converge uniformly to \( T(Z) \). If \( \Delta \) is the domain in the \( Z \)-plane defined by \( 1 < |T(Z)| < e^{1/\tau} \), the closure \( \overline{\Delta} \) of \( \Delta \) contains none of the points \( b_j \), and thus in particular \( T_k(Z) \to T(Z) \) uniformly on \( \overline{\Delta} \). Thanks to Proposition 5, we can define \( \Delta \) as the limit of \( \Delta_k \) as \( k \to \infty \). The domain \( \Delta \) is connected, since no region of the
plane can be "pinched off" from $\Delta_k$ by the approach of boundary curves of $\Delta_k$ to the boundary curves of $\Delta$ (the latter are not disjoint). For if part of the plane were pinched off, the argument following Proposition 4, applied to an analytic Jordan curve $\gamma$ in $D$ which separates $\bigcup_{i=1}^{m} B_i$ from $\bigcup_{j=1}^{n} C_j$, establishes that such a region would be bounded wholly by points of the locus $|T(Z)| = 1$ or wholly by points of the locus $|T(Z)| = e^{1/r}$, and would contain no pole or zero of $T(Z)$, which is impossible; the two loci mentioned can have no point in common.

We now consider the image under $Z=f(z)$ of a compact subdomain $D^*$ of $D$. On every $D^*$, the function $f(z)$ is the uniform limit of $f_k(z)$, so that the image of $D^*$ is the limit of the domains $f_k(D^*)$. But the domain $f_k(D^*)$ is included in the domain $f_k(D_k)=\Delta_k$, hence the limit $f(D^*)$ is included in the limit $\Delta$ of the domains $\Delta_k$. We conclude that $D$ is mapped by $f(z)$ onto a subdomain $\Delta_0$ of $\Delta$. As in [3] we next consider the inverse functions $f_k^{-1}(z)$, defined and univalent in the regions $\Delta_k$ and thus, for $k$ sufficiently large, in an arbitrary compact subdomain of $\Delta$. The functions $f_k^{-1}(z)$ are bounded in $\Delta_k$, hence form a normal family in $\Delta$, and we may restrict ourselves to a subsequence which approaches a limit $F(Z)$ on $\Delta$, uniformly on every compact subset of $\Delta$; $F(Z)$ is either univalent or identically constant on $\Delta$. By the argument given above, $\Delta$ is mapped by $F(Z)$ onto a subset $D_0$ of $D$. For $Z_0 \in \Delta$, we have $\lim_{k \to \infty} f_k^{-1}(Z_0) = F(Z_0) \in D_0$, so that for $k$ sufficiently large and $Z_0$ fixed all the points $f_k^{-1}(Z_0)$ lie in a compact subdomain of $D$. Then by Carathéodory’s criterion for continuous convergence [1], we have $f[F(Z_0)] = \lim_{k \to \infty} f_k[f_k^{-1}(Z_0)] = Z_0$, hence $f(z)$ is the inverse function of $F(Z)$ for $Z \in \Delta$ and $z \in D_0$. This means that the image $\Delta_0$ of $D$ under $f(z)$ coincides with $\Delta$, for every value $Z_0$ in $\Delta$ is assumed by $f(z)$ at the corresponding point $F(Z_0) \in D$.

**Proposition 6.** The locus $|T(Z)| = 1$ is the limit of the loci $|T_k(Z)| = 1$ as $k \to \infty$ and consists of $\mu$ Jordan curves $B_1^*$, definable as $B_1^* = \lim_{k \to \infty} B_1^*; B_1^*$ separates $\Delta$ from the point $a_1$. Similarly, the locus $|T(Z)| = e^{1/r}$ is the limit of the loci $|T_k(Z)| = e^{1/r_k}$ and consists of $\nu$ Jordan curves $C_1^*$, definable as $C_1^* = \lim_{k \to \infty} C_1^*; C_1^*$ separates $\Delta$ from the point $b_1$.

Every critical point of $T(Z)$ (of which the total number is $\mu + \nu - 2$) is a limit point of critical points of $T_k(Z)$, by Hurwitz’s Theorem. Every point of the locus $|T(Z)| = 1$ is a limit point of points of the loci $|T_k(Z)| = 1$ as $k \to \infty$.

We show next that $B_1^* = \lim_{k \to \infty} B_1^*$ is a Jordan curve. $B_1^*$ forms a part of the locus $|T(Z)| = 1$. We prove first that a closed arc $A$ of the locus $|T(Z)| = 1$ not containing a critical point of $T(Z)$ cannot be the limit of more than one arc of the locus $|T_k(Z)| = 1$. Let $N(A)$ be a neighborhood of $A$ bounded by arcs of $|T(Z)| = 1 + \varepsilon$, of $|T(Z)| = 1 - \varepsilon$ (with $0 < \varepsilon < 1/2$), and by arcs of two orthogonal trajectories of $|T(Z)| = \text{constant}$. If $N(A)$ is suitably chosen, throughout $N(A)$ we have $T'(Z) \neq 0$ and $|d \log T(Z)/dZ| \geq 2\delta > 0$, with
\[ \frac{\partial \log |T(Z)|}{\partial s} = 0 \] and \[ \frac{\partial \log |T(Z)|}{\partial n} \geq 2\delta > 0, \] where the latter directions are taken respectively along and orthogonal to the loci \( |T(Z)| = \text{constant} \). For \( k \) sufficiently large in \( N(A) \) we have \[ \frac{\partial \log |T_k(Z)|}{\partial n} \geq \delta. \] The loci \( |T_k(Z)| = \lambda \) approach uniformly the loci \( |T(Z)| = \lambda \), in both position and direction throughout \( N(A) \), so if \( N(A) \) contains one arc of \( |T_k(Z)| = 1 \) it cannot contain a second; thus \( A \) cannot be the limit of more than one arc of the locus \( |T_k(Z)| = 1 \). This establishes that in tracing the curve \( B^*_k \) no arc of it is traversed more than once. Furthermore, every closed Jordan curve on which \( |T(Z)| = 1 \) must separate \( \Delta \) from one of the points \( a_i \), otherwise in one of the regions into which the curve divides the plane the function \( \log |T(Z)| \) is harmonic and has constant boundary values, which is impossible. Thus \( B^*_k \) cannot be made up of more than one closed curve. We conclude that \( B^*_k = \lim_{k \to \infty} B^*_k \) is a Jordan curve, separating \( \Delta \) from the point \( a_i \).

Proceeding identically for the curves \( C^*_r \) we obtain Proposition 6.

We have already shown, in establishing the connectedness of \( \Delta \), that no subset of the curves \( B^*_k \) and \( C^*_r \) can form a closed cycle. Thus the domain \( \Delta \) is bounded by a Jordan configuration. The argument following Proposition 4 may be applied in \( D \) and in \( \Delta \) to yield that a set of curves \( B^*_m \) or \( C^*_n \) forms a continuum if and only if the corresponding curves \( B^*_m \) or \( C^*_n \) form a continuum; we conclude that the boundary of \( \Delta \) consists of a one to one transform of the curves (not points) of \( D \), with each continuum transformed into a continuum. It will be noted that the locus \( |T(Z)| = 1 \) has a critical point only where two or more of the curves \( B^*_k \) intersect, and the order of the critical point corresponds to the number of curves intersecting; otherwise an arc of that locus penetrates a region bounded by part of that locus and throughout which we have either \( |T(Z)| < 1 \) or \( |T(Z)| > 1 \). Similarly for the locus \( |T(Z)| = e^{1/r} \).

It remains to study the boundary correspondence effected by the map \( f(z) \).

**Proposition 7.** \( Z = f(z) \) maps the curve \( B^*_r \) onto the curve \( B^*_r \), and the curve \( C^*_r \) onto the curve \( C^*_r \), \( r = 1, \ldots, \mu; s = 1, \ldots, \nu \).

The Jordan curve \( B^*_r \) has at most a finite number of points in common with the remaining boundary curves of \( D \); denote these points by \( p_1, \ldots, p_m \). Similarly, the curve \( B^*_r \) has at most a finite number of points in common with the remaining boundary curves of \( \Delta \); denote these by \( P_1, \ldots, P_n \). The results of Carathéodory on boundary correspondence may be applied to the map \( f(z) \) [1]: they establish that \( Z = f(z) \) and \( z = f^{-1}(Z) \) are one to one and continuous on the boundary, provided that a boundary point is construed as an equivalence class of arcs drawn in the region \( D \) or \( \Delta \) from some fixed interior point to the boundary. In the present case this implies ordinary continuity and one to oneness of \( f(z) \) in the neighborhood of any point \( z_0 \) for which both \( z_0 \) and \( f(z_0) \) lie on only one boundary curve.
Let \( v_k(z) \) be the function harmonic in \( D_k \), equal to 0 on \( B^*_k \) and to 1 on the remaining boundary curves of \( D_k \), and let \( v(z) \) be the function harmonic and bounded in \( D \), equal to 0 on \( B_r \) and to 1 on the remaining boundary curves of \( D \), with the exception of the points \( p_1, p_2, \ldots, p_m \), where it is undefined. Similarly, let \( V_k(Z) \) be the function harmonic in \( \Delta_k \), equal to 0 on \( B^*_r \) and to 1 on the remaining boundary curves of \( \Delta_k \), and let \( V(Z) \) be the function harmonic and bounded in \( \Delta \), equal to 0 on \( B_r \) and to 1 on the remaining boundary curves of \( \Delta \), with the exception of the points \( P_1, \ldots, P_n \), where it is undefined. We shall prove that \( V_k(Z) \to V(Z) \), uniformly on any closed subset \( \Delta_0 \) of \( \Delta \), indeed on any closed subset \( \Delta_0 \) of \( \Delta \) containing no point \( P_j \); an analogous proof establishes \( v_k(z) \to v(z) \), uniformly on any closed subset \( D_0 \) of \( D \) containing no point \( p_j \).

Let \( V^\delta(Z) \) and \( V^\delta_k(Z) \) be functions defined similarly to \( V(Z) \) and \( V_k(Z) \) respectively, with boundary values as before except that those values in the circle \( |Z - p_j| \leq \delta \) are now to be replaced by \( |Z - p_j|/\delta \) on \( U \cup \overline{B'_r} \) and \( U \cup \overline{B'_r} \).

Let \( R \) be twice the diameter of \( \Delta \), and set

\[
Q^\delta(Z) = \sum_{j=1}^{n} \frac{\log R - \log |Z - P_j|}{\log R - \log \delta}, \quad \delta < R,
\]

whence \( Q^\delta(Z) > 0 \) in \( \Delta \), and \( \lim_{\delta \to 0} Q^\delta(Z) = 0 \), uniformly for \( Z \) in \( \Delta_0 \).

Let \( \Delta_0 \) be given, and let \( \epsilon (\epsilon > 0) \) be arbitrary. Fix \( \delta \) so small that we have \( Q^\delta(Z) < \epsilon/3 \) for \( Z \) in \( \Delta_0 \), and that \( \bigcup_{j=1}^{n} C^*_j \) and \( \bigcup_{j=1}^{n} C^*_j \) have no point in any \( |Z - P_j| < \delta \). From a study of the respective boundary values we conclude, except for \( Z = P_j \),

\[
0 < V(Z) - V^\delta(Z) < Q^\delta(Z), \quad Z \text{ in } \overline{\Delta},
\]
\[
0 < V_k(Z) - V^\delta_k(Z) < Q^\delta(Z), \quad Z \text{ in } \overline{\Delta_k},
\]
whence for \( Z \) in \( \Delta_0 \)

\[
0 < V(Z) - V^\delta(Z) < \epsilon/3,
\]
\[
0 < V_k(Z) - V^\delta_k(Z) < \epsilon/3.
\]

(11)

The boundary of \( \Delta_k \) approaches that of \( \Delta \) as \( k \to \infty \), and the assigned boundary values of \( V^\delta_k(Z) \) on \( \Delta_k \) are continuous and approach those of \( V^\delta(Z) \) on \( \Delta \) uniformly, so by a theorem of Lebesgue [2] for \( k \) sufficiently large we have uniformly in \( \Delta \),

\[
| V^\delta(Z) - V^\delta_k(Z) | < \epsilon/3;
\]

this holds in particular for \( Z \) in \( \Delta_0 \), whence by (11)

\[
| V(Z) - V_k(Z) | < \epsilon, \quad Z \text{ in } \Delta_0,
\]
provided only \( k \) is sufficiently large. Proceeding analogously in the \( z \)-plane, we obtain

\[
v(z) = \lim_{k \to \infty} v_k(z),
\]

uniformly on any closed subset of \( \overline{D} \) containing no point \( p_j \).

The function \( V_k[f_k(z)] \) is harmonic in \( D_k \) and coincides with \( v_k(z) \) on the boundary curves of \( D_k \), thus \( V_k[f_k(z)] = v_k(z) \). By (12) we have

\[
\lim_{k \to \infty} V_k[f_k(z)] = v(z),
\]

uniformly on any closed subdomain of \( D \).

Given \( z_0 \in B_r \), suppose \( f(z_0) \notin B_r^{*} \). Then by the Carathéodory continuity of \( f(z) \), there is a whole arc of \( B_r \) taken by \( f(z) \) onto some \( B_{q}^{*} \) for \( q \neq r \), and thus \( V[f(z)] = 1 \) on an arc of \( B_r \). Consequently, \( V[f(z)] \neq v(z) \) on a whole arc of the boundary of \( D \), and thus there exists \( z_1 \) interior to \( D \) with \( V[f(z_1)] \neq v(z_1) \).

Since \( z_1 \) lies interior to \( D \) we have \( Z_k = f_k(z_1) \to f(z_1) = Z_1 \), and in a neighborhood of \( Z_1 \) the functions \( V_k \) converge uniformly to \( V \); by Carathéodory's criterion for continuous convergence [1], it follows that \( V_k(Z_k) \to V(Z_1) \). But this, together with (13), yields \( v(z_1) = \lim_{k \to \infty} V_k(Z_k) = V(Z_1) = V[f(z_1)] \), and we have a contradiction.

The identical argument applied to the inverse functions \( f^{-1}(Z) \) and \( f^{-1}_k(Z) \) shows that the image of \( B_r \) under \( f(z) \) covers all of \( B_r^{*} \).

Proceeding as above for the curves \( C_s \), we obtain Proposition 7.

**Proposition 8.** The multiple points \( p_1, \ldots, p_m \) of \( B_r \) are mapped by \( f(z) \) in a one to one way onto the multiple points \( P_1, \ldots, P_n \) of \( B_r^{*} \), \( r = 1, \ldots, \mu \); similarly for the multiple points of \( C_s \), \( s = 1, \ldots, \nu \).

Since \( \bigcup_{i=1}^{\nu} B_i \) forms a Jordan configuration, a multiple point \( p_i \) of \( B_r \) is described uniquely by specifying the constituent curves on which it lies. If \( p_i \in B_r \cap B_{i_1} \cdots \cap B_{i_q} \), by Proposition 7 every image \( f(p_i) \in B_{i_q}^{*} \), \( B_{i_1}^{*} \), \ldots, \( B_{i_q}^{*} \), but, since \( \bigcup_{i=1}^{\nu} B_i^{*} \) is likewise a Jordan configuration, \( f(p_i) \) is thereby determined to be a single multiple point of \( B_r^{*} \). Applying the identical argument to \( f^{-1}(Z) \) we obtain the desired one to one correspondence.

Proceeding as above for the curves \( C_s \), we obtain Proposition 8.

This completes the proof of the one to oneness and continuity of \( f(z) \) on the boundary of \( D \). For given \( z_0 \in B_i \) not a multiple point, \( f(z_0) \in B_i^{*} \) is also not a multiple point (Proposition 8), and continuity and one-to-oneness of \( f(z) \) in a neighborhood of \( z_0 \) follow from Carathéodory continuity and one-to-oneness. If \( z_0 \) is a multiple point of \( B_i \), \( z_0 \in B_{i_1}, B_{i_2}, \ldots, B_{i_q} \), by Proposition 8 the point \( f(z_0) \in B_{i_q}^{*} \), \( B_{i_1}^{*} \), \ldots, \( B_{i_q}^{*} \). Let \( \gamma \) be an arc in \( \overline{D} \) joining a fixed interior point \( Q \) to a point \( \xi \) on the boundary, and let \( \xi \) trace the curves \( B_{i_1}, B_{i_2}, \ldots, B_{i_q} \) in such a way that \( \gamma \) is continuously deformed and that every curve is covered only once; let the curves \( B_{i_1}, B_{i_2}, \ldots, B_{i_q} \) be numbered sequentially in order of being transversely by \( \xi \). Then for any two con-
secutive ones \( B_{i_m}, B_{i_{m+1}} \) there exist arcs in \( D \) from \( Q \) to \( z_0 \) which have in their neighborhoods arcs joining \( Q \) both to points on \( B_{i_m} \) and to points on \( B_{i_{m+1}} \). Carathéodory continuity now implies that \( f(z) \) is continuous and one to one in any neighborhood of \( z_0 \). Theorem 2 is established.

As in [3], the validity of Theorem 2 may be extended to the limiting case in which the curves \( B_i \) are allowed to shrink to points:

**Theorem 3.** Let \( D \) be a region of the extended \( z \)-plane bounded by a Jordan configuration consisting of Jordan curves \( C_1, \ldots, C_r \), let \( \alpha_1, \ldots, \alpha_r \) be arbitrary distinct points of \( D \), and let \( M_1, \ldots, M_r \) be arbitrary positive numbers, with \( \sum M_i = 1 \). Then there exists a conformal map of \( D \) onto a region \( \Delta \) of the extended \( Z \)-plane, one to one and continuous in the closures of the two regions where \( \Delta \) is defined by

\[
| T(Z) | < 1, \quad T(Z) = \frac{A(Z - a_1)^{M_1} \cdots (Z - a_r)^{M_r}}{(Z - b_1)^{N_1} \cdots (Z - b_r)^{N_r}}, \quad \text{with } N_j > 0, \sum N_j = 1.
\]

The \( a_i \) are respective images of the \( \alpha_i \); the locus \( | T(Z) | = 1 \) is a Jordan configuration composed of \( r \) Jordan curves \( C_i^* \), respective images of the \( C_i \), which separate \( \Delta \) from the \( b_j \).

The proof follows step by step that of Theorem 2, so that we may omit most of the details.

As before, take the curves \( C_j \) to be analytic in the neighborhood of any point not a multiple point of the configuration, and the domain \( D \) to be bounded. Consider a sequence of approximating domains \( D_k \), defined as in Theorem 2, and bounded by disjoint contours \( C_j^k \); by Theorem 3 of [3], the \( D_k \) may be mapped by \( Z = f_k(z) \) onto a sequence of domains \( \Delta_k \) in the \( Z \)-plane defined by

\[
| T_k(Z) | < 1, \quad T_k(Z) = \frac{A_k(Z - a_{ik})^{M_1} \cdots (Z - a_{rk})^{M_r}}{(Z - b_{i1})^{N_1} \cdots (Z - b_{rk})^{N_r}}, \quad \text{with } N_{jk} > 0, \sum N_{jk} = 1, a_{ik} = f_k(\alpha_i), b_{jk} \text{ separated from } \Delta_k \text{ by } C_j^*, \text{ the image under } f_k(z) \text{ of } C_j.
\]

We again choose \( a_{1k} = 0, b_{1k} = 1, b_{2k} = \infty \) in \( T_k(Z) \); the functions \( f_k(z) \) admit in \( D - \alpha_1 \) the exceptional values \( 0, 1, \infty \), thus form a normal family in \( D - \alpha_1 \). We restrict ourselves to a subsequence of values \( k \) for which \( f_k(z) \) converges uniformly on every compact subdomain of \( D - \alpha_1 \) to \( f(z) \), and for which all the numbers \( A_k, a_{ik}, b_{jk}, N_{jk} \) approach respective limits \( A, a_i, b_j, N_j \).

Proposition 1 follows as before, establishing that \( f(z) \) is not identically constant, hence a univalent map of \( D - \alpha_1 \). This implies univalence in \( D \) as well, for if \( f(z_1) = f(z_2) = Z_0 \), with \( z_1, z_2 \in D \) and \( z_1 \neq z_2 \), the function \( f(z) \) takes on every value in a neighborhood of \( Z_0 \) for \( z \) in a neighborhood both of \( z_1 \) and of \( z_2 \), contradicting the univalence of \( f(z) \) in \( D - \alpha_1 \).

If \( D \) is simply connected, we modify this proof as before.

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In Proposition 3, we have
\[ N_{rk} = -\frac{1}{2\pi} \int_{c_r} \frac{\partial}{\partial n} \log |T_k(Z)| \, ds = -\frac{1}{2\pi} \int_{c_r} \frac{\partial u_k(z)}{\partial n} \, ds, \]
where \( u_k(z) \) is the function harmonic in \( D_k \) except for singularities \( M_i \log |z - \alpha_i| \) in a neighborhood of each \( z = \alpha_i \), and which takes on the value 0 for \( z \in \bigcup_{j=1}^{r} C_j \). Precisely as before we obtain
\[ N_{rk} > -\frac{1}{2\pi} \int_{c_i} \frac{\partial}{\partial n} w_r(z) \, ds > 0, \]
where \( w_r(z) \) is the function harmonic in the domain bounded by \( C_i \) and by \( \bigcup_{j=1}^{r} C_j \), except for singularities \( M_i \log |z - \alpha_i| \) in a neighborhood of each \( z = \alpha_i \), and which has the value 0 for \( z \in C_i \), \( \bigcup_{j=1}^{r} C_j \).

Proposition 4 follows with no change.

In Proposition 5 we have immediately \( a_r \neq a_s \) for \( r \neq s \), since \( a_r = f(a_r) \) and \( a_s = f(a_s) \) with \( f(z) \) univalent, \( z \in D \). The proof that \( b_r \neq b_s \) if \( r \neq s \) proceeds without change.

Letting
\[ T(Z) = \frac{A(Z - a_1)^{M_1} \cdots (Z - a_\mu)^{M_\mu}}{(Z - b_1)^{N_1} \cdots (Z - b_\nu)^{N_\nu}}, \]
we have that the domains \( \Delta_k: |T_k(Z)| < 1 \) converge uniformly to \( \Delta: |T(Z)| < 1 \). By the same argument as in Theorem 2, \( \Delta \) is connected and the locus \( |T(Z)| = 1 \) is a Jordan configuration consisting of \( \nu \) Jordan curves, possibly with multiple points, given by \( C_j^* = \lim_{k \to \infty} C_j^* \), with \( C_j^* \) separating \( \Delta \) from \( b_j \). The discussion which shows that \( D \) is mapped by \( f(z) \) onto \( \Delta \) follows precisely as before.

The discussion of Propositions 7 and 8 now applies without change.

We conclude that \( f(z) \) maps the domain \( D \) conformally onto \( \Delta \), where \( \Delta \) is defined by \( |T(Z)| < 1 \), and is one to one and continuous in the closures of the two regions. Theorem 3 is established.

Under the conditions of Theorems 2 and 3, the maps whose existence is asserted are essentially unique. We outline the proof of

**Theorem 4.** Let \( D \), defined by (2) with \( Z \) replaced by \( z \), be a region of the extended \( z \)-plane bounded by a Jordan configuration consisting of Jordan curves \( B_1, B_2, \ldots, B_\mu, C_1, C_2, \ldots, C_\nu \), \( \mu \nu \neq 0 \), with \( \bigcup_{j=1}^{r} B_i \) disjoint from \( \bigcup_{j=1}^{r} C_j \). Let \( B_i \) separate \( a_i \) from \( D \), and \( C_j \) separate \( b_j \) from \( D \). Let \( D \) be mapped conformally onto a region \( \Delta \) of the extended \( Z \)-plane, one to one in the corresponding closed regions, where \( \Delta \) is defined by

\[ 1 < |T^*(Z)| < e^{1/1\ast}, \quad T^*(Z) = \frac{A^*(Z - a_1^*)^{M_1^*} \cdots (Z - a_\mu^*)^{M_\mu^*}}{(Z - b_1^*)^{N_1^*} \cdots (Z - b_\nu^*)^{N_\nu^*}}, \]
with \( M^*, N^*, \tau^* > 0 \), \( \sum M^* = \sum N^* = 1 \), and let \( \Delta \) be bounded by a Jordan configuration consisting of Jordan curves \( B^*_1, B^*_2, \ldots, B^*_\mu, C^*_1, C^*_2, \ldots, C^*_v \), respective images of the \( B_i \) and \( C_j \). Let \( B^*_i \) separate \( a^*_i \) from \( \Delta \) and \( C^*_j \) separate \( b^*_j \) from \( \Delta \). Then the transformation \( Z = Z(z) \) is defined by a linear transformation of the complex variable \( z \).

The proof of Theorem 4 is quite similar to that of [3, Theorem 2]. The harmonic measure of \( U_{j-1} C_j \) with respect to \( D \) is the transform of the harmonic measure of \( U_{j-1} C^*_j \) with respect to \( \Delta \), and each is readily expressible in terms of \( T(z) \) and \( T^*(Z) \). Consideration of the variation of the conjugate of this harmonic measure on corresponding parts of the boundaries of \( D \) and \( \Delta \) yields at once \( \tau = \tau^* \), \( M_i = M_i^* \), \( N_j = N_j^* \). No region exterior to \( D \) bounded by a Jordan curve belonging, say, to the locus \( |T(z)| = 1 \) can contain a critical point of \( T(z) \), for any locus \( |T(z)| = c, 0 < c < 1 \), divides the given region into precisely two subregions and cannot pass through a critical point of \( T(z) \). It can now be established, as in [3], that the assumed map \( Z = Z(z) \) of \( D \) onto \( \Delta \) can be enlarged so as to map the extended \( z \)-plane one to one and conformally onto the extended \( Z \)-plane, hence is a linear transformation.

In Theorem 4, it is essential to assume that the given map defines a one to one correspondence of boundary points, not merely that \( U_{i-1} B_i \) is transformed into \( U_{i-1} B^*_i \). For let \( D \) be a doubly connected region with \( \mu > 1, v > 1 \). Then \( D \) admits a one to one map onto itself corresponding to any nontrivial rotation of the plane about 0 when \( D \) is mapped onto an annulus bounded by two circles with common center 0. Such a map of \( D \) onto itself carries \( B = U_{i-1} B_i \) onto itself, and also \( C = U_{j-1} C_j \), but need not be a linear transformation of the \( z \)-plane; it cannot be a linear transformation of the \( z \)-plane if we have \( \mu = 2, M_1 \neq M_2 \), for then the map does not transform the double point of \( B \) into a double point.

We add some general remarks as background material for Theorems 2 and 3 above, which in particular place in perspective the case that the given domain \( D \) is simply connected, and exhibit the canonical nature of the image domains \( \Delta \).

**Case I.** Let \( D \) be a simply connected region bounded by a Jordan configuration composed of two Jordan curves \( B_1 \) and \( B_2 \) with precisely one multiple point. By the Riemann mapping theorem, \( D \) may be mapped conformally onto the interior of the unit circle, with the multiple point corresponding to two separate points on the circumference. Since there exists a map of the interior of the unit circle onto itself that carries three given points on the circumference onto any prescribed three points on the circumference which have the same order with respect to the positive sense of the boundary, it follows that any two domains \( D \) and \( D' \) of the type described may be mapped conformally one on the other by functions one to one and continuous in the closures of the two regions. The mapping function is not uniquely determined, but there is not enough freedom to ensure the correspondence of
two arbitrary interior points. Letting \( h(z) \) be the harmonic measure of \( B_1 \) with respect to \( D \), and \( H(Z) \) the transformed harmonic measure in \( D' \), an arbitrary point \( z_1 \) in \( D \) will necessarily be mapped onto a point of the locus \( H(Z) = h(z_1) \) in \( D' \); the mapping function may be fixed completely by requiring the point \( z_1 \) to correspond to a given point of that locus.

**Case II.** Let \( D \) be a simply connected region bounded by a Jordan configuration composed of 3 Jordan curves \( B_1, B_2, B_3 \) with a single (common) multiple point. By the reasoning of Case I, any two such domains are mappable conformally one onto the other by a function one to one and continuous in the closures of the two domains. The mapping function is completely determined.

**Case III.** Let \( D \) be a simply connected region of the \( z \)-plane bounded by any other Jordan configuration composed of Jordan curves \( B_1, B_2, \ldots, B_m \) with multiple points \( p_1, p_2, \ldots, p_r \); similarly, let \( D' \) be a simply connected region of the \( Z \)-plane bounded by a Jordan configuration composed of Jordan curves \( C_1, C_2, \ldots, C_m \), with multiple points \( q_1, q_2, \ldots, q_r \). Let \( P \) be any point interior to \( D \), and consider the conjugate function \( v_P(z) \) of the Green’s function of \( D \) with pole at \( P \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_r \) be all the level loci of \( v_P(z) \) joining \( P \) to the multiple points \( p_j, 1 \leq j \leq r \), and let \( 0 \leq \alpha_i < 2\pi \) be the angle with the horizontal, say, which \( \gamma_i \) forms at \( P \). Similarly, let \( Q \) be any point interior to \( D' \), let \( \delta_1, \ldots, \delta_r \) be the level loci of the conjugate of the Green’s function of \( D' \) with pole at \( Q \) which join \( Q \) to the multiple points \( q_j, 1 \leq j \leq r \), and let \( \beta_i \) be the angle with the horizontal, say, which \( \delta_i \) forms at \( Q \). The map of the interior of \( D \) onto the interior of the unit circle which takes \( P \) onto zero is completely determined up to a rotation; when appropriately normalized, it associates with the points \( p_1, \ldots, p_r \) the points \( e^{i\alpha_1}, \ldots, e^{i\alpha_r} \). Similarly, the map, properly normalized, of the interior of \( D' \) onto the interior of the unit circle which takes \( Q \) onto zero associates with the points \( q_1, \ldots, q_r \) the points \( e^{i\beta_1}, \ldots, e^{i\beta_r} \). There exists a conformal map \( Z = f(z) \) of \( D \) onto \( D' \), one to one and continuous in the closures of the two regions and taking \( p_1, \ldots, p_r \) onto \( q_1, \ldots, q_r \), if and only if there exists a map of the interior of the unit circle onto itself, taking the corresponding points on the unit circumference one on another, that is, renaming the points if necessary, one which takes \( e^{i\alpha_1}, \ldots, e^{i\alpha_r} \) onto \( e^{i\beta_1}, \ldots, e^{i\beta_r} \) respectively. A necessary and sufficient condition that there exist a conformal map of the unit circle onto itself which takes \( e^{i\alpha_i} \) onto \( e^{i\beta_i} \), \( i = 1, 2, 3, 4 \), is the equality of the cross-ratios, one of which is easily seen to be equal to

\[
X(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\sin \left( \frac{\alpha_3 - \alpha_1}{2} \right) \sin \left( \frac{\alpha_4 - \alpha_2}{2} \right)}{\sin \left( \frac{\alpha_3 - \alpha_2}{2} \right) \sin \left( \frac{\alpha_4 - \alpha_1}{2} \right)}.
\]
Thus a necessary and sufficient condition for the existence of the desired map of $D$ onto $D'$ is that

$$X(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}) = X(\beta_i, \beta_{i+1}, \beta_{i+2}, \beta_{i+3}), \quad 1 \leq i \leq s - 3,$$

with $X$ as defined in (14). We note that the condition is independent of the points $P, Q$ which were chosen to be the poles of the respective Green's functions.

**Case IV.** Let each of $D$ and $D'$ be a doubly connected region, of which one boundary component, $B$ and $B'$ respectively, is a single Jordan curve, and the other boundary component, $C$ and $C'$ respectively, is each a union of two Jordan curves with a single common point. Let $u(z)$ and $v(z)$ be conjugate functions of the harmonic measures of $C$ with respect to $D$ and of $C'$ with respect to $D'$ respectively. $D$ and $D'$ may be mapped one to one and conformally onto circular annuli $R$ and $R'$ respectively, centered at zero, with $B$ and $B'$ taken onto the inner boundary, which we may assume to be the unit circumference. Under this mapping, and multiple point of $C$ corresponds to two distinct points $p_1$ and $p_2$ on the outer circumference of $R$; similarly, the multiple point of $C'$ corresponds to two distinct points $p'_1$ and $p'_2$ on the outer circumference of $R'$. The functions $u(z)$ and $v(z)$ represent the argument of the image of a point $z \in D$ or $D'$ under the map. There exists a conformal map of $D$ onto $D'$, one to one and continuous in the closures of the two regions, if and only if there exists a conformal map of $R$ onto $R'$ which takes the points $p_1$ and $p_2$ onto the points $p'_1$ and $p'_2$. A necessary and sufficient condition for the above is that

1. the moduli of $D$ and $D'$ are equal (the quantity $1/\tau$ of Theorem 2 gives the modulus of a doubly connected domain),
2. the variation of $u(z)$ over an arc of $C$, both of whose endpoints are double point of $C$, equals the variation of $v(z)$ over an arc of $C'$, both of whose endpoints are the double point of $C'$.

**Case V.** Let each of $D$ and $D'$ be a doubly connected region of which one boundary component $B \ [B']$ is a single Jordan curve, and the other boundary component $C \ [C']$ is a Jordan configuration composed of three Jordan curves, all with a single common multiple point. Let $u(z)$ and $v(z)$ be as defined in Case IV. By the argument of Case IV, a necessary and sufficient condition for the existence of a conformal map of $D$ onto $D'$, one to one and continuous in the closures of the two regions, is

1. the moduli of $D$ and $D'$ are equal,
2. the variations of $u(z)$ over the three Jordan curves comprising $C$ equal respectively the variations of $v(z)$ over the three Jordan curves comprising $C'$, taken in the same sense with respect to $D'$.

**Case VI.** Let each of $D$ and $D'$ be a doubly connected region of which each boundary component $B$ and $C \ [B', C']$ is the union of two Jordan curves with a single common point. Let $u(z)$ and $v(z)$ be as defined in Case IV. A
necessary and sufficient condition for the existence of a conformal map of
D onto D', one to one and continuous in the closures of the two regions, with
B and B' corresponding to each other, is

(1) the moduli of D and D' are equal,

(2) measured positively with respect to D from a fixed locus λ: \( u(z) = \) constant, not passing through a multiple point of B or C, the variations of \( u(z) \)
over the arcs of B and C from λ to the multiple points of B and C, and over
the Jordan curves of B and C which do not meet λ, beginning and ending
at the multiple points of B and C, equal respectively the variations of \( v(z) \)
measured positively with respect to D' from some corresponding fixed locus
λ': \( v(z) = \) constant over the corresponding arcs of B' and C'.

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