GRAPHS AND SUBGRAPHS, II[1]

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The present paper is a direct continuation of a previous study by the same title. The terminology is unchanged and the enumeration from the first paper is continued. We suppose as before that the basic graph G is finite and without loops. We observe that by small reformulations in certain statements loops could have been included in the theory, while it is essential for several results that G be finite.

Our starting point in Chapter 4 is the theorem of Petersen about the interrelation between conformal subgraphs (subgraphs with the same local degrees). The choice available in the determination of the edges in the desired subgraph H leads to the concept of free equivalence as well as to a unique decomposition of the graph into a bound and a free part. Criteria are established to determine when an edge is free or bound. These are applied, in particular, to the subgraphs with constant proportions for the local degrees. The existence of such subgraphs was established in Chapter 3. Here it is shown that for these all edges are free equivalent; hence the same is true for the regular graphs and subgraphs discussed in §3.2. A special case is a well known result by Petersen for subgraphs of first degree in regular graphs of degree 3 without peninsulas. It is of interest to note that this particular theorem has an important application for the method of alternating paths in general graph theory. In §4.4 it is shown that the accessible characters of vertices under alternating H-paths is invariant, that is, do not depend on H but only upon the class of conformal subgraphs to which H belongs.

In Chapter 5 the concept of free equivalence is discussed in greater detail. Its relation to the so-called cursal equivalence is examined. Among the results are criteria for two vertices to have the same accessible set with identical cursal properties. There exist a considerable number of problems related to those analysed, but these may be left to others.

Chapter 6 contains observations on regular graphs which are completely decomposable, that is, are the sum of subgraphs of first degree.

CHAPTER 4. PROPERTIES OF ALTERNATING PATHS

4.1. **Conformal subgraphs.** Two subgraphs H and K of the graph G are said to be *conformal* if they have the same local degrees.

$$\rho_H(v) = \rho_K(v) = \kappa(v) \quad (4.1.1)$$

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at every vertex v in the vertex set S. Let us take a_0 to be some vertex at which there exists an edge $E_0=(a_0, a_1)$ which is in K but not in H. We construct a path from a_0 beginning in E_0 such that the edges alternately belong to K but not to H and to H but not to K. If at the vertex a_1 one disregards the K-edge E_0 there will be one more edge in H than in K at this vertex. Thus one can select an edge $E_1=(a_1, a_2)$ in H but not in K. At the vertex a_2 the same process is repeated and an edge $E_2=(a_2, a_3)$ in K but not in H is obtained. We continue this selection of edges as far as possible. One may return to vertices previously encountered and then the edges already used are disregarded. For a finite graph one sees that this process can only stop by returning to a_0 in an H-edge and so we have constructed a cyclic path $C(a_0)$ of even length consisting alternately of edges in K but not in H and in H but not in K.

With respect to this path $C(a_0)$ we can perform a *cyclic deformation* of H, i.e., we construct a new conformal subgraph H_1 which coincides with H in all edges not belonging to C while any K-edge in C is assigned to H_1 while no H-edge in C belongs to H_1. If H_1 is not identical with K the same process is repeated and eventually one arrives at K. Thus we have the theorem due to Petersen.

THEOREM 4.1.1. *Let H and K be conformal subgraphs. Then H can be transformed into K by a series of cyclic deformations with respect to even cyclic paths whose edges alternately belong to K but not to H and to H but not to K.*

We have assumed G finite. The theorem is valid also when G is infinite, but locally finite, but in this case one may also have to use two-way infinite alternating paths and perform a sequence of deformations.

We say for short that a subgraph H is a *κ-graph* if it has the local degrees (4.1.1). We define an edge E to be a *free edge* if there exists some κ-graph H to which it belongs and also some κ-graph K to which it does not belong. A *bound edge* is an edge which is not free. The latter fall into two categories: The *bound H-edges* belonging to every κ-graph and the *bound $\overline{H}$-edges* belonging to no such graph. Let

$$B(H),\ B(\overline{H}),\ F$$

respectively be the subgraphs consisting of the bound H-edges, the bound $\overline{H}$-edges and the free edges. Then there exists a unique disjoint decomposition

$$G = B(H) + B(\overline{H}) + F. \tag{4.1.2}$$

A *free vertex* is the endpoint of a free edge; otherwise it is *bound*. The preceding discussion shows:

THEOREM 4.1.2. *Let H be a κ-graph. An edge is free if and only if it belongs to an even cyclic alternating H-path C. A vertex is free if and only if it lies on such a path.*

We notice that when an edge is free for the multiplicities κ it is also free for the complementary multiplicities $\tilde{\kappa}$. Two vertices $a=a_0$ and $b=a_n$ are *free equivalent* when they are connected by a series of free edges.

$$(4.1.3) \qquad (a_0, a_1)(a_1, a_2) \cdots (a_{n-1}, a_n).$$

This equivalence introduces a decomposition of the vertex set into a family of disjoint *free connected blocks* Ψ. The singular blocks consisting of a single vertex are the bound vertices. We notice that these blocks Ψ are *κ-invariant*; that is, they are independent of the choice of the particular κ-graph H. Two edges

$$(4.1.4) \qquad A = (a_0, a_1), \qquad B = (a_{n-1}, a_n)$$

may also be called free equivalent if there exists a series (4.1.3) of free edges including them.

These definitions show that the edges in an even alternating cyclic path C for any κ-graph H are free equivalent. We shall say that a set is *free closed* when it has the property that if it contains a vertex v then it contains every vertex free equivalent to v. The free connected blocks Ψ are free closed and all free closed sets are the sums of such blocks.

In the discussion in Chapter 2 of alternating paths we introduced the accessible sets $W(c_0)$ under α and β-paths from a center c_0. We shall prove:

THEOREM 4.1.3. *The accessible sets $W(c_0)$ from a center c_0 are free closed.*

Proof. It is sufficient to show that when v is a vertex accessible from c_0, for instance in a β-path, then all vertices lying on an even cyclic path C with v are also accessible. Let u be such a vertex. Since v is accessible there is a path $P(c_0, v)$; let v_1 be the first vertex in P lying on C. Then $P(c_0, v_1)$ can be continued in one or the other direction along C to u.

We saw in Chapter 2 that all vertices bicursally connected with c_0 formed the central block $\mathfrak{B}_0$. We have:

THEOREM 4.1.4. *The central bicursal equivalence block $\mathfrak{B}_0$ is free closed.*

Proof. Let an even cyclic path C have vertices in common with $\mathfrak{B}_0$. If a vertex u on C did not belong to $\mathfrak{B}_0$ there would be sections

$$C(u, x),\ C(u, y)$$

to the nearest vertices x and y lying in $\mathfrak{B}_0$. But since x and y are bicursal from c_0 within $\mathfrak{B}_0$ also u will become bicursal from c_0.

There exists a block $\mathfrak{P}_0$ of all vertices which are bicursally point equivalent to c_0 (§2.3). A similar argument shows that also $\mathfrak{P}_0$ is free closed.

4.2. **Characterization of free edges.** In Chapter 2 we analysed the properties of the accessible sets under alternating paths from a center c_0. We consider first the β-paths for a fixed κ-graph H and use the previous notations. All edges from a vertex c_0 to the set

$$W_\alpha + \overline{W} \tag{4.2.1}$$

must be bound α-edges. If namely such an edge should be a β-edge there would either be an entering β-edge to an α-vertex or an inaccessible vertex. No edge to $\overline{W}$ can be cursal. Nor can an α-edge to W_α be part of a returning even cyclic path to c_0 because it would imply the existence of an entering β-edge to W_α. We conclude that all free edges from c_0 must go to the complement

$$W_\beta + V_\alpha + V_\beta \tag{4.2.2}$$

of the set (4.2.1). We shall show:

THEOREM 4.2.1. *An α-edge $E=(c_0, u)$ is free if and only if its endpoint u belongs to the set* (4.2.2).

Proof. 1. $c_0 \in \mathfrak{B}_0$ is bicursal. Excluding the edges to the set (4.2.1) there can only exist edges from c_0 to $\mathfrak{B}_0$ and W_β. Theorem 2.2.4 shows that α-edges to $\mathfrak{B}_0$ are free. No α-edge can be cursal to a β-vertex in W_β. Thus when $u \in W_\beta$ there is a β-path from c_0 to u which can be continued through the α-edge E back to c_0, hence E is free.

2. c_0 is an α-vertex. No α-edge E can be cursal from c_0 since c_0 is not bicursal. But there is a β-path $P(c_0, u)$ to any vertex u in the set (4.2.2) ending in a β-edge and this can be continued through E to c_0.

If one considers α-paths from c_0 the analogous results must hold. All edges from c_0 to the set

$$W'_\beta + \overline{W}' \tag{4.2.3}$$

are bound β-edges. The free β-edges from c_0 must therefore go to the complement

$$W'_\alpha + V'_\alpha + V'_\beta \tag{4.2.4}$$

of (4.2.3). One verifies as in Theorem 4.2.1 that any β-edge from c_0 with its end point in the set (4.2.4) is free. When these results are combined with the preceding ones it follows:

THEOREM 4.2.2. *Let $E=(c_0, u)$ be an edge from the vertex c_0. Then E is a bound α-edge or a bound β-edge according to*

$$u \in W_\alpha + \overline{W}, \qquad u \in W'_\beta + V'_\beta.$$

It is a free edge when

$$u \in (W_\beta + V_\alpha + V_\beta)\cdot(W'_\alpha + V'_\alpha + V'_\beta).$$

4.3. **Regular graphs.** We shall apply this analysis to regular graphs or somewhat more generally, to graphs where the multiplicities have constant proportions. This means according to §3.1 that there are constants λ and $\bar{\lambda}$ such that

$$(4.3.1) \qquad \lambda = \frac{\kappa(v)}{\rho(v)}, \qquad \tilde{\lambda} = \frac{\tilde{\kappa}(v)}{\rho(v)}, \qquad \lambda + \lambda = 1$$

independent of the vertex v. The effective deficiency for a set A in such a graph was given by the expression (3.1.8).

The conditions (4.3.1) are satisfied when the local degrees for G, H and $\overline{H}$ have the forms

$$(4.3.2) \qquad \begin{aligned} \rho(v) &= n\cdot\rho_1(v), \\ \kappa(v) &= m\cdot\rho_1(v), \qquad \tilde{\kappa}(v) = \tilde{m}\cdot\rho_1(v) \end{aligned}$$

where the notations are such that

$$(4.3.3) \qquad \lambda = \frac{m}{n} \leqq \frac{1}{2}, \qquad \tilde{\lambda} = \frac{\tilde{m}}{n} \geqq \frac{1}{2}, \qquad m + \tilde{m} = n.$$

In Theorem 3.2.1 a condition was established under which a subgraph corresponding to the multiplicities (4.3.2) exists:

1. G has no peninsulas.
2. No set C with

$$\kappa(C) \equiv 1 \pmod 2$$

shall be a peninsula of rank

$$\rho(C, \overline{C}) < n/m.$$

The proof shows that under these conditions all terms in the sums (3.1.9) are non-negative.

We shall investigate the form of sets A of zero effective deficiency. We take G to be connected. From the formula (3.1.8) we conclude that for such a set A one must have

$$(4.3.4) \qquad \rho(I_1, I_1) = \rho(\tilde{I}_1, \tilde{I}_1) = 0$$

and

$$(4.3.5) \qquad \Sigma_1 = \Sigma_2 = 0;$$

with each term in the sums vanishing.

Suppose first that A has an even inner overfilled component. Then $\epsilon = 0$ for the corresponding term in (3.1.9) so that the condition (4.3.5) gives

$$\tilde{\lambda}\cdot\rho(I_1, C) + \lambda\cdot\rho(\overline{A}, C) = 0$$

or

$$\rho(I_1, C) = \rho(\overline{A}, C) = 0.$$

Since G is connected this is only possible when

$$A = C = S, \qquad \bar{A} = \varnothing. \tag{4.3.6}$$

The above argument applies also to an outer even overfilled component. Thus we may assume that A has no such components. Next let C be an inner odd overfilled component. Then $\epsilon = 1$ and for the corresponding term in (4.3.5) we have

$$\tilde{\lambda} \cdot \rho(I_1, C) + \lambda \cdot \rho(\bar{A}, C) = 1. \tag{4.3.7}$$

This can be fulfilled only in the cases:

$$\begin{aligned} &\rho(I_1, C) = 0, \quad \rho(\bar{A}, C) = \frac{1}{\lambda} = \frac{n}{m}, \\ &\rho(I_1, C) = 1, \quad \rho(\bar{A}, C) = 1, \\ &\rho(I_1, C) = 2, \quad \rho(\bar{A}, C) = 0, \quad \lambda = \tilde{\lambda} = \frac{1}{2}. \end{aligned} \tag{4.3.8}$$

The last two cases are excluded if G has no peninsulas of rank 2.

The same analysis applies to an outer overfilled component $\tilde{C}$. Corresponding to (4.3.7) one finds

$$\tilde{\lambda} \cdot \rho(A, \tilde{C}) + \lambda \cdot \rho(\tilde{I}_1, \tilde{C}) = 1.$$

This condition can be satisfied only when

$$\begin{aligned} &\rho(A, \tilde{C}) = 0, \quad \rho(\tilde{I}_1, \tilde{C}) = \frac{1}{\lambda}, \\ &\rho(A, \tilde{C}) = 1, \quad \rho(\tilde{I}_1, \tilde{C}) = 1, \\ &\rho(A, \tilde{C}) = 2, \quad \rho(\tilde{I}_1, \tilde{C}) = 0, \quad \lambda = \tilde{\lambda} = \frac{1}{2}. \end{aligned} \tag{4.3.9}$$

We apply these facts to the accessible set $W(C_0)$ under β-paths from a center c_0 with respect to some κ-graph H. In (2.4.5) we defined the *α-component* as the set

$$A = W_\alpha + V_\alpha \tag{4.3.10}$$

consisting of all α-vertices and all vertices in the bicursal α-blocks P_α. A has zero effective deficiency by Theorem 2.4.4.

We separate two cases:

1. c_0 is bicursal so that there exists a central bicursal block $\mathfrak{P}_0 \neq \varnothing$. Then by Theorem 2.4.2 $\mathfrak{P}_0$ is an inner even overfilled component of A and (4.3.6) gives

$$A = \mathfrak{P}_0 = S. \tag{4.3.11}$$

Theorem 4.2.1 shows that every α-edge from c_0 is free.

2. c_0 is an α-vertex. Theorem 2.4.2 shows that there are no even inner overfilled components of A and the odd components are the α-blocks $\mathfrak{P}_\alpha$. Theorem 2.4.3 establishes that the odd outer components are the β-blocks $\mathfrak{P}_\beta$. There are no even outer components since as in (4.3.6) it would lead to

$$\overline{A} = S, \qquad A = \varnothing.$$

Let us prove that there can be no inaccessible vertices from c_0. Theorem 2.4.3 shows that in general an inaccessible vertex $\bar{w}$ must belong either to an even outer component or it must be exactly filled from A. Here only the latter case is possible. But $\bar{w}$ can have no edges connecting it with any $\mathfrak{P}_\beta$ by Theorem 2.1.3, nor can there be any edges connecting $\bar{w}$ with $\bar{I}_1$ according to (4.3.4). We conclude that all edges from $\bar{w}$ go to A, but this makes $\bar{w}$ overfilled from A contrary to assumption. We conclude that

$$W(c_0) = S, \qquad \overline{W} = \varnothing. \tag{4.3.12}$$

In our case $I_1 = W_\alpha$. We conclude from Theorem 4.2.1 by means of (4.3.12) and (4.3.4) that also in this case every α-edge from c_0 is free.

A similar argument applies to the sets of zero effective deficiency with respect to the complementary multiplicities. We recall that these sets are the complements of sets of zero effective deficiency. Thus we conclude that all vertices are accessible also under α-paths from an arbitrary vertex c_0 and that all β-edges are free. This gives:

THEOREM 4.3.1. *Let G be a connected graph without peninsulas and multiplicities of the form* (4.3.2); *for vertex sets C with*

$$\kappa(C) \equiv 1 \pmod 2,$$

the condition

$$\rho(C, \overline{C}) \geq \frac{n}{m}$$

is fulfilled. Then there exist κ-graphs H for these multiplicities and all edges in G are free equivalent.

All decomposition theorems derived in §3.2 were special cases of graph decompositions of this type. We conclude that for the subgraphs whose existence were established in Theorems 3.2.2 and 3.2.3 all edges must be free. It follows in particular that for the regular subgraphs of regular graphs exhibited in Theorems 3.2.4 and 3.2.5 all edges must be free. A very special case is the result also due to Petersen that a connected regular graph of degree 3 without peninsulas has subgraphs of first degree and all its edges are free. In the next section we shall make use of this theorem to derive an important fact about the alternating paths in a general graph.

4.4. Invariance of the accessible characters. In Chapter 2 we investigated the properties of the alternating paths from a center c_0 with respect to a κ-subgraph H of G. These paths were separated into two families, the α-paths beginning in an edge in H and the β-paths beginning in an edge in the complementary graph $\overline{H}$. With respect to any one of these families of paths, for instance the β-paths, all vertices $v \in S$ were assigned certain *accessible characters*. First, v is *accessible* if it appears in some such path, *inaccessible* otherwise. Secondly, the accessible vertices were divided into three classes:

1. α-vertices accessible only in paths ending in an α-edge at v.
2. β-vertices accessible only by paths ending in a β-edge.
3. Bicursal vertices accessible from c_0 in paths ending both in α and β-edges.

The center c_0 is always accessible. It is bicursal if there are returning β-paths ending in a β-edge; otherwise c_0 is an α-vertex.

In this section we shall prove:

THEOREM 4.4.1. *The accessible character of a vertex is κ-invariant, that is, it is independent of the particular choice of the κ-graph H.*

We notice to begin with that by Theorem 4.1.1 it is sufficient to show that the accessible characters are unchanged by a single cyclic deformation of H into another κ-graph H'. We denote by C the even cycle with respect to which the deformation is performed. We shall show that if $P(c_0, v)$ is any path from c_0 to v in H then there exists a path $P'(c_0, v)$ with the same initial and terminal characters with respect to H'. This is evident if P and C have no edges in common. Let b_1 be the first vertex from which P and C have a common edge and denote by

$$D_1 = (b_1, b_2)$$

the first common section of these two paths. If there is only one such section one can, after the deformation, continue P from b_1 through the other part of C to b_2 and from b_2 as before. Thus we may assume that P and C have several sections in common. Our theorem will then be a consequence of the

LEMMA. *Let $P(b_1, b_{2n})$ be a path from b_1 to b_{2n} having a certain number of sections*

$$D_1 = (b_1, b_2),\ D_2,\ \cdots,\ D_n = (b_{2n-1}, b_{2n}) \tag{4.4.1}$$

in common with the even cycle C. The initial character of D_1 may be γ and the terminal character of D_n is δ. Then after the deformation of H with respect to C there still exists an H'-path with the same end characters and consisting of sections of P and C.

Instead of using the terminology of alternating paths in a graph we can introduce an equivalent formulation in terms of two intertwining continuous curves C and P. Here C is closed and the two curves have a number of links or common sections D_i indicated by their endpoints as in (4.4.1). Each curve C and P then consists of an alternating sequence of links D_i and noncoinciding sections C_j and P_k. We shall indicate this by writing

(4.4.2) $$P = D_1, P_1, D_2, P_2, \cdots, P_{n-1}, D_n,$$

and

(4.4.3) $$C = C_1, D_{k_1}, C_2, D_{k_2}, \cdots, C_{n-1}, D_1,$$

where $C_1 = (b_1, c_1)$ is the section of C from b_1 in the opposite direction of D_1. Through (4.4.2) and (4.4.3) we have defined directions on P and C. This gives each section D_i two assigned directions, one as member of P and the other as member of C. One sees that the intertwining of the curves P and C is described completely by the indices

$$k_1, k_2, \cdots$$

in (4.4.3) together with the information for each D_i whether its two directions coincide or are opposite.

The curve P may also be made closed by adding a section

$$P_0 = (b_1, b_{2n})$$

to it. Then the configuration consisting of the vertices b_i and the sections

(4.4.4 $$\{C_i\}, \{P_i\}, \{D_i\}$$

connecting them may be considered to be a regular graph G_0 of degree 3 with the edges (4.4.4). The removal of any edge cannot make G_0 disconnected, thus G_0 has no peninsulas. Each of the three families in (4.4.4) are the edges of a subgraph of first degree. The paths P and C in (4.4.2) and (4.4.3) are Hamilton circuits in G_0 passing through each vertex once. The graphs

(4.4.5) $$K = \{C_i\}, \qquad \overline{K} = P = \{D_i\} + \{P_i\}$$

are complementary subgraphs of G_0 with degrees 1 and 2 respectively. The result of Petersen, mentioned at the end of the previous section, on subgraphs of first degree in regular graphs of third degree without peninsulas, shall now be applied to the subgraphs K and $\overline{K}$ in (4.4.5). All edges in G_0 are free. Thus we conclude that the edge P_0 in $\overline{K}$ lies on an even cyclic path

(4.4.6) $$P_0, C_1, \overline{K}_1, C_{a_2}, \overline{K}_2, \cdots, C_{a_t}$$

where the $\overline{K}_i$ indicate sections D_i or P_i and C_{a_t} has the endpoint b_{2n}. When P_0 is omitted in (4.4.6) we obtain the desired path in G_0. We again interpret the sections (4.4.4) as alternating paths. One then readily verifies that (4.4.6), beginning with C_1, is a path cursal from b_1 to b_{2n} with respect to the graph H' obtained from H by deformation with respect to C. In regard to H' this path also has the end characters γ and δ as required by the lemma.

Theorem 4.4.1 can also be expressed:

THEOREM 4.4.2. *For a given center c_0 all the sets*

$$W,\ \overline{W},\ W_\alpha,\ W_\beta,\ V_\alpha + V_\beta$$

are κ-invariant, i.e., independent of the κ-graph H with respect to which the paths from c_0 are constructed.

One readily verifies from their definition that also the central bicursal blocks $\mathfrak{B}_0$ and $\mathfrak{P}_0$ are κ-invariant.

All noncentral bicursal blocks $\mathfrak{B}$ and $\mathfrak{P}$ are also invariant, but their entrances and entering characters may depend on the choice of the κ-graph H.

CHAPTER 5. ALTERNATING PATH PROBLEMS

5.1. **Cyclic connectivity.** We shall consider various types of problems concerning alternating paths. To abbreviate we shall use the terminology that a path $P(a_0, a_n)$ connecting two vertices a_0 and a_n is a (γ, δ)-*path* when the first edge in P is a γ-edge ($\gamma=\alpha$ or $\gamma=\beta$) and the last edge is a δ-edge ($\delta=\alpha$ or $\delta=\beta$).

Two vertices a_0 and b_0 are *cyclically connected* for the κ-graph H if there exists a series of even cyclic paths

$$C_1, C_2, \cdots, C_k \tag{5.1.1}$$

such that a_0 lies on C_1 and b_0 on C_k with each pair of consecutive cycles C_i and C_{i+1} having at least one vertex in common. Similarly two edges A and B are cyclically connected if A lies on C_1 and B on C_k.

THEOREM 5.1.1. *Two edges or vertices are free equivalent if and only if they are cyclically connected.*

Proof. Evidently the condition is sufficient from the definition of free equivalence introduced in §4.1. To prove the necessity we need only observe that if E is a free edge it belongs to an even cyclic path for every κ-graph H. This follows from the construction used in the proof of Theorem 4.1.1. Thus in the chain of free edges (4.1.3) each edge lies on an even H-cycle.

If in (5.1.1) a cycle C_i has a vertex in common with a preceding cycle C_j, $j<i$, then all cycles lying between C_j and C_i are superfluous in establishing the cyclic connection and may be omitted. Thus we can reduce the sequence such that each cycle C_i contains only vertices of C_{i-1} and C_{i+1}. We can suppose that C_i has an edge in common with each of them, since two cycles with vertices, but not edges in common may be combined into a single cycle.

We shall point out a certain analogy between free equivalence and bicursal equivalence as defined in §2.2. A vertex a_0 shall be called a γ-*apex* if there is a closed odd (γ, γ)-path $L(c_0, c_0)$ returning to c_0; such a path $L(c_0, c_0)$ shall be called a γ-*loop*. We say that a vertex b_0 is β-*loop-connected* to c_0 when the following conditions are fulfilled: There exists a sequence (5.5.1) of cyclic paths such that the C_i are either α or β-loops or even cyclic paths. The first C_1 shall be a β-loop with the apex c_0 while b_0 lies on C_k. The paths (5.1.1) are connected such that each has at least one vertex in common with the preceding; in the case of a loop its apex shall belong to one of the preceding paths.

THEOREM 5.1.2. *A vertex b_0 is β-bicursally equivalent to c_0 if and only if it is β-loop-connected with this vertex.*

Proof. We show first that every edge in the paths (5.1.1) is bicursal from c_0 in paths containing only these edges. This is clear for C_1 and so we may prove the theorem by induction on k. Let $C_k(x, y)$ be a section of C_k such that the ends x and y belong to previous C_i while the edges in $C_k(x, y)$ do not belong to any such C_i. Then by the induction assumption there exist β-paths from c_0 to the bicursal vertices x and y such that they may be continued through $C_k(x, y)$ both ways.

Next suppose that the set L of loop-connected edges were only a part of the set of bicursal edges connected with c_0. Then there would exist an edge $E=(c, d)$ bicursal from c_0 where c is loop-connected to c_0 while d is not. This leads to a contradiction. By the preceding there would exist a β-path $P(c_0,c)$ consisting of loop-connected edges such that P could be continued through E. Since E is bicursal from c_0 there exists some β-path $Q(c_0, d, c)$. Let e be the last vertex common to P and Q. Then $e\in L$ and E lies either on an even cycle with e or e is the apex of a loop containing E.

5.2. **Mutual connections.** Two vertices a_0 and b_0 shall be said to be *mutually connected* if there exist paths

$$(5.2.1) \qquad P(a_0, b_0), \qquad Q(a_0, b_0)$$

connecting them, having the end characters (α, γ) and $(\beta, \bar{\gamma})$ where $\bar{\gamma}$ is the opposite character of γ. We shall prove:

THEOREM 5.2.1. *Free equivalent vertices are mutually connected by paths consisting of free edges.*

Proof. The theorem is evident when a_0 and b_0 lie on the same even cycle. Thus it may be proved by induction on the number k of cycles in (5.1.1). Then if b_0 lies on C_k we may assume that b_0 does not belong to any of the previous cycles C_i. From b_0 we can follow C_k in two directions

$$(5.2.2) \qquad C_k(b_0, x), \qquad C_k(b_0, y)$$

to the first vertices x and y belonging to C_{k-1}. We suppose the paths (5.2.2) have the characters $(\alpha, \bar{\delta})$ and $(\beta, \bar{\epsilon})$.

According to the induction assumption there exist two paths

$$(5.2.3) \qquad P(a_0, x), \qquad Q(a_0, x)$$

consisting of edges in the first $k-1$ cycles and having opposite characters at a_0 and at x. To y there exist paths

$$(5.2.4) \qquad P_1(a_0, y), \qquad Q_1(a_0, y)$$

of the same kind. One of the paths (5.2.3) must have the end character δ and can be continued through $C_k(x, b_0)$ to b_0. Suppose for instance that this path is P and that it is a β-path from a_0. Let P_1 be the path in (5.2.4) which has the initial character α and d the first vertex in P_1 lying on C_k. If $P_1(a_0, d)$ can be continued on C_k through y to b_0 the theorem is proved. If $P_1(a_0, d)$ can be continued on C_k in the other direction then both paths

$$P' = P(a_0, x), \qquad P'' = P_1(a_0, d) + C_k(d, x)$$

can be continued through $C_k(x, b_0)$ to b_0. These paths are respectively (β, δ) and (α, δ)-paths. But one of the paths (5.2.4) can be continued through $C_k(y, b_0)$. By pairing it with that path P' or P'' which has the opposite character at a_0 a mutual connection of the desired kind has been obtained.

THEOREM 5.2.2. *If a_0 and b_0 are free equivalent and $E=(b_0, c)$ any edge from b_0 then E is cursal from b_0 to c in some path from a_0.*

Proof. When E is free this is a consequence of the proof of the preceding theorem. When E is not free there is a path of free edges $P(a_0, b_0)$ with a suitable terminal character so that it may be continued through E.

Under certain conditions one can conclude conversely to free equivalence from the existence of a mutual connection.

THEOREM 5.2.3. *Let a_0 and b_0 be two vertices mutually connected by a (γ, δ)-path $P(a_0, b_0)$ and a $(\bar{\gamma}, \bar{\delta})$-path $Q(a_0, b_0)$. If there are no $(\gamma, \bar{\delta})$-paths from a_0 to b_0 consisting of edges in P and Q, then a_0 and b_0 are free equivalent.*

Proof. The theorem is obvious when P and Q have no common edges, because together they form an even cyclic path. Thus it may be proved by induction with respect to the number of common edges in the two paths. Let $E=(c, d)$ be the first edge in P which also belongs to Q. Then E cannot be cursal in the same direction in the two paths because $P(a_0, c, d)$ could be continued in $Q(d, b_0)$ giving a $(\gamma, \bar{\delta})$-path to b_0.

We assume therefore that the ϵ-edge E has opposite directions in P and Q. Since

$$P(a_0, c, d) + Q(d, a_0)$$

is an even cyclic path the vertices a_0, c and d are free equivalent. Furthermore, c and b_0 are mutually connected by the (ϵ, γ)-path $P(c, b_0)$ and the $(\bar{\epsilon}, \bar{\gamma})$-path $Q(c, b_0)$. No $(\epsilon, \bar{\delta})$-path from c to b_0 consisting of edges in these paths can exist because it would lead to a $(\gamma, \bar{\delta})$-path from a_0 to b_0. Thus the theorem follows from the induction assumption.

We shall need also the following auxiliary result:

THEOREM 5.2.4. *Let Q be some γ-loop with the apex e and q some vertex on Q. If there exists some path $P(q, e)$ with the end character $\bar{\gamma}$ then e and q are free equivalent.*

Proof. If P has no edges in common with Q there are two possible returning paths

$$P(q, e) + Q(e, q)$$

to q. One of these is seen to be an even cyclic path. Thus one may base the proof upon induction with regard to the number of common edges in P and Q. Let $A=(a, b)$ be the last edge in P before e which lies on Q. Two cases may occur. First the direction from a to b on Q is such that the section $Q(a, b, e)$ does not contain q. Then

$$Q(e, q, a, b) + P(b, e)$$

will be an even cyclic path containing e and q. Secondly, q is included in $Q(a, b, e)$. Then the path

$$Q_1 = P(b, e) + Q(e, q, b)$$

is a loop with the apex b and containing q. If Q_1 is a δ-loop $P(q, a, b)$ has the end character $\bar{\delta}$ and so by the induction assumption q and b are free equivalent. But since

$$C = P(b, e) + Q(e, a, b)$$

is an even cycle also e and b are free equivalent.

5.3. **Cursal equivalence.** For some center c_0 we have defined the accessible sets

$$W(c_0, \alpha), \qquad W(c_0, \beta) \tag{5.3.1}$$

consisting of all vertices accessible from c_0 under α-paths and β-paths respectively. We shall introduce also the sets

$$\begin{aligned} W(c_0, \alpha + \beta) &= W(c_0, \alpha) + W(c_0, \beta), \\ W(c_0, \alpha \cdot \beta) &= W(c_0, \alpha) \cdot W(c_0, \beta). \end{aligned} \tag{5.3.2}$$

The first of these sets (5.3.2) consists of the vertices accessible by either α or β-paths, the second of the vertices accessible by both α and β-paths.

Let c_1 be a vertex which is accessible from c_0 in a $(\gamma, \bar{\delta})$-path. We shall say that c_1 is (γ, δ)-*cursally included from* c_0 if every edge E which is cursal from c_1 in a δ-path is cursal in the same direction from c_0 in a γ-path. Furthermore, c_0 and c_1 are (γ, δ)-*cursally equivalent* if c_1 is (γ, δ)-cursally included from c_0 and $c_0(\delta, \gamma)$-cursally included from c_1.

We extend this terminology also to the symbols $\alpha+\beta$ and $\alpha\cdot\beta$ as in (5.3.2).

We say for instance that c is $(\alpha+\beta, \gamma)$-cursally included from c_0 if c_1 is accessible from c_0 in some path ending in a $\bar{\gamma}$-edge and every edge E cursal from c_1 in a γ-path is cursal in the same direction in some path from c_0. The two vertices are $(\alpha+\beta, \gamma)$-cursally equivalent if c_1 is $(\alpha+\beta, \gamma)$-cursally included from c_0 and c_0 is $(\gamma, \alpha+\beta)$-cursally included from c_1.

We shall illustrate these definitions on a few simple examples:

THEOREM 5.3.1. *Two free equivalent vertices a_0 and b_0 are $(\alpha+\beta, \alpha+\beta)$-cursally equivalent.*

Proof. We saw in Theorem 5.2.1 that the two vertices are connected by paths. Let $A=(c, d)$ be an edge which is cursal in this direction in some path $P(a_0, c, d)$. Denote by $E=(e, f)$ the last edge in P whose initial vertex e is free equivalent to a_0. Then by Theorem 5.2.2 there exists a free path $Q(b_0, e)$ which can be continued from e through P.

THEOREM 5.3.2. *Let e be the apex of a β-loop Q. Then every vertex q on Q is $(\beta, \alpha+\beta)$-included from e.*

Proof. Let $A=(c, d)$ be an edge cursal in this direction from q in a path $P(q, c, d)$. The last vertex on P lying on Q shall be f. Then there is a β-path $Q(e, f)$ which can be continued in $P(f, c, d)$.

A consequence is:

THEOREM 5.3.3. *A vertex a_0 is $(\beta, \alpha+\beta)$-cursally equivalent to itself if and only if it is a β-apex.*

Proof. Theorem 5.3.2 shows that a β-apex is $(\beta, \alpha+\beta)$-cursally equivalent to itself. Conversely, let a_0 have this property. Then there exists a β-path returning to a_0. If this is not a β-loop it is an even cyclic path ending in an α-edge. This must also be cursal in a β-path from a_0, hence a_0 is a β-apex. A consequence of Theorems 5.3.1 and 5.3.3 is:

THEOREM 5.3.4. *Two free equivalent γ-apexes are (γ, γ)-cursally equivalent.*

An α-apex and a β-apex which are free equivalent are (α, β)-cursally equivalent. Theorems 5.1.2 and 5.3.1 and 5.3.2 combined yield:

THEOREM 5.3.5. *For a β-apex c_0 let $B(c_0)$ be the central block of vertices bicursally equivalent to c_0. Then every vertex b in $B_0(c_0)$ is $(\beta, \alpha+\beta)$-cursally included from c_0.*

For the proof of a following theorem we need the observation:

THEOREM 5.3.6. *Let e be the apex of a β-loop Q and q a vertex on Q. If q is $(\beta, \alpha+\beta)$-cursally equivalent to e then e and q are free equivalent.*

Proof. Under these conditions there must be some path $P(q, e)$ ending in an α-edge at e so that it can be continued through the two β-edges

$$E_1 = (e, q_1), \qquad E_2 = (e, q_2)$$

in Q at e. The theorem is therefore a consequence of Theorem 5.2.4.

We are now prepared to prove:

THEOREM 5.3.7. *Two vertices a_0 and b_0 are free equivalent if and only if they are $(\alpha+\beta, \alpha+\beta)$-cursally equivalent.*

Proof. According to Theorem 5.3.1 it is sufficient to show that $(\alpha+\beta, \alpha+\beta)$-cursally equivalent vertices are free equivalent. Then there exists some path $P(a_0, b_0)$. We denote the edges in P by

$$A_i = (a_i, a_{i+1}), \qquad i = 0, 1, \cdots, n, a_n = b_0.$$

Each of these edges is cursal in P from b_0 in the direction from a_{i+1} to a_i, hence they must be cursal from a_0 in the same direction. Thus for each i there exists a path

$$Q_i(a_0, a_{i+1}, a_i).$$

If Q_i has no edges in common with $P(a_0, a_i)$ the closed path

$$P(a_0, a_i) + Q_i(a_i, a_{i+1}, a_0)$$

is even cyclic or a loop with the apex a_0. When P and Q_i have edges in common let

$$A_j = (a_j, a_{j+1})$$

be the last such edge in $P(a_0, a_i)$ before a_i. If A_j appears in the same direction from a_0 in P and Q_i then

$$P(a_{j+1}, a_i) + Q_i(a_i, a_{i+1}, a_{j+1})$$

is a loop with the apex a_{j+1}. If A_j has opposite directions the path

$$P(a_j, a_{j+1}, a_i) + Q(a_i, a_{i+1}, a_j)$$

is an even cycle. Thus one sees that every vertex $a_i \neq a_0$ on P is connected either with an even cycle to some previous vertex a_j or by a loop with a_j as its apex. This means that b_0 is loop-connected with a_0 and so by Theorem 5.1.2 b_0 lies in the bicursal block $\mathfrak{B}_0(a_0)$ defined by a_0. If in the preceding a_i and a_j lie on the same even cycle they are $(\alpha+\beta, \alpha+\beta)$-cursally equivalent according to Theorem 5.3.1. If a_i lies on a loop with the apex a_j then a_i is $(\alpha+\beta, \alpha+\beta)$-cursally included from a_j by Theorem 5.3.2. But since a_0 and b_0 were $(\alpha+\beta, \alpha+\beta)$-cursally equivalent it follows that in each such step a_i must be $(\alpha+\beta, \alpha+\beta)$-cursally equivalent to a_j and so from Theorem 5.3.6 we obtain that a_i and a_j are free equivalent. This completes the proof.

5.4. **(β, β)-cursal equivalence.** We shall determine when two vertices a_0 and b_0 are (β, β)-cursally equivalent, that is, when the two vertices have the same accessible set with the same cursal properties for all edges. We examine first when b_0 is (β, β)-cursally included from a_0.

THEOREM 5.4.1. *The vertices which are (β, β)-cursally included from the vertex a_0 are those which belong to the set*

$$\mathfrak{B}_0(a_0) + W_\alpha(a_0) \tag{5.4.1}$$

consisting of the α-vertices from a_0 and the vertices belonging to the central bicursal equivalence block.

Proof. We show first that no vertices outside of the set (5.4.1) can be (β, β)-cursally included from a_0. The inaccessible vertices are excluded by the definition of cursal inclusion. No β-vertex can be (β, β)-cursally included from a_0 since the unicursal β-edges to it would become cursal in the opposite direction. Nor can any vertex v in a noncentral bicursal block $\mathfrak{B}(e)$ have this property since the unicursal entering edge to the entrance e would become cursal in the opposite direction in a β-path from v.

Assume now $b_0 \in W_\alpha$ and let $E = (c, d)$ be an edge cursal from b_0 in a β-path $P(b_0, c, d)$. Since b_0 is an α-vertex there exists a (β, α)-path $Q(a_0, b_0)$. If P and Q have no edges in common Q may be continued through P and E. Next assume that $D = (d_1, d_2)$ is the first edge in Q also in P. Then D cannot appear in opposite directions in P and Q because

$$Q(a_0, d_1, d_2) + P(d_2, b_0)$$

would be a (β, β)-path from a_0 to b_0. Thus D appears in the same direction in P and Q and the path $Q(a_0, d_1)$ can be continued in $P(d_1, d_2, c, d)$.

Finally assume $b_0 \in \mathfrak{B}_0(a_0)$ and as before let $E = (c, d)$ be an edge which is cursal from b_0 in a β-path $P(b_0, c, d)$. If E has both end points in $\mathfrak{B}_0(a_0)$ it is bicursal from a_0 and there is nothing to prove. Assume therefore that f is the last vertex in P before d which belongs to $\mathfrak{B}_0(a_0)$. Since f is bicursal from a_0 there exists a path $Q(a_0, f)$ within $\mathfrak{B}_0(a_0)$ which can be continued in $P(f, c, d)$.

One can also determine the vertices which are (β, α)-cursally included from a_0. Through an analogous analysis one finds:

THEOREM 5.4.2. *The set*

$$\mathfrak{B}_0(a_0) + W_\beta \tag{5.4.2}$$

consists of all vertices (β, α)-cursally included from a_0.

Since the intersection of the two sets (5.4.1) and (5.4.2) is $\mathfrak{B}_0(a_0)$ we can also state:

THEOREM 5.4.3. *The central block $\mathfrak{B}_0(a_0)$ consists of all vertices which are both (β, β) and (β, α)-cursally included from a_0.*

We prove next:

THEOREM 5.4.4. *If two vertices a_0 and b_0 are (β, β)-cursally equivalent then they are free equivalent; in addition either both are β-apexes or neither vertex is a β-apex and in the latter case b_0 is an α-vertex from a_0 and a_0 an α-vertex from b_0. Conversely, when these conditions are fulfilled a_0 and b_0 are (β, β)-cursally equivalent.*

Proof. We point out first that if $b_0 \in W_\alpha(a_0)$ in (5.4.1) then b_0 cannot be a β-apex. There exists some (β, α)-path $P(a_0, b_0)$. If Q were a β-loop with the apex b_0 there would be a first vertex q in P lying on Q. But then $P(a_0, q)$ could be continued in one or the other direction along Q giving a (β, β)-path from a_0 to b_0.

Next we show that if a_0 is a β-apex it cannot be (β, β)-cursally equivalent to an α-vertex b_0. If this were the case there would exist some β-path $Q(b_0, a_0)$ with the first vertex d in $\mathfrak{B}_0(a_0)$. Since d is bicursal from a_0 within $\mathfrak{B}_0(a_0)$ there would be some β-path from a_0 which could be continued through $Q(d, b_0)$ to give a (β, β)-path to b_0.

These observations show that to be (β, β)-cursally equivalent a_0 and b_0 must either both be β-apexes or both not β-apexes.

If a_0 and b_0 are both β-apexes each is $(\beta, \alpha+\beta)$-cursally equivalent to itself according to Theorem 5.3.3 and so they are free equivalent by Theorem 5.3.7. Conversely, if this is the case they are (β, β)-cursally equivalent by Theorem 5.3.1 and

$$\mathfrak{B}_0(a_0) = \mathfrak{B}_0(b_0).$$

If a_0 and b_0 are not β-apexes Theorem 5.4.1 shows that they are mutually α-vertices and from Theorem 5.2.3 one concludes that they are free equivalent. Theorem 5.4.1 shows conversely that if these conditions are fulfilled the two vertices are (β, β)-cursally equivalent.

We shall not pursue this topic of cursal equivalence any further although there are other results of interest. We shall only mention that instead of cursal inclusion and cursal equivalence one can introduce a wider concept of accessible inclusion and accessible equivalence. For instance, two vertices a_0 and b_0 are (β, β)-accessibly equivalent if every vertex v which can be reached from a_0 in a (β, γ)-path can be reached from b_0 in the same manner and vice versa. There are a number of results which may be obtained but we shall not discuss these questions here.

CHAPTER 6. SIMULTANEOUS SUBGRAPHS

6.1. Subgraphs of first degree. As before let G be a finite graph and H a subgraph of *first degree*. We write G as a direct sum

$$G = H + \overline{H}. \tag{6.1.1}$$

In the following we shall prefer to call an edge of H a β-edge and an edge of $\overline{H}$ an α-edge.

To an arbitrary vertex $c_0 \in S$ we construct the accessible set $W(c_0)$ under β-paths. Since there is only a single β-edge at each vertex G can have no β-apexes, hence in $W(c_0)$ there is no central bicursal set. Furthermore, since any noncentral α-block $\mathfrak{P}_\alpha$ has an entrance e which is a β-apex there can be no such blocks. Thus the α-component of the accessible set $W(c_0)$ is simply the set $A = W_\alpha$ of α-vertices.

Any β-edge from W_α must go to W_β or be the entering edge of some β-block $\mathfrak{P}_\beta$ whose entrance e is an α-apex. Each block $\mathfrak{P}_\beta$ is composed of a certain number of bicursal equivalence blocks $\mathfrak{B}$. Since their entrances cannot be β-apexes all $\mathfrak{B}$ are β-blocks.

Take $v \in \mathfrak{P}_\beta$. According to the general theory of alternating paths there are both α and β-edges at v belonging to the section graph $G(\mathfrak{P}_\beta)$ except possibly at the entrance e of $\mathfrak{P}_\beta$. Thus except for the entering β-edge to $\mathfrak{P}_\beta$ coming from W_α there are no β-edges touching $\mathfrak{P}_\beta$. It follows that there are no β-edges from $\mathfrak{P}_\beta$ to β-vertices while all α-edges from $\mathfrak{P}_\beta$ go to W_α. We conclude that each block $\mathfrak{P}_\beta$ is a disjoint component within the section graph $G(\overline{A})$ of the complement $\overline{A}$ of $A = W_\alpha$. Similarly, any β-vertex b_0 is isolated in $G(\overline{A})$ because the β-edge to b_0 must come from A and the α-edges at b_0 must be cursal to A.

We denote by

$$\mathfrak{F}_\beta = \{\mathfrak{P}_\beta\} + \{b\} \tag{6.1.2}$$

the family of all β-blocks $\mathfrak{P}_\beta$ and all β-vertices b. Since there is exactly one β-edge from each vertex in W_α to a single component in $\mathfrak{F}_\beta$ we conclude that W_α and $\mathfrak{F}_\beta$ contain the same number of terms

$$\nu(\mathfrak{F}_\beta) = \nu(W_\alpha). \tag{6.1.3}$$

There may be other connected components in $G(\overline{A})$ but these must consist of inaccessible vertices in the complement $\overline{W}$ of $W(c_0)$. There can be no edges from $\overline{W}$ to $\mathfrak{F}_\beta$ and the edges connecting $\overline{W}$ with A must all be α-edges.

We make the observation:

THEOREM 6.1.1. *Let H be a subgraph of first degree for which every edge is free. Then the blocks $\mathfrak{P}_\beta$ consist of a single bicursal equivalence block $\mathfrak{B}_\beta$.*

Proof. If $\mathfrak{P}_\beta$ consists of more than one block $\mathfrak{B}$ the section graph $G(\mathfrak{P}_\beta)$ has bridges which are entering edges to the various $\mathfrak{B}$. Thus they are bound β-edges as we have seen, contrary to assumption.

Two or more subgraphs shall be called *simultaneous* if no two of them have any edge in common. Let us suppose that G has a family of $k \geqq 2$ simultaneous subgraphs H_i of first degree. Then any edge in such a graph is free and Theorem 6.1.1 holds. The previous discussion also shows:

THEOREM 6.1.2. *Let G be a graph with $k \geqq 2$ simultaneous subgraphs H_i of first degree. Then each bicursal block $\mathfrak{B}$ and each β-vertex b is connected to W_α by a single edge in each graph H_i.*

Thus for any $a_i \in W$ there is exactly one edge belonging to each H_j and these edges connect a_i with β-vertices b and blocks $\mathfrak{B}$. The proof of Theorem 6.1.2 follows immediately from Theorem 4.4.1.

6.2. **Completely decomposable graphs.** A graph G is *completely decomposable* if it is the sum of n simultaneous subgraphs of first degree

$$G_n = \Sigma H_i, \qquad H_1 = H. \tag{6.2.1}$$

Such a graph is regular of degree n. By means of the preceding observations a certain reduction can be achieved in the problem of determining when a regular graph G_n is completely decomposable.

As before we construct the accessible set $W(c_0)$ with respect to H-paths from a vertex c_0. Since there are at least two graphs H_i in (6.2.1) we conclude from Theorem 6.1.1 that $\mathfrak{P}_\beta = \mathfrak{B}$ for each bicursal equivalence block. Theorem 6.1.2 shows that there is a single edge of H_i connecting a block $\mathfrak{B}$ with W_α. Since there are no other types of edges in G_n we conclude that each $\mathfrak{B}$ and each β-vertex b is connected to W_α by n edges, one from each H_i. There can be no inaccessible vertices when G_n is connected, because as we saw, the edges from W_α to $\overline{W}$ were α-edges, that is, edges not in H. Since this must be true for every H_i in (6.2.1) there can be no such edges.

Let us denote by

$$E_i = (a_i, b_i), \qquad a_i \in W_\alpha, \qquad b_j \in \mathfrak{B}, E_i \subset H_i \tag{6.2.2}$$

the n edges connecting a block $\mathfrak{B}$ with W_α. To each $\mathfrak{B}$ there is a section graph $G(\mathfrak{B})$ consisting of the edges in G connecting vertices in $\mathfrak{B}$. To $G(\mathfrak{B})$ we shall construct an enlarged graph as follows. To the set $\mathfrak{B}$ we adjoin a single new vertex b^* and to the edges in $G(\mathfrak{B})$ we add the new edges

$$E_i^* = (b^*, b_i) \tag{6.2.3}$$

defined by the endpoints b_i of the connecting edges E_i in (6.2.2). This new graph we shall call the *block graph* of $\mathfrak{B}$ and denote by

$$G(\mathfrak{B}^*), \mathfrak{B}^* = \mathfrak{B} + b^*. \tag{6.2.4}$$

The block graph (6.2.4) is seen to be regular of degree n; it is connected and has no bridges. We shall show:

THEOREM 6.2.1. *Let the regular connected graph G_n have a subgraph H of first degree such that all edges in G_n are free for H. Then G_n is completely decomposable if and only if each block $\mathfrak{B}$ defined by H has n connecting edges* (6.2.2) *and the block graphs* (6.2.4) *are completely decomposable.*

Proof. It is evident from their definition that the block graphs (6.2.4) are completely decomposable if G_n has this property. Conversely let there exist a subgraph H of G_n of first degree with the properties indicated. Then there can be no inaccessible vertices and $\mathfrak{P}_\beta = \mathfrak{B}$ for each bicursal block.

We introduce a bipartite graph

$$G' = (W'_\alpha, \mathfrak{F}'_\beta) \tag{6.2.5}$$

where the vertex set W'_α is in one-to-one correspondence with the set W_α of α-vertices and $\mathfrak{F}'_\beta$ in one-to-one correspondence with the terms in the family $\mathfrak{F}_\beta$ in (6.1.2) consisting of the β-vertices and the β-blocks. We connect vertices a'_i in W'_α by an edge (a'_i, b'_j) to a term b'_j in $\mathfrak{F}'_\beta$ if and only if there is an edge in G connecting the corresponding quantities a_i in W_α and $\mathfrak{B}_j$ in F_β. We notice that this construction may result in multiple edges in G'. Evidently G' is regular of degree n by our assumptions.

From the general theory of bipartite graphs we conclude that

$$G' = \Sigma H'_k \qquad (k = 1, 2, \cdots, n) \tag{6.2.6}$$

is the sum of n simultaneous subgraphs of first degree. The decomposition (6.2.6) gives a corresponding decomposition of the graph of edges in G connecting W_α and $\mathfrak{F}_\beta$ into subgraphs H_k, so that there is a single edge of each H_k at every vertex $a_i \in W_\alpha$ and a single edge of H_k to each $\mathfrak{B}$ or β-vertex b. This classification of the edges from W_α to $\mathfrak{F}_\beta$ can be combined with the postulated complete decomposition of each block graph (6.2.4) to give a complete decomposition of G.

Theorem 6.2.1 shows in particular that the graph G is completely decomposable if there are no β-blocks $\mathfrak{B}$, that is, the family $\mathfrak{F}_\beta$ in (6.1.2) consists only of β-vertices.

Theorem 6.2.1 may be used for repeated reductions of the problem of determining when a regular graph is completely decomposable. Such reductions are possible until one reaches the point where $W_\alpha = c_0$ consists of a single vertex and this holds for every vertex c_0. When this stage has been reached every vertex in G is an α-apex.

In case G is planar it is readily verified that all block graphs $G^*(\mathfrak{B})$ in (6.2.4) are planar. Thus the preceding discussion applies to such graphs and the problem of determining when a complete decomposition exists may be reduced to the case where G has a subgraph of first degree for which all edges in G are free and all vertices are α-apexes. This applies in particular to regular planar graphs of degree 3.

Bibliography

1. O. Ore, *Graphs and subgraphs*, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 109–136.
2. J. Petersen, *Die Theorie der regulären Graphen*, Acta Math. vol. 15 (1891) pp. 193–220.

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