ON A SOJOURN TIME PROBLEM IN THE THEORY OF STOCHASTIC PROCESSES(1)

BY

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1. **Introduction.** Let us consider a stochastic process $\{\xi(t), 0 \leqq t < \infty\}$ with values in an abstract space X. Let $X = A + B$ where A and B are fixed disjoint sets. Suppose that $\xi(0) \in A$. Then the process $\{\xi(t)\}$ assumes the states A and B alternately. Denote by $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ the successive sojourn times spent in states A and B respectively.

We suppose that $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of positive random variables. Let $\zeta_n = \xi_0 + \xi_1 + \cdots + \xi_n$ and $\chi_n = \eta_0 + \eta_1 + \cdots + \eta_n$ $(n = 0, 1, 2, \cdots)$ where $\xi_0 \equiv \eta_0 \equiv 0$. Further it is assumed that

$$\lim_{n \to \infty} P\left\{\frac{\zeta_n - A_1 n}{A_2 n^a} \leqq x\right\} = G(x) \tag{1}$$

and

$$\lim_{n \to \infty} P\left\{\frac{\chi_n - B_1 n}{B_2 n^b} \leqq x\right\} = H(x) \tag{2}$$

where $G(x)$ and $H(x)$ are nondegenerate distribution functions, $A_1 \geqq 0$, $A_2 > 0$, $a > 0$, $B_1 \geqq 0$, $B_2 > 0$, $b > 0$. If $A_1 > 0$ $(B_1 > 0)$ then $a < 1$ $(b < 1)$ holds(2).

Denote by $\alpha(t)$ and $\beta(t)$ the total sojourn times spent in states A and B respectively during the time interval $(0, t)$. Clearly $\alpha(t)$ and $\beta(t)$ are random variables and $\alpha(t) + \beta(t) = t$ for all $t \geqq 0$.

In this paper we shall determine the asymptotic distribution of the random variable $\beta(t)$ as $t \to \infty$.

2. **The asymptotic distribution of $\beta(t)$.** We shall prove the following

THEOREM. *If $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of positive random variables with properties* (1) *and* (2), *then we have*

$$\lim_{t \to \infty} P\left\{\frac{\beta(t) - M_1 t}{M_2 t^m} \leqq x\right\} = P(x) \tag{3}$$

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(2) We say that a sequence of distribution functions converges to a limiting distribution function if it converges in every continuity point of the limiting distribution. Further if a distribution function in its every continuity point agrees with a certain function, then we use the same notation for these two functions.

where $P(x)$ is a nondegenerate distribution function and M_1, M_2, m are constants which depend on (1) *and* (2).

Table I *contains the appropriate distribution functions $P(x)$ and constants M_1, M_2, m.*

In Table I *ζ and χ are independent random variables with distribution functions $\mathbf{P}\{\zeta \leqq x\} = G(x)$ and $\mathbf{P}\{\chi \leqq x\} = H(x)$, further ρ is a random variable with $\mathbf{P}\{\rho \leqq x\} = P(x)$.*

Remarks. The random variable $\beta(t)$ can be expressed always as a sum of a random number of random variables, where in addition the number of the random variables depends on the random variables themselves. So in some particular cases we can determine the asymptotic distribution of $\beta(t)$ by a theorem of F. J. Anscombe [1]. If specifically we suppose that $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed independent positive random variables, then the asymptotic distribution of $\beta(t)$ can be obtained also in some particular cases by using a theorem of W. L. Smith [10]. Below we shall show that the determination of the asymptotic distribution of $\beta(t)$ may be reduced in all cases to that of a sum of a random number of random variables, where the number of the variables is independent of the variables themselves. To obtain the asymptotic distribution we can apply in all cases a theorem of R. L. Dobrushin [5], or in some particular cases a theorem of H. Robbins [9].

We remark that more general problems for Markov chains have been treated by W. Doeblin [6] (cf. D. G. Kendall [7]), K. L. Chung [2], R. L. Dobrushin [4], J. Lamperti [8] and for Markov-processes by D. A. Darling and M. Kac [3].

The case when $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed independent positive random variables has been investigated by the author [11].

We need some auxiliary theorems to the proof.

3. **Auxiliary theorems.** Let us correspond a stochastic process $\{\nu(t),\ 0 \leqq t < \infty\}$ to the sequence of random variables $\{\zeta_n\}$ supposing that $\nu(t)$ assumes only non-negative integer values and

$$\{\nu(t) < n\} \equiv \{\zeta_n \geqq t\} \tag{4}$$

holds for all $t \geqq 0$ and $n = 1, 2, \cdots$.

Lemma 1. *If the random variables ξ_n ($n = 0, 1, 2, \cdots$) satisfy* (1) *then the limiting distribution*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu(t) - C_1 t}{C_2 t^c} \leqq x\right\} = F(x) \tag{5}$$

TABLE I

	A_1	B_1	(a,b)	M_1	M_2	m	$P(x)$	ρ	x
1.	0	0	$a>b$	0	$B_2A_2^{-b/a}$	$\frac{b}{a}$	$P\left\{\frac{\chi^a}{\zeta^b}\leqq x^a\right\}$	$\chi\zeta^{-b/a}$	$(0, \infty)$
2.	0	0	$a=b$	0	1	1	$P\left\{\frac{\chi}{\zeta}\leqq\frac{A_2}{B_2}\frac{x}{(1-x)}\right\}$	$\frac{B_2\chi}{A_2\zeta+B_2\chi}$	$(0, 1)$
3.	0	0	$a<b$	1	$A_2B_2^{-a/b}$	$\frac{a}{b}$	$P\left\{\frac{\zeta^b}{\chi^a}\geqq(-x)^b\right\}$	$-\zeta\chi^{-a/b}$	$(-\infty, 0)$
4.	0	>0	$a>1$	0	$B_1A_2^{-1/a}$	$\frac{1}{a}$	$P\{\zeta\geqq x^{-a}\}$	$\zeta^{-1/a}$	$(0, \infty)$
5.	0	>0	$a=1$	0	1	1	$P\left\{\zeta\geqq\frac{B_1}{A_2}\frac{(1-x)}{x}\right\}$	$\frac{B_1}{B_1+A_2\zeta}$	$(0, 1)$
6.	0	>0	$a<1$	1	$A_2B_1^{-a}$	a	$P\{\zeta\geqq -x\}$	$-\zeta$	$(-\infty, 0)$
7.	>0	0	$b>1$	1	$A_1B_2^{-1/b}$	$\frac{1}{b}$	$P\{\chi\leqq(-x)^{-b}\}$	$-\chi^{-1/b}$	$(-\infty, 0)$
8.	>0	0	$b=1$	0	1	1	$P\left\{\chi\leqq\frac{A_1}{B_2}\frac{x}{(1-x)}\right\}$	$\frac{B_2\chi}{A_1+B_2\chi}$	$(0, 1)$
9.	>0	0	$b<1$	0	$B_2A_1^{-b}$	b	$P\{\chi\leqq x\}$	χ	$(0, \infty)$
10.	>0	>0	$a>b$	$\frac{B_1}{A_1+B_1}$	$\frac{B_1A_2}{(A_1+B_1)^{1+a}}$	a	$P\{\zeta\geqq -x\}$	$-\zeta$	$(-\infty, \infty)$
11.	>0	>0	$a=b$	$\frac{B_1}{A_1+B_1}$	$\left(\frac{A_1}{A_1+B_1}\right)^{1+a}$	a	$P\left\{\frac{A_1B_2\chi-B_1A_2\zeta}{A_1^{1+a}}\leqq x\right\}$	$\frac{A_1B_2\chi-B_1A_2\zeta}{A_1^{1+a}}$	$(-\infty, \infty)$
12.	>0	>0	$a<b$	$\frac{B_1}{A_1+B_1}$	$\frac{A_1B_2}{(A_1+B_1)^{1+b}}$	b	$P\{\chi\leqq x\}$	χ	$(-\infty, \infty)$

exists. Here $F(x)$ is a nondegenerate distribution function and C_1, C_2, c are constants, for which we have

A_1	C_1	C_2	c	$F(x)$	ν
0	0	$1/A_2^{1/a}$	$\dfrac{1}{a}$	$1 - G(x^{-a})$	$\zeta^{-1/a}$
>0	$1/A_1$	A_2/A_1^{1+a}	a	$1 - G(-x)$	$-\zeta$

where ν and ζ are random variables with distribution function $\mathbf{P}\{\nu \leqq x\} = F(x)$ and $\mathbf{P}\{\zeta \leqq x\} = G(x)$.

Proof. First, consider the case $A_1 > 0$. By (4) we have

$$\mathbf{P}\{\nu(t) < n\} = \mathbf{P}\{\zeta_n \geqq t\}$$

for all $t \geqq 0$ and $n = 1, 2, \cdots$. Hence also

$$(6) \qquad \mathbf{P}\left\{\frac{\nu(t) - \dfrac{t}{A_1}}{\dfrac{A_2}{A_1}\left(\dfrac{t}{A_1}\right)^a} < \frac{n - \dfrac{t}{A_1}}{\dfrac{A_2}{A_1}\left(\dfrac{t}{A_1}\right)^a}\right\} = \mathbf{P}\left\{\frac{\zeta_n - A_1 n}{A_2 n^a} \geqq \frac{t - A_1 n}{A_2 n^a}\right\}$$

holds for all $t > 0$ and $n = 1, 2, \cdots$. Now in (6) put

$$n = \left[\frac{t}{A_1} + \frac{xA_2}{A_1}\left(\frac{t}{A_1}\right)^a\right].$$

In this case we have

$$\lim_{t\to\infty} \frac{n - \dfrac{t}{A_1}}{\dfrac{A_2}{A_1}\left(\dfrac{t}{A_1}\right)^a} = x$$

and

$$\lim_{t\to\infty} \frac{t - A_1 n}{A_2 n^a} = -x.$$

If $t \to \infty$ (and thus $n \to \infty$) in (6), then we have

$$\lim_{t\to\infty} P\left\{ \frac{\nu(t) - \dfrac{t}{A_1}}{\dfrac{A_2}{A_1}\left(\dfrac{t}{A_1}\right)^a} < x \right\} = \lim_{n\to\infty} P\left\{\frac{\zeta_n - A_1 n}{A_2 n^a} \geqq -x\right\} = 1 - G(-x)$$

in every continuity point of the limiting distribution. This proves (5) in case $A_1 > 0$.

Second, consider the case $A_1 = 0$. Then $x \geqq 0$. By (4) we have

$$P\left\{ \frac{\nu(t)}{\left(\dfrac{t}{A_2}\right)^{1/a}} < \frac{n}{\left(\dfrac{t}{A_2}\right)^{1/a}} \right\} = P\left\{\frac{\zeta_n}{A_2 n^a} \geqq \frac{t}{A_2 n^a}\right\}. \tag{7}$$

Putting

$$n = \left[x\left(\frac{t}{A_2}\right)^{1/a} \right]$$

in (7) and taking the limit $t \to \infty$ (and thus $n \to \infty$), we have

$$\lim_{t\to\infty} P\left\{ \frac{\nu(t)}{\left(\dfrac{t}{A_2}\right)^{1/a}} \leqq x \right\} = \lim_{n\to\infty} P\left\{\frac{\zeta_n}{A_2 n^a} \geqq \frac{1}{x^a}\right\} = 1 - G(x^{-a})$$

in every continuity point of the limiting distribution. This proves (5) in case $A_1 = 0$.

Next we introduce a new stochastic process $\{\vartheta(t),\ 0 \leqq t < \infty\}$ as follows

$$\vartheta(t) = \chi_{\nu(t)} = \eta_0 + \eta_1 + \cdots + \eta_{\nu(t)}. \tag{8}$$

LEMMA 2. *For all* $t \geqq 0$ *and* $0 \leqq x \leqq t$ *we have*

$$P\{\beta(t) \leqq x\} = P\{\vartheta(t - x) \leqq x\}. \tag{9}$$

Proof. Denote by τ the least instant for which $\alpha(\tau) = t - x$. Clearly $\xi(\tau) \in A$ and τ is a random variable defined uniquely. Now we shall show that

$$P\{\beta(t) \leqq x\} = P\{\beta(\tau) \leqq x\}. \tag{10}$$

Taking into consideration that $\alpha(t) + \beta(t) = t$ for all $t \geqq 0$ and $\alpha(t)$ and $\beta(t)$ are nondecreasing functions of t we can easily prove the following identities

$$\{\beta(t) \leqq x\} \equiv \{\alpha(\tau) \leqq \alpha(t)\} \equiv \{\tau \leqq t\} \equiv \{\alpha(\tau) + \beta(\tau) \leqq t\} \equiv \{\beta(\tau) \leqq x\}$$

which proves (10).

Under the condition that the instant τ takes place on the $(n+1)$st

"A-interval" ($n=0, 1, 2, \cdots$) we have $\beta(\tau)=\chi_n$ and in this case $\zeta_n < t-x \leqq \zeta_{n+1}$, that is $\nu(t-x)=n$. Consequently

$$\beta(\tau) = \chi_{\nu(t-x)} = \vartheta(t-x)$$

holds and thus (9) is proved by (10).

LEMMA 3. *If* (2) *and* (5) *hold, then the limiting distribution*

$$\lim_{t\to\infty} P\left\{\frac{\vartheta(t) - D_1 t}{D_2 t^d} \leqq x\right\} = Q(x) \tag{11}$$

exists, with a nondegenerate distribution function $Q(x)$ *and with constants* D_1, D_2, d. *We have*

C_1	B_1	(c,b)	D_1	D_2	d	ϑ
0	0	—	0	$B^2C_2^b$	cb	$\chi\nu^b$
0	>0	—	0	B_1C_2	c	ν
>0	0	—	0	$B_2C_1^b$	b	χ
>0	>0	$c>b$	B_1C_1	B_1C_2	c	ν
>0	>0	$c=b$	B_1C_1	1	b	$B_1C_2\nu+B_2C_1^b\chi$
>0	>0	$c<b$	B_1C_1	$B_2C_1^b$	b	χ

where ϑ *is a random variable with distribution function* $Q(x)$ *and* χ *and* ν *are independent random variables with distribution functions* $P\{\chi\leqq x\}=H(x)$ *and* $P\{\nu\leqq x\}=F(x)$.

Proof. Knowing the asymptotic distributions of χ_n and $\nu(t)$ as $n\to\infty$ and $t\to\infty$ respectively, we can easily determine the asymptotic distribution of $\beta(t)$ as $t\to\infty$ by the theorem of R. L. Dobrushin [5]. Particularly it follows from Dobrushin's theorem, that the asymptotic distribution of $\vartheta(t)$ is independent of the special forms of $\{\chi_n\}$ and $\{\nu(t)\}$, it depends only on the asymptotic behaviours of $\{\chi_n\}$ and $\{\nu(t)\}$. Accordingly in order to get the asymptotic distribution of $\vartheta(t)$ we can suppose specifically that $\chi_n=B_1 n + B_2 n^b\chi$ and $\nu(t)=C_1+C_2t^c\nu$ where χ and ν are independent random variables with distribution function $H(x)$ and $F(x)$ respectively. In this case we have

$$\vartheta(t) = \chi_{\nu(t)} = B_1(C_1t + C_2t^c\nu) + B_2\chi(C_1t + C_2t^c\nu)^b.$$

Applying the above formula we can easily see that in every case the asymptotic distribution of $\vartheta(t)$ as $t\to\infty$ exists and the appropriate limiting distribution $Q(x)$ and the corresponding parameters D_1, D_2, d can be easily obtained.

4. **Proof of the theorem.** Taking into consideration the relation (9) and knowing the asymptotic distribution of $\vartheta(t)$ as $t\to\infty$ we can determine the asymptotic distribution of $\beta(t)$ as $t\to\infty$.

LEMMA 4. *If* (11) *holds then the limiting distribution*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t)-M_1t}{M_2t^m} \leqq x\right\} = P(x) \tag{12}$$

exists, where $P(x)$ *is an appropriate nondegenerate distribution function and* M_1, M_2, m *are constants. We have*

d	M_1	M_2	m	$P(x)$	x	ρ
$d>1$	1	$D_2^{-1/d}$	$\frac{1}{d}$	$Q(-x^{-d})$	$(-\infty, 0)$	$-\vartheta^{-1/d}$
$d=1$	0	1	1	$Q\left(\frac{1}{D_2}\frac{x}{(1-x)}\right)$	$(0, 1)$	$\frac{D_2\vartheta}{1+D_2\vartheta}$
$d<1$	$\frac{D_1}{1+D_1}$	$\frac{D_2}{(1+D_1)^{1+d}}$	d	$Q(x)$	$(-\infty, \infty)$	ϑ

where ρ *and* ϑ *are random variables with distribution functions* $P(x)$ *and* $Q(x)$ *respectively.*

Proof. For the sake of brevity let us define

$$u = t + D_1t + xD_2t^d \tag{13}$$

then by (9) we have

$$\mathbf{P}\{\vartheta(t) \leqq D_1t + xD_2t^d\} = \mathbf{P}\{\beta(u) \leqq u - t\}. \tag{14}$$

Since in all cases $t\to\infty$ if $u\to\infty$, consequently by (11) we obtain from (14) that

$$\lim_{u\to\infty} \mathbf{P}\{\beta(u) \leqq u - t\} = Q(x) \tag{15}$$

where $t=t(u)$ can be determined from (13).

If $d>1$ then $D_1=0$, $x\geqq 0$ and

$$t = \left(\frac{u}{xD_2}\right)^{1/d} + o(u^{1/d}), \qquad (u\to\infty),$$

if $d=1$ then $D_1=0$, $0\leqq x<\infty$ and

$$t = \frac{u}{1+xD_2},$$

and finally if $d<1$, then $-\infty<x<\infty$ and

$$t = \frac{u}{1+D_1} - \frac{xD_2}{1+D_1}\left(\frac{u}{1+D_1}\right)^d + o(u^d), \qquad (u\to\infty).$$

Hence by (15) we obtain the following limiting distributions

$$\lim_{u\to\infty} \boldsymbol{P}\left\{\frac{\beta(u)-u}{\left(\dfrac{u}{D_2}\right)^{1/d}} \leqq -x^{-1/d}\right\} = Q(x), \qquad \text{if } d>1,$$

$$\lim_{u\to\infty} \boldsymbol{P}\left\{\frac{\beta(u)}{u} \leqq \frac{xD_2}{1+xD_2}\right\} = Q(x), \qquad \text{if } d=1,$$

and

$$\lim_{u\to\infty} \boldsymbol{P}\left\{\frac{\beta(u)-\dfrac{D_1}{1+D_1}u}{\dfrac{D_2}{1+D_1}\left(\dfrac{u}{1+D_1}\right)^d} \leqq x\right\} = Q(x), \qquad \text{if } d<1.$$

This completes the proof of Lemma 4.

Applying Lemma 4 we can easily obtain all statements of the theorem.

5. **Examples.** Throughout this section we shall suppose that $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed independent positive random variables.

1. Suppose that $\boldsymbol{E}\{\xi_n\}=\alpha$, $\boldsymbol{E}\{\eta_n\}=\beta$, $\boldsymbol{D}\{\xi_n\}=\sigma_\alpha$, $\boldsymbol{D}\{\eta_n\}=\sigma_\beta$. Then the limiting distributions (1) and (2) exist and $A_1=\alpha$, $B_1=\beta$, $A_2=\sigma_\alpha$, $B_2=\sigma_\beta$, $a=b=1/2$ and $G(x)=H(x)=\Phi(x)$ where

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

In this case by the 11th statement of the Theorem we have

$$\lim_{t\to\infty} P\left\{\frac{\beta(t) - \dfrac{\beta}{\alpha+\beta}t}{\left(\dfrac{\alpha}{\alpha+\beta}\right)^{3/2} t^{1/2}} \leqq x\right\} = P\left\{\frac{\alpha\sigma_\beta\chi - \beta\sigma_\alpha\zeta}{\alpha^{3/2}} \leqq x\right\}$$

where χ and ζ are independent random variables with distribution function $\Phi(x)$. Hence

$$\lim_{t\to\infty} P\left\{\frac{\beta(t) - \dfrac{\beta}{\alpha+\beta}t}{\left(\dfrac{t}{\alpha+\beta}\right)^{1/2}} \leqq x\right\} = \Phi\left(\frac{(\alpha+\beta)x}{(\alpha^2\sigma_\beta^2 + \beta^2\sigma_\alpha^2)^{1/2}}\right). \tag{16}$$

if $-\infty < x < \infty$.

2. Suppose that $\lim_{x\to\infty} x^{1/2} P\{\xi_n > x\} = A$ and $\lim_{x\to\infty} x^{1/2} P\{\eta_n > x\} = B$, where A and B are positive constants. Then the limiting distributions (1) and (2) exist with $A_1 = B_1 = 0$, $A_2 = A^2$, $B_2 = B^2$, $a = b = 2$ and $G(x) = H(x) = F_{1/2}(x)$ where

$$F_{1/2}(x) = \begin{cases} 2\left[1 - \Phi\left(\left(\dfrac{\pi}{2x}\right)^{1/2}\right)\right] & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

In this case by the 2nd statement of the Theorem for $0 \leqq x \leqq 1$ we obtain

$$\lim_{t\to\infty} P\left\{\frac{\beta(t)}{t} \leqq x\right\} = P\left\{\frac{\chi}{\zeta} \leqq \frac{A^2}{B^2}\,\frac{x}{(1-x)}\right\},$$

where χ and ζ are independent random variables with distribution function $F_{1/2}(x)$. We can write $\chi = \pi/2\chi^{*2}$ and $\zeta = \pi/2\zeta^{*2}$ where χ^* and ζ^* are independent random variables with the same distribution function $\Phi(x)$. If we take into consideration that

$$P\left\{\frac{\zeta^*}{\chi^*} \leqq x\right\} = \frac{1}{2} + \frac{1}{\chi}\operatorname{arc\,tan} x$$

then we obtain

$$\begin{aligned} \lim_{t\to\infty} P\left\{\frac{\beta(t)}{t} \leqq x\right\} &= \frac{2}{\pi}\operatorname{arc\,tan}\frac{A}{B}\left(\frac{x}{1-x}\right)^{1/2} \\ &= \frac{2}{\pi}\operatorname{arc\,sin}\left(\frac{A^2x}{A^2x + B^2(1-x)}\right)^{1/2} \end{aligned} \tag{17}$$

if $0 \leqq x \leqq 1$.

3. Suppose that $\lim_{x\to\infty} x^{1/2}\mathbf{P}\{\xi_n > x\} = A$ and (2) holds with $\mathbf{E}\{\eta_n\} = \beta > 0$. In this case $A_1 = 0$, $A_2 = A^2$, $a = 2$, $B_1 = \beta$ and $G(x) = F_{1/2}(x)$. By the 4th statement of the Theorem we have

$$\lim_{t\to\infty} \mathbf{P}\left\{ \frac{\beta(t)}{\dfrac{\beta t^{1/2}}{A}} \leqq x \right\} = 1 - F_{1/2}(x^{-2})$$

that is

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t)}{t^{1/2}} \leqq x\right\} = 2\Phi\left(\left(\frac{\pi}{2}\right)^{1/2} \frac{xA}{\beta}\right) - 1 \tag{18}$$

if $0 \leqq x < \infty$.

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