A UNIQUENESS THEOREM FOR HAAR MEASURE(1)

BY

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Introduction. In our work in surface area theory we had occasion to use Haar measure in the space of oriented lines in Euclidean 3-space (see [1]; numbers in square brackets refer to the Bibliography at the end of this paper). We assumed at that time, as a matter of course, that the uniqueness of Haar measure for this classical situation in Integral Geometry should follow directly from the general results on Haar measure. However, we became aware later on (see [2]) of certain peculiar features of this special situation which seem to preclude the use of any of the general uniqueness theorems known to us. The purpose of this paper is to prove a uniqueness theorem for Haar measure which covers not only the special case referred to above but also various other cases occurring in Integral Geometry. We shall formulate presently our Uniqueness Theorem.

Let X be a separable metric space. For $x \in X$, $0 < r < \infty$ we denote by $c(x, r)$ the set of those points $y \in X$ whose distance from x does not exceed r (the *closed sphere* with center x and radius r). Let there be given in X a family $\mathfrak{H} = \{h\}$ of homeomorphisms h of X onto itself, such that the following condition is satisfied: if x_1, x_2 are any two points in X, then there exist finite positive constants l, L, ρ, as well as an $h \in \mathfrak{H}$ (where l, L, ρ, h depend upon upon x_1, x_2) such that $h(x_1) = x_2$ and

$$(1) \qquad c(x_2, lr) \subset h[c(x_1, r)] \subset c(x_2, Lr) \qquad \text{for } 0 < r < \rho.$$

Note that this condition is weaker than the requirement that distance in X should be invariant under the homeomorphisms $h \in \mathfrak{H}$. Actually, we were compelled to operate with the weaker condition (1) by the fact that in the special situation referred to above no invariant distance is available.

By a Haar measure in X (relative to $\mathfrak{H}$) we mean a measure μ on the family of the Borel subsets of X such that $\mu[h(B)] = \mu(B)$ for every Borel set $B \subset X$ and for every $h \in \mathfrak{H}$. A Haar measure μ is *nontrivial* if there exists a bounded open set O such that $0 < \mu(O) < \infty$.

In the special situation referred to above, no invariant distance is available, as we already noted. On the other hand, there is available a nontrivial Haar measure μ which satisfies an additional strong requirement. Due to this fact, the special situation is covered by a general theorem which we now shall formulate.

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UNIQUENESS THEOREM. *Let μ_1, μ_2 be nontrivial Haar measures in the separable metric space X, relative to a family $\mathfrak{H}=\{h\}$ of homeomorphisms h of X onto itself satisfying the condition* (1) *stated above. Assume that for every bounded set $E\subset X$ there exist finite positive constants $k(E)$, $K(E)$ such that*

$$\mu_2[c(x, 5r)] < K(E)\mu_2[c(x, r)] \qquad \text{for } x \in E,\ 0 < r < k(E). \tag{2}$$

Then there exists a finite positive constant γ such that $\mu_1(B)=\gamma\mu_2(B)$ for every Borel set $B\subset X$.

The proof is given in §4, while in §5 and §6 we discuss, in the way of illustration, the application to the space of oriented lines in Euclidean 3-space. In making the proof of the Uniqueness Theorem, it is convenient to extend first the measures μ to Borel-regular Carathéodory outer measures. In §1 we collect the needed background material on such outer measures. Several density theorems are needed in the proof of the uniqueness theorem. Since we need these density theorems in a more general form than available in the literature, we prove them explicitly in §2. The *Covering Theorem* established in §3 is perhaps of independent interest.

1. **Preliminaries.** 1.1. A Carathéodory outer measure $\Gamma(E)$ in a metric space X is a real-valued function defined for all the subsets E of X, such that the following holds.

(i) Always $0\leqq\Gamma(E)\leqq\infty$.

(ii) $\Gamma(\emptyset)=0$, where $\emptyset$ is the empty set.

(iii) If $E_1\subset E_2$, then $\Gamma(E_1)\leqq\Gamma(E_2)$.

(iv) If $E=\bigcup E_n$, then $\Gamma(E)\leqq\sum\Gamma(E_n)$.

(v) If E_1, E_2 are nonempty sets whose distance is positive, then $\Gamma(E_1\cup E_2)=\Gamma(E_1)+\Gamma(E_2)$.

1.2. A set E is termed Γ-measurable (or measurable with respect to the Carathéodory outer measure Γ) if for every choice of a pair of sets E', E'' such that $E'\subset E$, $E''\subset X-E$ one has $\Gamma(E'\cup E'')=\Gamma(E')+\Gamma(E'')$. Every Borel subset of X is Γ-measurable. For a comprehensive presentation of the theory of Carathéodory outer measure, the reader may consult [5].

1.3. A Carathéodory outer measure Γ is termed *locally finite* if for every point $x\in X$ there exists a bounded open set O_x such that $x\in O_x$ and $\Gamma(O_x)<\infty$. If Γ is locally finite and X is separable, then (since the family of these open sets O_x covers X) there follows the existence of a sequence of bounded open sets O_n such that

$$X=\bigcup O_n, \qquad \Gamma(O_n)<\infty, \qquad n=1, 2, \cdots.$$

1.4. A Carathéodory outer measure Γ will be termed *Borel-regular* if for every set $E\subset X$ there exists a Borel set $B\subset X$ such that $E\subset B$ and $\Gamma(E)=\Gamma(B)$.

1.5. A measure $\mu(B)$ on the family of the Borel sets $B\subset X$ is a non-negative set function such that the following holds.

(i) Always $0 \leqq \mu(B) \leqq \infty$.

(ii) $\mu(\emptyset) = 0$.

(iii) If B_n, $n = 1, 2, \cdots$, is a sequence of pair-wise disjoint Borel sets, then $\mu(\bigcup B_n) = \sum \mu(B_n)$.

Given such a measure μ, we define for an arbitrary set $E \subset X$,

$$\Gamma(E) = \underset{B \supset E}{\text{gr.l.b.}}\, \mu(B),$$

where B is a generic notation for a Borel set. It is easy to see that Γ is then a Borel-regular Carathéodory outer measure which agrees with μ on Borel sets.

2. **Density lemmas.** 2.1. Throughout the present §2 we shall consider, in a separable metric space X, a pair of Carathéodory outer measures Λ, σ with the following properties.

(i) Λ and σ are both locally finite and Borel-regular (see 1.3, 1.4).

(ii) For every bounded set $E \subset X$ there exist two positive finite constants $k(E)$, $K(E)$ such that

$$\sigma[c(x, 5r)] < K(E)\sigma[c(x, r)] \qquad \text{for } x \in E,\ 0 < r < k(E).$$

We note that obviously $0 < \sigma[c(x, r)] < \infty$ for each point $x \in X$ and for $0 < r < k_x$, where k_x is a constant depending on x, as a consequence of (i) and (ii).

2.2. LEMMA. *Let t be a finite positive number and E a bounded subset of X. Let $\overline{E}$ be a bounded closed set containing E. Put*

$$G_t(E) = \left\{ x \,\middle|\, \limsup_{r \to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > t \right\}. \tag{1}$$

Then (*see* 2.1 (ii))

$$\sigma[G_t(E)] \leqq \frac{K(\overline{E})}{t} \Lambda(E). \tag{2}$$

Proof. Consider the family of those closed spheres $c(x, r)$ for which

$$x \in \overline{E}, \qquad 0 < r < k(\overline{E}), \qquad \frac{\Lambda[c(x, r) \cap E]}{\sigma | c(x, r) |} > t.$$

Clearly these closed spheres cover $G_t(E)$. Accordingly (see [3; 4]) there exists a sequence $c(x_i, r_i)$, $i = 1, 2, \cdots$, of these closed spheres such that $c(x_i, r_i) \cap c(x_j, r_j) = \emptyset$ for $i \neq j$ and

$$G_t(E) \subset \bigcup_i c(x_i, 5r_i).$$

It follows that

$$\sigma[G_t(E)] \leqq \sum_i \sigma[c(x_i, 5r_i)] < K(\overline{E}) \sum_i \sigma[c(x_i, r_i)] \tag{3}$$
$$< \frac{K(\overline{E})}{t} \sum_i \Lambda[c(x_i, r_i) \cap E].$$

Now since the closed spheres $c(x_i, r_i)$ are pair-wise disjoint and Λ-measurable, we have (see [5, p. 44])

$$\sum_i \Lambda[c(x_i, r_i) \cap E] = \Lambda\left[\left(\bigcup_i c(x_i, r_i)\right) \cap E\right] \leqq \Lambda(E),$$

and (2) follows in view of (3).

2.3. Lemma. *Let E be a Λ-measurable subset of X. Then*

$$\lim_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} = 0 \qquad \textit{for } \sigma\textit{-a.e. } x \in \mathcal{C}E.$$

Proof. Case 1. E is bounded and $\Lambda(E) < \infty$. For $0 < t < \infty$ put

$$H_t(E) = \left\{x \mid x \in \mathcal{C}E, \limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > t\right\}. \tag{4}$$

Keep t fixed, and assign $\eta > 0$. Then (see [6]) there exists a closed set C such that

$$C \subset E, \qquad \Lambda(E - C) < \eta. \tag{5}$$

Since C is a closed subset of E, we have

$$c(x, r) \cap E = c(x, r) \cap (E - C) \quad \text{for } x \in \mathcal{C}E, r \text{ small.}$$

Hence clearly

$$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap (E - C)]}{\sigma[c(x, r)]} > t \qquad \text{for } x \in H_t(E).$$

Thus we have, in the notations of 2.2, the inclusion

$$H_t(E) \subset G_t(E - C).$$

Consequently, application of 2.2(2) to the set $E - C$ (with $\overline{E}$ taken as the closure of E) yields (see (5) and 2.1(ii))

$$\sigma[H_t(E)] \leqq \frac{K(\overline{E})}{t} \Lambda(E - C) < \frac{K(\overline{E})}{t} \eta.$$

As $\eta > 0$ was arbitrary, we conclude that

$$H_t(E) = 0 \qquad \text{for } 0 < t < \infty,$$

and the lemma follows since

$$\left\{x \mid x \in \complement E, \limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > 0\right\} = \bigcup_n H_{1/n}(E).$$

CASE 2. E is any Λ-measurable subset of X. Since Λ is locally finite, there exists (see 1.3) a sequence of bounded open sets O_n such that $X = \bigcup O_n$ and $\Lambda(O_n) < \infty$, $n = 1, 2, \cdots$. Put

$$E_n = E \cap O_n.$$

Clearly the set

$$\left\{x \mid x \in \complement E, \limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > 0\right\}$$

is a subset of the union of the sets

$$\left\{x \mid x \in \complement E_n, \limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap E_n]}{\sigma[c(x, r)]} > 0\right\}, \quad n = 1, 2, \cdots.$$

Since all these latter sets are of σ-measure zero by Case 1, the lemma follows.

2.4. LEMMA. *Let E be a subset of X such that*

$$\liminf_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} = 0 \qquad \text{for } x \in E. \tag{6}$$

Then $\Lambda(E) = 0$.

Proof. CASE 1. Assume first that there exists an open set O such that

$$E \subset O, \quad \sigma(O) < \infty, \quad O \text{ bounded}. \tag{7}$$

Then E itself is bounded. Hence the constants $k(E)$, $K(E)$ are available (see 2.1). Assign now $\eta > 0$. In view of (6) and (7) the family of closed spheres $c(x, r)$ satisfying the conditions (see 2.1)

$$x \in E, \quad 0 < r < k(E), \quad c(x, r) \subset O, \quad \frac{\Lambda[c(x, 5r) \cap E]}{\sigma[c(x, 5r)]} < \eta, \tag{8}$$

cover E. Accordingly (see [3; 4]) there exists a sequence $c(x_i, r_i)$, $i = 1, 2, \cdots$, of these closed spheres such that $c(x_i, r_i) \cap c(x_j, r_j) = \emptyset$ for $i \neq j$ and

$$E \subset \bigcup_i c(x_i, 5r_i).$$

It follows that

$$E \subset \bigcup_i [c(x_i, 5r_i) \cap E],$$

and hence (see (8) and 2.1)

$$\Lambda(E) \leqq \sum_i \Lambda[c(x_i, 5r_i) \cap E] < \eta \sum_i \sigma[c(x_i, 5r_i)]$$
$$< \eta K(E) \sum_i \sigma[c(x_i, r_i)] \leqq \eta K(E)\sigma(O).$$

Since $\sigma(O) < \infty$ and η was arbitrary, it follows that $\Lambda(E) = 0$.

CASE 2. E is an arbitrary subset of X satisfying (6). Since σ is locally finite, there exists a sequence of bounded open sets O_n such that $X = \bigcup O_n$, $\sigma(O_n) < \infty$, $n = 1, 2, \cdots$ (see 1.3). Put

$$E_n = E \cap O_n.$$

Then we have *a fortiori*

$$\liminf_{r\to 0} \frac{\Lambda[c(x, r) \cap E_n]}{\sigma[c(x, r)]} = 0 \qquad \text{for } x \in E_n,$$

and hence $\Lambda(E_n) = 0$ by Case 1. Since $E = \bigcup E_n$, it follows that $\Lambda(E) = 0$.

3. **A Covering Theorem.** 3.1. Throughout the present §3 we consider, in a separable metric space X, a pair of Carathéodory outer measures Λ, σ with the properties stated in 2.1.

3.2. LEMMA. *Let B^* be a Borel subset of X such that $\Lambda(B^*) < \infty$ and $\sigma(B^*) > 0$, and let $\mathfrak{F}$ be a family of Borel subsets of X which cover B^* in the following manner: for every point $x \in B^*$ there exists a set $S_x \in \mathfrak{F}$ such that $x \in S_x$ and*

(1) $$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap S_x]}{\sigma[c(x, r)]} > 0.$$

Then there exists a set $S \in \mathfrak{F}$ such that

(2) $$\Lambda(B^* - S) < \Lambda(B^*).$$

Proof. Since $\sigma(B^*) > 0$, by 2.3 (applied with $E = \mathcal{C}B^*$) we can select a point $x \in B^*$ such that

(3) $$\lim_{r\to 0} \frac{\Lambda[c(x, r) \cap \mathcal{C}B^*]}{\sigma[c(x, r)]} = 0.$$

For this point $x \in B^*$ we have then

$$\frac{\Lambda[c(x, r) \cap S_x]}{\sigma[c(x, r)]} \leqq \frac{\Lambda[c(x, r) \cap S_x \cap B^*]}{\sigma[c(x, r)]} + \frac{\Lambda[c(x, r) \cap S_x \cap \mathcal{C}B^*]}{\sigma[c(x, r)]}$$
$$\leqq \frac{\Lambda[c(x, r) \cap S_x \cap B^*]}{\sigma[c(x, r)]} + \frac{\Lambda[c(x, r) \cap \mathcal{C}B^*]}{\sigma[c(x, r)]}.$$

By (1) and (3) it follows that

$$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap S_x \cap B^*]}{\sigma[c(x, r)]} > 0,$$

and thus $\Lambda(S_x \cap B^*) > 0$. Hence

$$\Lambda(B^* - S_x) = \Lambda[B^* - (S_x \cap B^*)] = \Lambda(B^*) - \Lambda(S_x \cap B^*) < \Lambda(B^*),$$

and thus (2) holds for $S = S_x$.

3.3. COVERING THEOREM. *Let B be a Borel subset of X, and let $\mathfrak{F}$ be a family of Borel subsets of X which cover B in the following manner: for every point $x \in B$ there exists a set $S_x \in \mathfrak{F}$ such that $x \in S_x$ and*

$$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap S_x]}{\sigma[c(x, r)]} > 0.$$

Then there exists a sequence S_i, $i = 1, 2, \cdots$, of sets of the family $\mathfrak{F}$ such that

$$\sigma\left(B - \bigcup_i S_i\right) = 0. \tag{4}$$

Proof. CASE 1. $\Lambda(B) < \infty$. Let us put

$$\alpha = \text{gr.l.b. } \Lambda\left(B - \bigcup_i S_i\right), \tag{5}$$

where the greatest lower bound is taken with respect to all sequences $\{S_i\}$ of sets of $\mathfrak{F}$. Clearly $\alpha < \infty$ since $\Lambda(B) < \infty$. Obviously, there exists a sequence $\{S_i^*\}$ of sets of $\mathfrak{F}$ such that

$$\alpha = \Lambda\left(\mathrm{B} - \bigcup_i S_i^*\right). \tag{6}$$

We assert that

$$\sigma\left(B - \bigcup_i S_i^*\right) = 0. \tag{7}$$

Indeed, assume that $\sigma(B - \bigcup_i S_i^*) > 0$, and put $B^* = B - \bigcup_i S_i^*$. Then we would have $\sigma(B^*) > 0$, and hence by 3.2 there should exist a set $S \in \mathfrak{F}$ such that $\Lambda(B^* - S) < \Lambda(B^*)$. In view of (6) this would yield, since

$$B^* - S = \left(B - \bigcup_i S_i^*\right) - S = B - \left[\left(\bigcup_i S_i^*\right) \cup S\right],$$

the inequality

$$\Lambda\left\{B - \left[\left(\bigcup_i S_i^*\right) \cup S\right]\right\} < \Lambda\left(B - \bigcup_i S_i^*\right) = \alpha,$$

in contradiction with (5). Thus (7) holds, and the theorem is proved for the case when $\Lambda(B) < \infty$.

CASE 2. B is an arbitrary Borel set. Since Λ is locally finite, there exists a sequence O_n of bounded open sets such that $X = \bigcup O_n$, $\Lambda(O_n) < \infty$. Put

$$B_n = B \cap O_n, \qquad n = 1, 2, \cdots.$$

By Case 1, for each n we have then a sequence S_i^n, $i = 1, 2, \cdots$, of sets of $\mathfrak{F}$ such that

$$\sigma\left(B_n - \bigcup_i S_i^n\right) = 0.$$

Since $B = \bigcup B_n$, it follows that

$$\sigma\left(B - \bigcup_n \bigcup_i S_i^n\right) \leqq \sigma\left[\bigcup_n \left(B_n - \bigcup_i S_i^n\right)\right] \leqq \sum_n \sigma\left(B_n - \bigcup_i S_i^n\right) = 0,$$

and the proof is complete.

4. **Proof of the Uniqueness Theorem.** 4.1. Throughout the present §4, we consider a separable metric space X. We assume that there is given a family $\mathfrak{H} = \{h\}$ of homeomorphisms h of X onto itself such that the following condition is satisfied: if x_1, x_2 are any two points of X, then there exist finite positive constants l, L, ρ and a homeomorphism $h \in \mathfrak{H}$ (where l, L, ρ, h depend upon x_1, x_2), such that $h(x_1) = x_2$ and

$$c(x_2, lr) \subset h[c(x_1, r)] \subset c(x_2, Lr) \qquad \text{for } 0 < r < \rho.$$

4.2. A Carathéodory outer measure Γ in X will be termed *admissible* (with respect to the family $\mathfrak{H} = \{h\}$) if the following holds.

(i) Γ is Borel-regular.

(ii) There exists a nonempty, bounded open set O such that $0 < \Gamma(O) < \infty$.

(iii) $\Gamma(B) = \Gamma[h(B)]$ for every Borel subset B of X, and for every $h \in \mathfrak{H}$.

REMARK 1. Actually, (iii) implies that $\Gamma(E) = \Gamma[h(E)]$ for an arbitrary set $E \subset X$. Indeed, by (i) there exists a Borel set B such that $E \subset B$ and $\Gamma(E) = \Gamma(B)$. Then $h(E) \subset h(B)$, and hence

$$\Gamma(E) = \Gamma(B) = \Gamma[h(B)] \geqq \Gamma[h(E)]. \tag{1}$$

Similarly, there exists by (i) a Borel set B' such that $h(E) \subset B'$ and $\Gamma[h(E)] = \Gamma(B')$. Then $E \subset h^{-1}(B')$, and hence

$$\Gamma[h(E)] = \Gamma(B') = \Gamma[h^{-1}(B')] \geqq \Gamma(E).$$

Thus $\Gamma(E) = \Gamma[h(E)]$ in view of (1).

REMARK 2. If Γ is an admissible Carathéodory outer measure in X, then

Γ is locally finite. Indeed, fix a point x_0 in the open set O occurring in condition (ii) above. For each $x \in X$ there exists (see 4.1) a homeomorphism $h_x \in \mathfrak{H}$ such that $h(x_0) = x$. Then $h_x(O)$ is an open set such that $x \in h_x(O)$ and $\Gamma[h_x(O)] = \Gamma(O)$. Since $0 < \Gamma(O) < \infty$, it follows that $0 < \Gamma[h_x(O)] < \infty$. Hence Γ is locally finite.

REMARK 3. If Γ is an admissible Carathéodory outer measure in X and O is a nonempty open set, then $\Gamma(O) > 0$. Indeed, assume that $\Gamma(O) = 0$. Pick a point $x_0 \in O$. For each point $x \in X$ there exists then (see 4.1) an $h_x \in \mathfrak{H}$ such that $h_x(x_0) = x$. Then $h_x(O)$ is an open set such that $x \in h_x(O)$ and $\Gamma[h_x(O)] = \Gamma(O) = 0$. Since X is separable, there exists a sequence $h_{x_n}(O)$, $n = 1, 2, \cdots$, of these open sets O_x such that $X = \bigcup_n h_{x_n}(O)$. It follows that $\Gamma(X) = 0$, in contradiction with the admissibility of Γ.

REMARK 4. Let Γ_1, Γ_2 be admissible Carathéodory outer measures in X. Then there exist (see 4.2 (ii)) nonempty, bounded open sets O_1, O_2 such that $0 < \Gamma_1(O_1) < \infty$, $0 < \Gamma_2(O_2) < \infty$. Choose points $x_1 \in O_1$, $x_2 \in O_2$. By 4.1 there exists an $h \in \mathfrak{H}$ such that $h(x_1) = x_2$. Then $h(O_1)$ is an open set such that $x_2 \in h(O_1)$. On setting $O = h(O_1) \cap O_2$, it follows that O is a nonempty open set. Hence $0 < \Gamma_1(O)$, $0 < \Gamma_2(O)$ by Remark 3. Also, $\Gamma_1(O) \leqq \Gamma_1[h(O_1)] = \Gamma_1(O_1) < \infty$, $\Gamma_2(O) \leqq \Gamma_2(O_2) < \infty$. Thus it is shown that there exists a non-empty open set O such that $0 < \Gamma_1(O) < \infty$ and $0 < \Gamma_2(O) < \infty$.

4.3. UNIQUENESS THEOREM. *Let Λ and σ be admissible Carathéodory outer measures in X, where σ satisfies the following condition: for every bounded set $E \subset X$ there exist finite positive constants $k(E)$, $K(E)$ such that*

$$\sigma[c(x, 5r)] < K(E)\sigma[c(x, r)] \qquad \text{for } x \in E,\ 0 < r < k(E). \tag{2}$$

Then there exists a finite positive constant γ such that $\Lambda(E) = \gamma\sigma(E)$ for every set $E \subset X$.

The proof will be made in several steps.

4.4. LEMMA. *Let x_1 be a point of a set $E_1 \subset X$ such that*

$$\limsup_{r \to 0} \frac{\Lambda[c(x_1, r) \cap E_1]}{\sigma[c(x_1, r)]} > 0, \tag{3}$$

and let a point $x_2 \in X$ be assigned. Then there exists $h \in \mathfrak{H}$ such that $h(x_1) = x_2$ and

$$\limsup_{r \to 0} \frac{\Lambda[c(x_2, r) \cap h(E_1)]}{\sigma[c(x_2, r)]} > 0. \tag{4}$$

Proof. By condition (ii) in 4.1 there exist positive finite constants l, L, ρ, and an $h \in \mathfrak{H}$ such that $h(x_1) = x_2$ and

$$c(x_2, lr) \subset h[c(x_1, r)] \subset c(x_2, Lr) \qquad \text{for } 0 < r < \rho. \tag{5}$$

By (2), applied with $E = x_1$, there exist finite positive constants k, K such that

$$\sigma[c(x_1, 5r)] < K\sigma[c(x_1, r)] \qquad \text{for } 0 < r < k. \tag{6}$$

Let n be the smallest positive integer such that

$$\frac{L}{l} < 5^n. \tag{7}$$

Then by (6) we have

$$\sigma\left[c\left(x_1, \frac{r}{l}\right)\right] \leqq \sigma\left[c\left(x_1, 5^n \frac{r}{L}\right)\right] < K^n\sigma\left[c\left(x_1, \frac{r}{L}\right)\right] \text{ for } 0 < r < kL/5^n. \tag{8}$$

We restrict r by the inequalities

$$0 < r < L\rho, \qquad 0 < r < l\rho, \qquad 0 < r < kL/5^n. \tag{9}$$

Then we have, by (5) and (8)

$$\frac{\Lambda[c(x_2, r) \cap h(E_1)]}{\sigma[c(x_2, r)]} \geqq \frac{\Lambda\left\{h\left[c\left(x_1, \frac{r}{L}\right)\right] \cap h(E_1)\right\}}{\sigma\left\{h\left[c\left(x_1, \frac{r}{l}\right)\right]\right\}}$$

$$= \frac{\Lambda\left[c\left(x_1, \frac{r}{L}\right) \cap E_1\right]}{\sigma\left[c\left(x_1, \frac{r}{l}\right)\right]} = \frac{\Lambda\left[c\left(x_1, \frac{r}{L}\right) \cap E_1\right]}{\sigma\left[c\left(x_1, \frac{r}{L}\right)\right]} \cdot \frac{\sigma\left[c\left(x_1, \frac{r}{L}\right)\right]}{\sigma\left[c\left(x_1, \frac{r}{l}\right)\right]}$$

$$\geqq \frac{\Lambda\left[c\left(x_1, \frac{r}{L}\right) \cap E_1\right]}{\sigma\left[c\left(x_1, \frac{r}{L}\right)\right]} \cdot \frac{1}{K^n},$$

and (4) follows in view of (3).

4.5. LEMMA. *Let B_0 be a Borel subset of X such that $\Lambda(B_0) > 0$. Then there exists a sequence h_i, $i = 1, 2, \cdots$, of homeomorphisms in $\mathfrak{H}$ such that*

$$\sigma\left[X - \bigcup_i h_i(B_0)\right] = 0.$$

Proof. By the lemma in 2.4, the assumption $\Lambda(B_0) > 0$ implies the existence of a point $x_0 \in B_0$ such that

$$\limsup_{r \to 0} \frac{\Lambda[c(x_0, r) \cap B_0]}{\sigma[c(x_0, r)]} > 0. \tag{10}$$

Consider now any point $x \in X$. Then the lemma in 4.4 yields, in view of (10), the existence of an $h_x \in \mathfrak{H}$ such that $h_x(x_0) = x$ and

$$\limsup_{r \to 0} \frac{\Lambda[c(x, r) \cap h_x(B_0)]}{\sigma[c(x, r)]} > 0.$$

Thus the family of the sets $h_x(B_0)$ covers X in the manner required in the Covering Theorem of 3.3, and the present lemma follows.

4.6. Noting that σ itself satisfies all the conditions placed upon Λ, we can set $\Lambda = \sigma$ in the preceding lemma, obtaining the following corollary.

LEMMA. *Let B_0 be a Borel subset of X such that $\sigma(B_0) > 0$. Then there exists a sequence of homeomorphisms h_i, $i = 1, 2, \cdots$, in $\mathfrak{H}$ such that*

$$\sigma\left[X - \bigcup_i h_i(B_0)\right] = 0.$$

4.7. LEMMA. *Let B be a Borel subset of X such that $\sigma(B) = 0$. Then $\Lambda(B) = 0$.*

Proof. Assume that $\Lambda(B) > 0$. By the lemma in 4.5 there exists then a sequence $h_i \in \mathfrak{H}$, $i = 1, 2, \cdots$, such that

$$\sigma\left[X - \bigcup_i h_i(B)\right] = 0. \tag{11}$$

Now since $\sigma[h_i(B)] = \sigma(B) = 0$, we have also

$$\sigma\left[\bigcup_i h_i(B)\right] = 0,$$

and hence in view of (11) it follows that $\sigma(X) = 0$, in contradiction with the admissibility of σ (see 4.2). Thus the lemma is proved.

4.8. LEMMA. *Let B be a Borel subset of X such that $\Lambda(B) = 0$. Then $\sigma(B) = 0$.*

Proof. Assume that $\sigma(B) > 0$. By the lemma in 4.6 there exists then a sequence $h_i \in \mathfrak{H}$, $i = 1, 2, \cdots$, such that

$$\sigma\left[X - \bigcup_i h_i(B)\right] = 0.$$

By 4.7 it follows that

$$\Lambda\left[X - \bigcup_i h_i(B)\right] = 0. \tag{12}$$

Now since $\Lambda[h_i(B)] = \Lambda(B) = 0$, we have also

$$\Lambda\left[\bigcup_i h_i(B)\right] = 0.$$

In view of (12) it follows that $\Lambda(X)=0$, in contradiction with the admissibility of Λ. Thus the lemma is proved.

4.9. Considered on the family of the Borel subsets of X, Λ is a countably additive set function. The lemma in 4.7 means (see [5, p. 30]) that Λ is absolutely continuous with respect to σ. Furthermore, since σ is locally finite (see 4.2, Remark 2) and X is separable, X is a countable union of open sets of finite σ-measure. Accordingly, by the Radon-Nikodyn theorem (see [5, p. 36]) there exists in X a real-valued, Borel-measurable function $f(x)$ such that

$$\Lambda(B) = \int_B f d\sigma$$

for every Borel set $B \subset X$. By 4.2, Remark 4, there exists a nonempty, bounded open set O such that

$$0 < \sigma(O) < \infty, \qquad 0 < \Lambda(O) < \infty. \tag{13}$$

We put

$$\gamma = \frac{\Lambda(O)}{\sigma(O)} \cdot \tag{14}$$

Then γ is finite and positive in view of (13).

4.10. Lemma. *For* $0<t<\infty$, *let*

$$B_t = \{x \mid f(x) < t\}, \qquad B^t = \{x \mid f(x) > t\}.$$

Then one of B_t, B^t *is of* σ-*measure zero.*

Proof. Assume, for example, that $\sigma(B_t)>0$. Noting that B_t is a Borel set since f is Borel-measurable, by the lemma in 4.6 we infer the existence of a sequence $h_i \in \mathfrak{H}$ such that

$$\sigma\left[X - \bigcup_i h_i(B_t)\right] = 0,$$

and hence *a fortiori*

$$\sigma\left[\mathcal{C}B_t - \bigcup_i h_i(B_t) \cap \mathcal{C}B_t\right] = 0. \tag{15}$$

Now consider the set

$$b_i = h_i^{-1}[h_i(B_t) \cap \mathcal{C}B_t] = B_t \cap h_i^{-1}(\mathcal{C}B_t).$$

Since h_i is a homeomorphism from X onto X, b_i is a Borel set. We assert that $\sigma(b_i)=0$. Indeed, assume that $\sigma(b_i)>0$. Since $b_i \subset B_t$ and $f<t$ on B_t, it follows that

(16) $$\Lambda[h_i(b_i)] = \Lambda(b_i) = \int_{b_i} f d\sigma < t\sigma(b_i).$$

Also, since

(17) $$h_i(b_i) = h_i(B_t) \cap \mathcal{C}B_t \subset \mathcal{C}B_t,$$

we have $f \leqq t$ on $h_i(b_i)$, and hence

(18) $$\Lambda[h_i(b_i)] = \int_{h_i(b_i)} f d\sigma \geqq t\sigma[h_i(b_i)] = t\sigma(b_i).$$

From (16) and (18) we obtain the contradiction $t\sigma(b_i) > t\sigma(b_i)$, and thus it is shown that $\sigma(b_i) = 0$. Hence, by (17), we have also

$$\sigma[h_i(b_i)] = \sigma[h_i(B_t) \cap \mathcal{C}B_t] = 0.$$

In view of (15) it follows that $\sigma(\mathcal{C}B_t) = 0$, and the lemma is proved, since $B^t \subset \mathcal{C}B_t$.

4.11. Lemma. *$f(x) = \gamma$ for σ-a.e. $x \in X$ (see* (14)).

Proof. Deny this assertion. Then clearly there should exist a real number t such that either

(19) $$\gamma < t < \infty \quad \text{and} \quad \sigma(B^t) > 0,$$

or else

(20) $$0 < t < \gamma \quad \text{and} \quad \sigma(B_t) > 0.$$

Since the argument is entirely analogous in both cases, we discuss explicitly only the situation indicated by (20). By the lemma in 4.10 we infer from (20) that $\sigma(B^t) = 0$, and hence

$$f(x) \leqq t < \gamma \qquad \text{for } \sigma\text{-a.e. } x \in X.$$

For the set O of 4.9 it follows that

$$\Lambda(O) = \int_O f d\sigma < \gamma\sigma(O),$$

in contradiction with 4.9 (14), and the lemma is proved.

4.12. The Uniqueness Theorem of 4.3 follows now directly. Indeed, the lemma of 4.11 yields

$$\Lambda(B) = \int_B f d\sigma = \gamma\sigma(B)$$

for every Borel set $B \subset X$. Since Λ and σ are both Borel-regular, it follows that $\Lambda(E) = \gamma\sigma(E)$ for every set $E \subset X$.

4.13. The Uniqueness Theorem stated in the introduction follows now readily. Indeed, let μ_1, μ_2 be two measures defined on the family of the Borel sets in X which satisfy the conditions of that theorem. Let Λ and σ be the Borel-regular extensions of μ_1, μ_2 respectively (see **1.5**). Then evidently Λ and σ satisfy the assumptions of the Uniqueness Theorem of **4.3**, and hence there exists a finite positive constant γ such that $\Lambda(E)=\gamma\sigma(E)$ for $E\subset X$. Since Λ and σ agree with μ_1 and μ_2 respectively on Borel sets, the Uniqueness Theorem stated in the introduction follows.

5. Space of oriented lines in Euclidean three-space. 5.1. Let $\mathfrak{G}$ be the family of oriented lines g in Euclidean three-space R^3. For $g\in\mathfrak{G}$ we denote by $[g]$ the nonoriented line in R^3 determined by g and by $u(g)$ the unit vector from the origin parallel to g and pointing in the direction of g. We use vector notation x to denote a point (x_1, x_2, x_3) in R^3. Let $\mathfrak{T}$ be the family of distance preserving transformations of R^3. Then the range of definition of each $T\in\mathfrak{T}$ can be extended to $\mathfrak{G}$ as follows: $\bar{g}=T(g)$ for $[\bar{g}]=T([g])$, $u(\bar{g})=T[u(g)]-T(O)$, where O is the origin in R^3. Then T is a one-to-one mapping from $\mathfrak{G}$ onto $\mathfrak{G}$ and the inverse of T on $\mathfrak{G}$ is the extension of the inverse of T on R^3.

5.2. As in [1] we introduce a metric on $\mathfrak{G}$ so that for $T\in\mathfrak{T}$, T is a homeomorphism on $\mathfrak{G}$. For $g\in\mathfrak{G}$ let $\phi(g)$ be the point on $[g]$ closest to the origin. Then it is easily verified that

$$d(g_1, g_2) = |\phi(g_2) - \phi(g_1)| + |u(g_2) - u(g_1)|$$

is a metric for $\mathfrak{G}$.

5.3. Before proceeding with the verification that each $T\in\mathfrak{T}$ is a homeomorphism on $\mathfrak{G}$ we consider the following facts given in [2] concerning the introduction of a metric in $\mathfrak{G}$. Given an ordered pair of distinct points (x, y) in R^3 let us denote by $\gamma(x, y)$ the unique oriented line g such that $[g]$ contains x and y and the orientation of g is in the direction from x to y. Clearly, every oriented line can be represented in the form $g=\gamma(x, y)$ in infinitely many ways. Given a sequence g_n, $n=0, 1, 2, \cdots$, of oriented lines in $\mathfrak{G}$ we shall say that g_n converges to g_0, in symbols $g_n\to g_0$, if and only if there exist representations $g_n=\gamma(x_n, y_n)$, $n=0, 1, 2, \cdots$, such that $|x_n-x_0|\to 0$ and $|y_n-y_0|\to 0$. This limit concept has the following properties. (i) If $g_n\to g_0$ then $T(g_n)\to T(g_0)$ for every $T\in\mathfrak{T}$. (ii) The metric introduced in 5.2 is such that $g_n\to g_0$ if and only if $d(g_n, g_0)\to 0$. (iii) If d^* is a metric on $\mathfrak{G}$ such that $g_n\to g_0$ if and only if $d^*(g_n, g_0)\to 0$, then d and d^* are equivalent metrics and d^* is *not invariant* under $T\in\mathfrak{T}$, i.e., $d^*(g_1, g_2)=d^*[T(g_1), T(g_2)]$ does *not* hold for every pair of oriented lines g_1, g_2 and every $T\in\mathfrak{T}$.

5.4. For $g\in\mathfrak{G}$ we note that

$$\phi(g)\cdot u(g) = 0 \tag{1}$$

where $\cdot$ indicates the scalar product of two vectors and

$$\phi(g) = x - [x\cdot u(g)]u(g) \qquad \text{for } x \in [g]. \tag{2}$$

For $T\in\mathfrak{T}$, $\bar{g}=T(g)$, $\bar{x}=T[\phi(g)]$, we have

$$[\bar{x} - T(O)]\cdot u(\bar{g}) = [\bar{x} - T(O)]\cdot\{T[u(g)] - T(O)\} = 0, \tag{3}$$

and from (2) and (3)

$$\phi(\bar{g}) = \bar{x} - [T(O)\cdot u(\bar{g})]u(\bar{g}). \tag{4}$$

From (4) for $g_1, g_2\in G$, $T\in\mathfrak{T}$, $\bar{g}_i=T(g_i)$, $\bar{x}_i=T[\phi(g_i)]$, $i=1, 2$, we obtain

$$\begin{aligned} |\phi(\bar{g}_2) - \phi(\bar{g}_1)| &\leqq |\bar{x}_2 - \bar{x}_1| + 2|T(O)|\,|u(\bar{g}_2) - u(\bar{g}_1)| \\ &= |\phi(g_2) - \phi(g_1)| + 2|T(O)|\,|u(g_2) - u(g_1)| \end{aligned} \tag{5}$$

and similarly from (4) we obtain

$$|\phi(g_2) - \phi(g_1)| \leqq |\phi(\bar{g}_2) - \phi(\bar{g}_1)| + 2|T(O)|\,|u(\bar{g}_2) - u(\bar{g}_1)|. \tag{6}$$

From (5) and (6) it follows that for $g_1, g_2\in\mathfrak{G}$, $T\in\mathfrak{T}$ and $\lambda(T)=1+2|T(O)|$

$$d[T(g_1), T(g_2)] \leqq \lambda(T)d(g_1, g_2), \tag{7}$$

$$d(g_1, g_2) \leqq \lambda(T)d[T(g_1), T(g_2)]. \tag{8}$$

Thus each $T\in\mathfrak{T}$ is a homeomorphism from $\mathfrak{G}$ onto $\mathfrak{G}$. We obtain easily from (7) and (8) the following result.

THEOREM. *For $g_1, g_2\in\mathfrak{G}$ and $T\in\mathfrak{T}$ such that $g_2=T(g_1)$ we have, for $r>0$ and $\lambda=\lambda(T)$,*

$$c(g_2, r/\lambda) \subset T[c(g_1, r)] \subset c(g_2, \lambda r).$$

6. **Haar measure in $\mathfrak{G}$.** 6.1. We denote by U the unit sphere $|x|=1$ in R^3. P will be a generic notation for a point on U. The transformation $P=\psi(g)$, where P is the end point of the vector $u(g)$, is a continuous transformation from $\mathfrak{G}$ onto U. For each $P\in U$ we set $\mathfrak{G}_P=\psi^{-1}(P)$. For $P\in U$ let $R^2(P)$ be the plane through the origin and perpendicular to the line through the origin and P. Now $\phi(g)$ (see 5.2) is a continuous transformation from $\mathfrak{G}$ onto R^3. For $P\in U$ we have

$$\phi(\mathfrak{G}_P) = R^2(P).$$

We denote by L_2 the Lebesgue planar exterior measure of a set lying in a plane and by m the Hausdorff two-dimensional measure on U. For a Borel set $B\subset\mathfrak{G}$ we set

$$\mu(B) = (1/2\pi)\int_U L_2[\phi(B\cap\mathfrak{G}_P)]dm.$$

In [1] it is shown that $\mu(B)$ is an invariant measure in $\mathfrak{G}$ for bounded Borel sets and the extension to unbounded Borel sets is immediate. As mentioned in the introduction, μ can be extended to an invariant Borel regular Carathéodory outer measure on $\mathfrak{G}$.

6.2. Let g_0 be the x_3-axis oriented in the direction from the origin O to the point $P_0=(0, 0, 1)$. For $r\leqq 1$ we have

$$L_2\phi[c(g_0, r) \cap \mathfrak{G}_P] \begin{cases} \geqq \pi r^2/4 & \text{for } |P - P_0| \leqq r/2, \\ \leqq \pi r^2 & \text{for } |P - P_0| \leqq r, \\ = 0 & \text{for } |P - P_0| > r. \end{cases} \tag{1}$$

Using polar coordinates θ, Φ and letting $\bar{\Phi}(r)$ be such that $r^2=2(1-\cos\bar{\Phi}(r))$, from (1) we obtain

$$\mu[c(g_0, r)] \leqq (r^2/2)\int_0^{2\pi}\int_0^{\bar{\Phi}(r)} \sin\Phi d\theta d\Phi = \pi r^4/2, \tag{2}$$

$$\mu[c(g_0, r)] \geqq (r^2/8)\int_0^{2\pi}\int_0^{\bar{\Phi}(r/2)} \sin\Phi d\theta d\Phi = \pi r^4/32. \tag{3}$$

6.3. THEOREM. *For $g\in\mathfrak{G}$, $\lambda(g)=1+2|\phi(g)|$, $0<r<1/5\lambda(g)$ we have*

$$\mu[c(g, 5r)] < [10\lambda(g)]^8\mu[c(g, r)].$$

Proof. Let g_0 be the x_3-axis oriented in the direction from the origin O to the point $(0, 0, 1)$ and let $T\in\mathfrak{T}$ be such that $g=T(g_0)$ and $T(O)=\phi(g)$. From the theorem in 5.4 and the inequalities (2) and (3) in 6.2, on setting $\lambda=\lambda(g)$, we obtain

$$\begin{aligned}\mu[c(g, 5r)] &\leqq \mu\{T[c(g_0, 5\lambda r)]\} = \mu[c(g_0, 5\lambda r)] \leqq (5\lambda r)^4\pi/2 \\ &< (10\lambda)^8\pi(r/\lambda)^4/32 \leqq (10\lambda)^8\mu[c(g_0, r/\lambda)] \\ &= (10\lambda)^8\mu\{T[c(g_0, r/\lambda)]\} \leqq (10\lambda)^8\mu[c(g, r)].\end{aligned}$$

6.4. The family of homeomorphisms $\mathfrak{T}$ satisfy the conditions of 4.1, since for $g_1, g_2\in\mathfrak{G}$ there is a $T\in\mathfrak{T}$ such that $g_2=T(g_1)$, and by the theorem in 5.4 we can take $l=1/\lambda(T)$ and $L=\lambda(T)$. If E is any bounded set in $\mathfrak{G}$ then there is a finite, positive constant $K(E)$ such that $[10\,\lambda(g)]^8\leqq K(E)$ for $g\in E$ and for $k(E)^8=1/K(E)$ by the theorem in 6.3 we have

$$\mu[c(g, 5r)] < K(E)\mu[c(g, r)] \qquad \text{for } g \in E,\ 0 < r < k(E).$$

Thus from the Uniqueness Theorem in 4.3 we have the following uniqueness theorem in $\mathfrak{G}$.

THEOREM. *If ν is a Borel regular Carathéodory outer measure in $\mathfrak{G}$ such that $\nu[T(E)]=\nu(E)$ for $E\subset\mathfrak{G}$, $T\in\mathfrak{T}$ and there is a nonempty bounded open set $O\subset\mathfrak{G}$ such that $0<\nu(O)<\infty$, then there exists a finite positive constant γ such that $\nu(E)=\gamma\mu(E)$ for $E\subset\mathfrak{G}$.*

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