A FIBERING OF A CLASS OF HOMOGENEOUS COMPLEX MANIFOLDS

BY

JUN-ICHI HANO AND SHOSHICHI KOBAYASHI(1)

1. **Introduction.** H. C. Wang has completely clarified the structure of compact homogeneous complex spaces with finite fundamental group. Such a space is always a homogeneous complex space of a connected compact semi-simple Lie group on account of a theorem of Montgomery, as it was shown by Wang. Then he reduces his problem to the study of compact complex coset spaces of complex semi-simple Lie groups, which he pursues by means of Lie algebra [7]. The purpose of this paper is to give a different (and more differential geometric) proof to the result of Wang. The main difference between his method and ours is that we make a strong use of the canonical 2-form associated to an invariant complex structure while he uses a theorem of Morozoff concerning the conjugateness of maximal solvable subalgebras of a complex semi-simple Lie algebra. We begin with the study of a wider class of homogeneous complex manifolds (not necessarily compact) by a purely differential geometric method (Theorem A). Then we restrict ourselves to the study of homogeneous complex manifolds of compact (not necessarily semi-simple) Lie groups and reduce it, by utilizing elementary properties of roots of compact Lie groups, to the study of homogeneous spaces of connected compact semi-simple Lie groups by the centralizers of tori (Theorems B and C). The invariant complex structures of a homogeneous space of a connected compact semisimple Lie group by the centralizer of a torus have been studied by various authors.

2. **Definitions and statements of results.** Let G/B be a homogeneous complex manifold of a connected Lie group G by a closed subgroup B and let $2n$ be the real dimension of G/B. Assume that G/B admits an invariant volume element (i.e., a nonzero exterior differential form V of degree $2n$ invariant by G). In terms of local coordinate system $z^1, \cdots, z^n$ on G/B, V is expressed by

$$V = K(z, \bar{z})dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

Set

$$R_{j\bar{k}} = \partial^2 \log K/\partial z^j \partial \bar{z}^k,$$

$$\rho = (-1)^{1/2} \sum R_{j\bar{k}} dz^j \wedge d\bar{z}^k, \qquad (\text{i.e., } \rho = (-1)^{1/2}\partial\bar{\partial} \log K).$$

Presented to the Society, April 19, 1958, under the title *On complex homogeneous spaces of Lie groups*; received by the editors January 19, 1959.

(1) The first named author was supported by an Office of Naval Research contract at the University of Washington during the preparation of this paper; the second named author was supported by National Science Foundation Grant at Massachusetts Institute of Technology.

The 2-form ρ is well defined (i.e., independent of choice of local coordinate system) and is invariant by G. We shall call ρ the canonical 2-form of G/B.

THEOREM A. *Let G/B be a homogeneous complex manifold of a connected Lie group G by a closed subgroup B. Assume that G/B admits an invariant volume element V and let ρ be the canonical 2-form. Then there exists a unique closed subgroup L (not necessarily connected) of G with the following properties*:

(a) *L contains B.*

(b) *L/B is connected.*

Consider G/B as a fibre bundle over G/L with typical fibre L/B and with projection p defined in a natural way. Then

(c) *The restriction of ρ to each fibre is identically zero.*

(d) *There exists a 2-form σ of maximal rank on the base space G/L such that $p^*(\sigma)=\rho$. (Hence, G/L is a homogeneous simplectic manifold.)*

The subgroup L, characterized by the properties (a), (b), (c) *and* (d), *possesses also the following properties*:

(e) *L contains the connected component of the identity of the center of G.*

(f) *Each fibre of G/B is a complex analytic submanifold of G/B. (In particular, L/B is a homogeneous complex manifold.)*

We shall show also that *if L/B is compact, then it is complex parallisable.* If G is compact, our result is more complete. If G is compact, B has only a finite number of connected components. Let B_0 be the connected component of the identity of B. Then G/B_0 is a covering space of G/B with a finite number of sheets. Without loss of generality we may, therefore, assume B to be connected.

THEOREM B. *Assume G to be compact and B to be connected in Theorem* A. *Then the subgroup L of G possesses the following additional properties*:

(g) *L/B is a complex torus.*

(h) *L is the centralizer of a toral subgroup of G.*

(i) *B is a C-subgroup of G. (In fact, the semi-simple part of B coincides with that of L.)*

(j) *Rank $G-$Rank $B=$the real dimension of L/B.*

(k) *G/L admits an invariant complex structure such that the projection p is a complex analytic map of G/B onto G/L. (Hence, G/B is a complex analytic principal fibre bundle over G/L with L/B (a complex torus) as structure group.)*

We recall the definition of C-subgroups due to Wang [7]. In general, a connected closed subgroup B of a connected compact Lie group G is called a *C-subgroup* if its semi-simple part coincides with that of the centralizer of a toral subgroup of G.

In general, let G be a connected compact Lie group and T a toral subgroup of G. Let $C(T)$ be the centralizer of T in G. The invariant complex structures of $G/C(T)$ can be described in terms of roots of the Lie algebra of G and they are finite in numbers [2; 7]. The space $G/C(T)$ is homogeneous Kaehlerian [1] and even rational algebraic [3].

Finally, we shall prove the converse of Theorem B (i).

THEOREM C. *Let G be a connected compact Lie group and B a C-subgroup of G. Then G/B admits an invariant complex structure provided that* dim G/B *is even. Moreover precisely,*

(i) *There exists a toral subgroup T of G such that the centralizer $C(T)$ of T contains B and that the semi-simple part of $C(T)$ coincides with that of B.*

(ii) *Given an invariant complex structure on each of $G/C(T)$ and $C(T)/B$, there exists a unique invariant complex structure on G/B such that G/B is a complex analytic principal fibre bundle over $G/C(T)$ with group $C(T)/B$.*

In the course of proof of Theorem B(k) and Theorem C, we describe completely the invariant complex structures of G/B in terms of roots of the Lie algebra of G.

Concerning the statement (k) of Theorem B, we do not know whether, without the assumption that G be compact, G/L admits an invariant complex structure such that G/B is a complex analytic fibre bundle over G/L with fibre L/B. In the proof of (k) we were unable to avoid the use of roots.

3. **Preliminary lemmas.** Let M be a complex manifold (not necessarily homogeneous) and V a volume element of M. Define the 2-form ρ as in §2. To each infinitesimal transformation (i.e., real vector field) X of M we assign a 1-form ρ_X as follows:

$$\rho_X = \iota(X)\rho,$$

where ι denotes the interior product. We define a real valued function δX, called the *divergence* of X by

$$\theta(X)\cdot V = \delta X \cdot V,$$

where $\theta(X)$ is the Lie derivation with respect to X.

Let X be an infinitesimal transformation of M leaving the complex structure J invariant. In terms of local coordinate system $z^1, \cdots, z^n$,

$$X = \sum \xi^j(\partial/\partial z^j) + \sum \bar{\xi}^j(\partial/\partial \bar{z}^j),$$

where each ξ^j is a holomorphic function in $z^1, \cdots, z^n$ and $\bar{\xi}^j$ is the complex conjugate of ξ^j. Let Y be an infinitesimal transformation of M. The following formula of Koszul [6] can be verified by a straightforward calculation:

$$2\iota(Y)\iota(X)\rho = \theta(Y)\cdot(\delta JX) - \theta(JY)\cdot(\delta X).$$

LEMMA 3.1. *Let X be an infinitesimal transformation of M leaving both the complex structure J and the volume element V invariant. Then*

$$2\rho_X = d(\delta JX).$$

Proof. Let Y be any infinitesimal transformation of M. Then

$$2\iota(Y)\rho_X = 2\iota(Y)\cdot\iota(X)\rho = \theta(Y)\cdot(\delta JX) = \iota(Y)d(\delta JX),$$

because of the Koszul's formula and $\delta X=0$. Hence, $2\rho_X=d(\delta JX)$. Q.E.D.

For any infinitesimal transformations X and Y of M, the following formula (the verification of which is trivial) holds:

$$\delta(J[X, Y]) = \theta(JX)\cdot(\delta Y) - \theta(Y)\cdot(\delta JX).$$

LEMMA 3.2. *If both X and Y leave J and V invariant, then*

$$2\iota(Y)\rho_X = -\delta(J[X, Y]).$$

Proof. $\delta(J[X, Y])=-\theta(Y)\cdot(\delta JX)=-2\iota(Y)\iota(X)\rho=-2\iota(Y)\rho_X$. Q.E.D.

4. **Proof of Theorem** A. Set $M=G/B$. Let $T_x(M)$ be the tangent space to M at x. Define a subspace T'_x of $T_x(M)$ by

$$T'_x = \{v \in T_x(M);\ \iota(v)\rho = 0\}.$$

As G is transitive on M, T'_x can be also defined as follows:

$$T'_x = \{v \in T_x(M);\ \rho_X(v) = 0 \text{ for all } X \text{ in } \mathfrak{g}\},$$

where $\mathfrak{g}$ is the Lie algebra of G. (We consider X as an infinitesimal transformation of M.) Let $F(x)$ be the maximal integral submanifold through x defined by the system of 1-forms $\{\rho_X;\ X\in\mathfrak{g}\}$. By Lemma 3.1, $F(x)$ is the connected component of x of the following set:

$$\{y \in M;\ f_X(y) = f_X(x) \text{ for all } X \in \mathfrak{g}\},$$

where $f_X=\delta JX$. Hence, $F(x)$ *is closed.* Let $\mathfrak{o}$ be the point of $M=G/B$ corresponding to B. Define

$$L = \{s \in G;\ s(F(\mathfrak{o})) = F(\mathfrak{o})\}.$$

As $F(\mathfrak{o})$ is closed, L is a closed subgroup of G. The invariance of ρ by G implies

LEMMA 4.1. $s(F(x))=F(sx)$ *for all s in G and x in M.*

As ρ is real and of type (1, 1), T'_x is stable under J, hence

LEMMA 4.2. *$F(x)$ is a complex analytic submanifold of M.*

Now, (a) follows from Lemma 4.1. Let x be any point of $F(\mathfrak{o})$. Take $s\in G$ such that $s(\mathfrak{o})=x$. By Lemma 4.1, s lies in L. Hence, L is transitive on $F(\mathfrak{o})$, i.e.,

$$F(\mathfrak{o}) = L/B,$$

which proves (b).

Consider G/B as a fibre bundle over G/L with fibre L/B. By Lemma 4.1, the fibre of G/B containing x is $F(x)$. Hence, the restriction of ρ to each fibre vanishes identically. Hence, (c).

Let Z be any infinitesimal transformation of M such that, at each $x \in M$, Z is tangent to the fibre $F(x)$. Then

$$\theta(Z)\rho = d \cdot \iota(Z)\rho + \iota(Z) \cdot d\rho = 0,$$

because $\iota(Z)\rho = 0$ and $d\rho = 0$. There exists, therefore, a (unique) 2-form σ on G/L such that $p^*(\sigma) = \rho$. From the definition of $F(x)$, it follows that σ is of maximal rank. As ρ is invariant by G, so is σ, thus proving (d).

Let Y be any element of the center of $\mathfrak{g}$. By virtue of Lemma 3.2, the 1-parameter group generated by Y maps each $F(x)$ into itself. Therefore, Y lies in the Lie algebra of L, thus proving (e). Finally, (f) follows from Lemma 4.2.

Now, we shall prove the uniqueness of L satisfying (a), (b), (c) and (d). Let L' be a closed subgroup of G satisfying (a)–(d). From (b) and (c), it follows that L'/B is contained in $F(\mathfrak{o})$. From (d) it follows that L'/B actually coincides with $F(\mathfrak{o})$. We can now easily verify that $L' = L$.

5. **The case when L/B is compact.** Let j be the injection of L/B into G/B. Then

$$j^*(\rho) = 0 \qquad (\text{i.e., } j^*(\partial\bar{\partial} \log K) = 0).$$

As j is complex analytic,

$$0 = j^*(\partial\bar{\partial} \log K) = \partial\bar{\partial}(j^* \log K) = \partial\bar{\partial}(\log j^*K).$$

Hence

$$j^*K = f \cdot \bar{f},$$

where f is a holomorphic function defined locally on L/B. Let $X_1, \cdots, X_n$ be arbitrarily chosen elements of the Lie algebra $\mathfrak{g}$ of G. Then

$$X_k = Z_k + \bar{Z}_k, \qquad k = 1, \cdots, n,$$

where each Z_k is a holomorphic vector field on G/B and $\bar{Z}_k$ is its complex conjugate. Define a non-negative function Ψ by

$$\Psi = \iota(\bar{Z}_n)\iota(Z_n) \cdots \iota(\bar{Z}_1)\iota(Z_1)V.$$

Then, we can write locally

$$j^*(\Psi) = \psi\bar{\psi}$$

where ψ is a holomorphic function defined in a coordinate neighborhood of L/B. Hence, $j^*(\Psi)$ is plurisubharmonic. If L/B is compact, then $j^*(\Psi)$ is necessarily a constant. Choose $X_1, \cdots, X_n$ in such a way that $Z_1, \cdots, Z_n$ are linearly independent (over the complex numbers) at the point $\mathfrak{o}$ of G/B and that $X_1, \cdots, X_m$ belong to the Lie algebra of L where m is the complex dimension of L/B. Then Ψ is a nonzero constant on $j(L/B)$. Hence $Z_1, \cdots, Z_m$, considered as holomorphic vector field on L/B, are linearly independent (over the complex numbers) at every point of L/B. Thus,

PROPOSITION 5.1. *If L/B is compact in Theorem* A, *then L/B is complex parallisable.*

Let N be the largest normal subgroup of L contained in B. Set $L' = L/N$ and $B' = B/N$. Then L' acts effectively on $L'/B' = L/B$. From a result of Wang [8], it follows that L' is a complex Lie group and B' is a discrete subgroup of L'. If L is compact, then L' is a complex torus and B' reduces to the identity element. Hence,

PROPOSITION 5.2. *If L is compact, then B is normal in L and L/B is a complex torus.*

6. **Proof of Theorem** B, (g), $\cdots$, (j). Note that (g) is an immediate consequence of Proposition 5.2. Let C be the connected component of the identity of the center of G. By (e), L contains C. Set

$$G^* = G/C \quad \text{and} \quad L^* = L/C.$$

Then, G^* is semi-simple and $G^*/L^* = G/L$. As G^*/L^* is homogeneous symplectic by (d), L^* is the centralizer of a certain toral subgroup T^* of G^* by a result of Borel [1]. Let T be the toral subgroup of G such that $T/C = T^*$. We shall show that L is the centralizer of T in G. Denote by $\mathfrak{g}$, $\mathfrak{l}$, $\mathfrak{c}$, etc. the Lie algebras of G, L, C, etc. As G is compact, $\mathfrak{g}$ (resp. $\mathfrak{l}$) is isomorphic to the direct sum of $\mathfrak{g}^*$ (resp. $\mathfrak{l}^*$) and $\mathfrak{c}$. From the fact that $\mathfrak{l}^*$ is the centralizer of $\mathfrak{t}^*$ in $\mathfrak{g}^*$, it follows easily that $\mathfrak{l} = \mathfrak{l}^* + \mathfrak{c}$ is the centralizer of $\mathfrak{t} = \mathfrak{t}^* + \mathfrak{c}$ in $\mathfrak{g} = \mathfrak{g}^* + \mathfrak{c}$. A theorem of H. Hopf states that the centralizer of a toral subgroup of a connected compact Lie group is always connected. On the other hand, L is connected. (Note that both B and L/B are connected.) Hence, L is exactly the centralizer of T in G, thus completing the proof of (h).

Both (i) and (j) are immediate consequences of (g).

7. **Determination of complex structure and proof of** (k). Let G, B and L be as in Theorem B. Let $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{b}$ be the Lie algebras of G, L and B respectively. As G is compact, there exists a positive definite symmetric bilinear form ϕ on g which is ad. G-invariant. Let

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{m}, \tag{1}$$

$$\mathfrak{l} = \mathfrak{b} + \mathfrak{n} \tag{2}$$

be the orthogonal decompositions with respect to ϕ. Then

$$[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}, \tag{3}$$

$$[\mathfrak{b}, \mathfrak{n}] = 0, \tag{4}$$

$$[\mathfrak{n}, \mathfrak{n}] = 0. \tag{5}$$

Observe that (3) is trivial and that (4) and (5) follow from the fact that B is normal in L and L/B is a (complex) toral group. Set

$$\mathfrak{m} = \mathfrak{n} + \mathfrak{w}. \text{ (Then, } \mathfrak{g} = \mathfrak{b} + \mathfrak{m},\ [\mathfrak{b}, \mathfrak{m}] \subset \mathfrak{m}.) \tag{6}$$

Let π be the natural projection of G onto G/B. Let π_* be the induced linear mapping of $\mathfrak{g}$ onto the tangent space $T_0(G/B)$ to G/B at $\mathfrak{o}=\pi(e)$, where e is the identity of G. Then π_* is a linear isomorphism of $\mathfrak{m}$ onto $T_0(G/B)$. Let J be the tensor field on G/B defining the complex structure of G/B. The restriction of J to the point $\mathfrak{o}$ induces an endomorphism I of $\mathfrak{m}$ with the following properties:

(7) $I^2 = -1$

(8) $[X, I\cdot Y] = I\cdot[X, Y]$ $\qquad X \in \mathfrak{b},\ Y \in \mathfrak{m}.$

(9) $I\cdot[X, Y]_\mathfrak{m} - [I\cdot X, Y]_\mathfrak{m} - [X, I\cdot Y]_\mathfrak{m} - I\cdot[I\cdot X, I\cdot Y]_\mathfrak{m} = 0$ for $X, Y \in \mathfrak{m}.$

Remark. In general, if G is a connected Lie group and B is a connected closed subgroup of G and if there exists a decomposition of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{b}+\mathfrak{m}$ and $[\mathfrak{b}, \mathfrak{m}]\subset\mathfrak{m}$, then every invariant complex structure J on G/B induces an endomorphism I of $\mathfrak{m}$ satisfying (7), (8) and (9); and, conversely, every endomorphism I of $\mathfrak{m}$ satisfying these conditions comes from an invariant complex structure J on G/B.

Let $\mathfrak{g}^c$ (resp. $\mathfrak{b}^c$, $\mathfrak{m}^c$, etc.) be the complexification of $\mathfrak{g}$ (resp. $\mathfrak{b}$, $\mathfrak{m}$, etc.). We extend I to a complex endomorphism of $\mathfrak{m}^c$ in a natural manner. Let $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) be the eigen space of I belonging to the eigen value $(-1)^{1/2}$ (resp. $-(-1)^{1/2}$). Then $\mathfrak{m}^c$ is a direct sum of $\mathfrak{m}^+$ and $\mathfrak{m}^-$. In terms of $\mathfrak{b}^c$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ we can express (8) and (9) as follows [5]:

$$[\mathfrak{b}^c, \mathfrak{m}^+] \subseteq \mathfrak{m}^+, \qquad [\mathfrak{b}^c, \mathfrak{m}^-] \subseteq \mathfrak{m}^-, \tag{8$'$}$$

(9′) Both $\mathfrak{b}^c + \mathfrak{m}^+$ and $\mathfrak{b}^c + \mathfrak{m}^-$ are subalgebras of $\mathfrak{g}^c$.

Since L/B is a complex analytic submanifold of G/B, $\mathfrak{n}$ is stable by I:

$$I(\mathfrak{n}) = \mathfrak{n}. \tag{10}$$

Hence

$$\mathfrak{n}^c = \mathfrak{n}^+ + \mathfrak{n}^-, \qquad \mathfrak{n}^+ = \mathfrak{m}^+ \cap \mathfrak{n}^c, \qquad \mathfrak{n}^- = \mathfrak{m}^- \cap \mathfrak{n}^c \tag{11}$$

Let $\mathfrak{h}_\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{b}$, i.e., a maximal abelian subalgebra of $\mathfrak{b}$. Set

$$\mathfrak{h} = \mathfrak{h}_\mathfrak{b} + \mathfrak{n}. \tag{12}$$

Then $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ by virtue of (4) and (5). On the other hand, dim $\mathfrak{h}$=rank G by (j). Hence $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Clearly, $\mathfrak{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. We shall fix this Cartan subalgebra once for all.

Let D be the set of nonzero roots α of $\mathfrak{g}^c$ (with respect to the Cartan subalgebra $\mathfrak{h}^c$). Let $D(\mathfrak{b}^c)$ (resp. $D(\mathfrak{w}^c)$) be the set of $\alpha \in D$ whose root vectors belong to $\mathfrak{b}^c$ (resp. $\mathfrak{w}^c$). Clearly, α lies in $D(\mathfrak{b}^c)$ if and only if $-\alpha$ does. We have

$$D = D(\mathfrak{b}^c) \cup D(\mathfrak{w}^c) \text{ (disjoint union).} \tag{13}$$

Because $[\mathfrak{h}, \mathfrak{b}] \subseteq \mathfrak{b}$, $[\mathfrak{h}, \mathfrak{n}] = 0$ and $[\mathfrak{h}, \mathfrak{w}] \subseteq \mathfrak{w}$.

We set

$$\mathfrak{h}^+ = \mathfrak{h}_{\mathfrak{b}}^c + \mathfrak{n}^+, \qquad \mathfrak{h}^- = \mathfrak{h}_{\mathfrak{b}}^c + \mathfrak{n}^-. \tag{14}$$

Then

$$[\mathfrak{h}^+, \mathfrak{b}^c] \subseteq \mathfrak{b}^c, \qquad [\mathfrak{h}^+, \mathfrak{m}^+] \subseteq \mathfrak{m}^+. \tag{15}$$

The first relation follows from (4). In order to prove the second one, it suffices to show that $[\mathfrak{n}^+, \mathfrak{m}^+] \subseteq \mathfrak{m}^+$ because of (8′). From (3) and (5) we obtain that $[\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}$. This together with (9′) implies that $[\mathfrak{n}^+, \mathfrak{m}^+] \subseteq \mathfrak{m}^+$.

We shall now prove the following

LEMMA 7.1. *If α and β are different roots with respect to the Cartan subalgebra $\mathfrak{h}^c$ of $\mathfrak{g}^c$, then the restrictions of α and β to $\mathfrak{h}^+$ give different linear functions.*

Proof. Otherwise, we would have

$$(\alpha - \beta)(Y - (-1)^{1/2} I \cdot Y) = 0 \qquad \text{for all } Y \text{ in } \mathfrak{n},$$
$$(\alpha - \beta)X = 0 \qquad \text{for all } X \text{ in } \mathfrak{h}_{\mathfrak{b}}.$$

Since G is compact, every root assumes only pure imaginary values on $\mathfrak{h}$. Hence

$$(\alpha - \beta) \cdot Y = (-1)^{1/2} (\alpha - \beta) \cdot I \cdot Y = 0 \qquad \text{for all } Y \text{ in } \mathfrak{n}.$$

As $\mathfrak{h}$ is a direct sum of $\mathfrak{n}$ and $\mathfrak{h}_{\mathfrak{b}}$, α would agree with β on $\mathfrak{h}$, which is a contradiction.

Choose, for each $\alpha \in D$, a root vector E_α in such a way that E_α and $E_{-\alpha}$ are complex conjugate to each other, i.e., both $E_\alpha + E_{-\alpha}$ and $(-1)^{1/2}(E_\alpha - E_{-\alpha})$ belong to $\mathfrak{g}$. Let $D(m^+)$ (resp. $D(\mathfrak{m}^-)$) be the set of $\alpha \in D$ such that $E_\alpha \in \mathfrak{m}^+$ (resp. $E_\alpha \in \mathfrak{m}^-$). Then

LEMMA 7.2. *$\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) is spanned by $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) and $\{E_\alpha; \alpha \in D(\mathfrak{m}^+)\}$ (resp. $\{E_\alpha; \alpha \in D(\mathfrak{m}^-)\}$).*

Proof. From Lemma 7.1 it follows that there exists an element H^* in $\mathfrak{h}^+$ such that

$$\alpha(H^*) \neq \beta(H^*) \qquad \text{if } \alpha \neq \beta.$$

The eigen space of ad. H^* belonging to any nonzero eigen value is of complex dimension 1.

The subspace $\mathfrak{m}^+$, stable under ad. H^* (by (15)), is therefore spanned by $\mathfrak{m}^+ \cap \mathfrak{h}^c = \mathfrak{n}^+$ and the root vectors E_α contained in $\mathfrak{m}^+$. Since $\mathfrak{m}^-$, $\mathfrak{n}^-$ and $E_{-\alpha}$ are complex conjugates of $\mathfrak{m}^+$, $\mathfrak{n}^+$ and E_α respectively, $\mathfrak{m}^-$ is spanned by $\mathfrak{n}^-$ and the E_α's such that $-\alpha \in D(\mathfrak{m}^+)$.

LEMMA 7.3. *If $\alpha \in D(\mathfrak{m}^+)$, then*

$$I \cdot (E_\alpha + E_{-\alpha}) = (-1)^{1/2}(E_\alpha - E_{-\alpha}),$$
$$I \cdot (-1)^{1/2}(E_\alpha - E_{-\alpha}) = -(E_\alpha + E_{-\alpha}).$$

Proof. A trivial consequence of $I \cdot E_\alpha = (-1)^{1/2} E_\alpha$ and $I \cdot E_{-\alpha} = -(-1)^{1/2} E_{-\alpha}$.

LEMMA 7.4. (1) $I \cdot \mathfrak{w} = \mathfrak{w}$.

(2) $\mathfrak{w}^+$ *(resp. $\mathfrak{w}^-$) is spanned by $\{E_\alpha; \alpha \in D(\mathfrak{m}^+)\}$ (resp. $\{E_\alpha; \alpha \in D(\mathfrak{m}^-)\}$), where $\mathfrak{w}^+ = \mathfrak{m}^+ \cap \mathfrak{w}$ (resp. $\mathfrak{w}^- = \mathfrak{m}^- \cap \mathfrak{w}$).*

Proof. An immediate consequence of (13).

Let I' be the restriction of I to $\mathfrak{w}$. By Lemma 7.4, I' is an endomorphism of $\mathfrak{w}$. We have

(16) $$I'^2 = -1,$$

(17) $$[X, I'Y] = I'[X, Y], \qquad X \in \mathfrak{l},\ Y \in \mathfrak{w}.$$

(18) $$I'[X, Y]_{\mathfrak{w}} - [I'X, Y]_{\mathfrak{w}} - [X, I'Y]_{\mathfrak{w}} - I'[I'X, I'Y]_{\mathfrak{w}} = 0, \qquad X, Y \in \mathfrak{w}.$$

The formula (16) follows from (7). The proof of (17) can be divided into two cases: $X \in \mathfrak{b}$ or $X \in \mathfrak{n}$. If $X \in \mathfrak{b}$, then (17) is a consequence of (8). If $X \in \mathfrak{n}$ and $Y = E_\alpha + E_{-\alpha}$ or $(-1)^{1/2}(E_\alpha - E_{-\alpha})$ where $\alpha \in D(\mathfrak{m}^+)$, then (17) can be verified easily. Hence (17) holds for $X \in \mathfrak{n}$ and $Y \in \mathfrak{w}$ by Lemma 7.4. The formula (18) follows from (9). Therefore, I' defines an invariant complex structure on G/L, thus completing the proof of (k).

We are now in position to complete the proof of the following

PROPOSITION 7.5. *There exists a fundamental system of roots $\alpha_1, \cdots, \alpha_l$ (where l is the rank of the semi-simple part of G) of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$ such that*

(1) α lies in $D(\mathfrak{b}^c)$ if and only if it is a linear combination of $\alpha_1, \cdots, \alpha_r$ where r is the rank of the semi-simple part of B.

(2) A root α belonging to $D - D(\mathfrak{b}^c) = D(\mathfrak{m}^+) \cup D(\mathfrak{m}^-)$ is positive with respect to the ordering of D defined by $\alpha_1, \cdots, \alpha_l$ if and only if it is in $D(\mathfrak{m}^+)$.

Proof. It is well known that if α, β and $\alpha+\beta$ are in D, then

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} \cdot E_{\alpha+\beta},$$

where $N_{\alpha,\beta}$ is a nonzero constant. On the other hand, $\mathfrak{b}^c + \mathfrak{m}^+$ is a subalgebra of $\mathfrak{g}^c$. Therefore, if α, $\beta \in D(\mathfrak{b}^c) \cup D(\mathfrak{m}^+)$ and if $\alpha+\beta \in D$, then $\alpha+\beta \in D(\mathfrak{b}^c) \cup D(\mathfrak{m}^+)$. It is easy to see that, for each $\alpha \in D$, at least one of the roots $\pm\alpha$ is in $D(\mathfrak{b}^c) \cup D(\mathfrak{m}^+)$. Hence, by a theorem of Borel [2, Corollary 4.10] and Harish-Chandra [4, Lemma 4], there exists an ordering of D with respect to which all positive roots belong to $D(\mathfrak{b}^c)\cup D(\mathfrak{m}^+)$. Let $\alpha_1, \cdots, \alpha_l$ be the simple roots with respect to this ordering of D. By a change of numbering, we may assume that $\alpha_1, \cdots, \alpha_r$ are the simple roots contained in $D(\mathfrak{b}^c)$. We shall show that each $\alpha\in D(\mathfrak{b}^c)$ is a linear combination of $\alpha_1, \cdots, \alpha_r$. Assuming the contrary, let α be the least positive root in $D(\mathfrak{b}^c)$ which can not be so written. Then α can not be simple. Hence, $\alpha=\beta+\gamma$ where β and γ are positive roots. Since β and γ are positive, they are in $D(\mathfrak{b}^c)\cup D(\mathfrak{m}^+)$. We shall prove that they are actually in $D(\mathfrak{b}^c)$. Assume that at least one of them, say γ, is in $D(\mathfrak{m}^+)$. Then $-\beta=-(\beta+\gamma)+\gamma$ would be in $D(\mathfrak{m}^+)$ because $-(\beta+\gamma)\in D(\mathfrak{b}^c)$ and $[\mathfrak{b}^c, \mathfrak{m}^+]\subset\mathfrak{m}^+$. Hence, $\beta\in D(\mathfrak{m}^-)$ which is a contradiction. As β and γ are less than α, they are linear combinations of $\alpha_1, \cdots, \alpha_r$, thus proving our assertion. Evidently, every linear combination of $\alpha_1, \cdots, \alpha_r$ belongs to $D(\mathfrak{b}^c)$. The second half of the proposition follows from the definition of $\alpha_1, \cdots, \alpha_l$. Q.E.D.

8. **Proof of Theorem** C. (i). The proof is due to Wang (cf. (7.1) of [7]), although we do not assume G to be semi-simple. Let U be the subgroup of G consisting of those elements which commute with every element of the semi-simple part B_s of B. Let T be a maximal torus of U containing the identity component B_a of the center of B. Evidently, both B_a and B_s are contained in $C(T)$. As B is a semi-direct product of B_a and B_s, B is contained in $C(T)$. We shall see that $(C(T))_s=B_s$. (We always denote by $(*)_s$ the semi-simple part of $(*)$.) As B is a C-subgroup, there exists a toral subgroup T' of G such that $(C(T'))_s=B_s$. Obviously, T' is a toral subgroup of U. By a theorem of Hopf, there exists an element u of U such that $uT'u^{-1}\subset T$. Then

$$uC(T')u^{-1} = C(uT'u^{-1}) \supset C(T),$$
$$B_s = uB_su^{-1} = u(C(T'))_su^{-1} = (C(uT'u^{-1}))_s \supset (C(T))_s.$$

On the other hand, $C(T)\supset B_s$ because $T\subset U$. Hence

$$(C(T))_s \supset B_s.$$

(ii) Let $L=C(T)$ and

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{w}, \qquad \mathfrak{l} = \mathfrak{b} + \mathfrak{n}, \qquad \mathfrak{m} = \mathfrak{w} + \mathfrak{n}$$

be the same decomposition as in §7. Then as the semi-simple part of B coincides with that of L, $\mathfrak{n}$ is contained in the center of $\mathfrak{l}$. Hence

$$[\mathfrak{b}, \mathfrak{n}] = 0, \qquad [\mathfrak{n}, \mathfrak{n}] = 0.$$

The proof of (ii) is reduced to that of the following statement:

Let I' be a linear endomorphism of $\mathfrak{w}$ *satisfying the conditions* (16), (17) *and* (18) *of* §7. *Let I'' be a linear endomorphism of* $\mathfrak{n}$ *such that* $I''^2=-1$. *Then the endomorphism* $I=I'+I''$ *of* $\mathfrak{m}=\mathfrak{w}+\mathfrak{n}$ *satisfies the conditions* (7), (8) *and* (9) *of* §7.

While the verification of (7) is trivial, that of (8) will be divided into two cases:

If $X\in\mathfrak{b}$ and $Y\in\mathfrak{w}$, then (8) is a consequence of (17).

If $X\in\mathfrak{b}$ and $Y\in\mathfrak{n}$, then (8) is a consequence of $[\mathfrak{b}, \mathfrak{n}]=0$.

We divide the verification of (9) into four cases: (i) X, $Y\in\mathfrak{n}$; (ii) $X\in\mathfrak{n}$ and $Y\in\mathfrak{w}$; (iii) $X\in\mathfrak{w}$ and $Y\in\mathfrak{n}$; (iv) X, $Y\in\mathfrak{w}$. The case (i) is trivial (note $[\mathfrak{n}, \mathfrak{n}]=0$). In the case (ii), (9) follows from (17); in fact, the left hand side of (9) is the sum of $I\cdot[X, Y]-[X, I\cdot Y]$ and $-(I\cdot[I\cdot X, I\cdot Y]+[I\cdot X, Y])$ both of which vanish because of (17). The case (iii) can be treated in the same way. Finally, we consider the case (iv). By virtue of (18), we have only to show that

$$I\cdot[X, Y]_{\mathfrak{n}} - [I\cdot X, Y]_{\mathfrak{n}} - [X, I\cdot Y]_{\mathfrak{n}} - I\cdot[I\cdot X, I\cdot Y]_{\mathfrak{n}} = 0 \quad \text{for } X,\ Y \in \mathfrak{w}.$$

Our Lemma 7.2, applied to the complex homogeneous manifold G/L, states that $\mathfrak{w}^+$ (resp. $\mathfrak{w}^-$) is spanned by $\{E_\alpha;\ \alpha\in D(\mathfrak{w}^+)\}$ (resp. $\{E_\alpha;\ \alpha\in D(\mathfrak{w}^-)\}$). Now, the above formula can be verified by a straightforward computation making use of Lemma 7.3 and of the fact that if α, $\beta\in D$ and $\alpha+\beta\neq 0$ then $[E_\alpha, E_\beta]=N_{\alpha,\beta}\cdot E_{\alpha+\beta}$.

Thus, the study of invariant complex structures on G/B is reduced to the study of invariant complex structures on $G/C(T)$. The existence of invariant complex structures on $G/T(C)$ can be proved by reversing the process of §7. We shall not give the complete proof here as we can find it in [7].

Bibliography

1. A. Borel, *Kaehlerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 1147–1151.

2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer. J. Math. (1958) pp.

3. M. Goto, *On algebraic homogeneous spaces*, Amer. J. Math. vol. 76 (1954) pp. 811–818.

4. Harish-Chandra, *Representations of semi-simple Lie groups* IV, Amer. J. Math. vol. 77 (1955) pp. 743–777.

5. A. Froelicher, *Zur Differentialgeometrie der komplexen Structuren*, Math. Ann. vol. 129 (1955) pp. 50–95.

6. J-L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canad. J. Math. vol. 7 (1955) pp. 562–576.

7. H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math. vol. 76 (1954) pp. 1–32.

8. ———, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 771–776.

9. H. Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen* II, III, Math. Z. vol. 24 (1925) pp. 328–395.

Nagoya University,
Nagoya, Japan
University of Washington,
Seattle, Washington
Massachusetts Institute of Technology,
Cambridge, Massachusetts