A noncommutative Jordan algebra $A$ over a field $F$ is a nodal algebra [9] in case every element of $A$ may be written in the form $\alpha 1 + z$ where $\alpha$ is in $F$, $1$ is the unity element of $A$, and $z$ is nilpotent, while the set $N$ of nilpotent elements is not a subalgebra of $A$. $F$ is necessarily of characteristic $p > 0$.

L. A. Kokoris gave in [5] the first examples of simple nodal noncommutative Jordan algebras. Generalizing these, he has constructed in [7] a class $K$ of nodal noncommutative Jordan algebras $A$ of dimension $p^n$, and has proved [6; 7] that each simple nodal noncommutative Jordan algebra of characteristic $p \neq 2$ is in $K$. Although not all of the algebras in $K$ are simple, it turns out that those algebras with which we are principally concerned here are indeed simple.

Any algebra $A$ of dimension $p^n$ in $K$ may be represented as follows: let $B_n = F[x_1, \ldots, x_n]$, $x_i^p = 0$, be a truncated polynomial ring with partial differentiation operators $\partial / \partial x_i$. Write $f \cdot g$ for the commutative associative product in $B_n$. Then $A$ is the same vector space as $B_n$, but multiplication in $A$ is defined by

\begin{equation}
fg = f \cdot g + \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij}, \quad c_{ij} = -c_{ji},
\end{equation}

where the $c_{ij}$ ($= -c_{ji}$) are elements in $B_n$ which are arbitrary except for the proviso that at least one of them has an inverse (equivalently, at least one of them is not in the radical $N$ of $B_n$). Then necessarily $n \geq 2$. Also we note that

\begin{equation}
c_{ij} = \frac{1}{2} [x_i, x_j], \quad i, j = 1, 2, \ldots, n.
\end{equation}

This representation of an algebra $A$ in $K$ is not unique. It is well-known [4, p. 108; or 8, §4.5, Proposition 6] that any representatives in $B_n$ of the elements of any basis of the $n$-dimensional space $N/N \cdot N$ will serve for the
By the chain rule \( \frac{df}{dx_i} = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial y_k} \cdot \left( \frac{\partial y_k}{\partial x_i} \right) \right) \), one obtains

\[
fg = f \cdot g + \sum_{k,l=1}^{n} \frac{\partial f}{\partial y_k} \cdot \frac{\partial g}{\partial y_l} \cdot d_{kl}
\]

with

\[
d_{kl} = \frac{1}{2} \left[ y_k, y_l \right] = \sum_{i,j=1}^{n} \frac{\partial y_k}{\partial x_i} \cdot \frac{\partial y_l}{\partial x_j} \cdot c_{ij},
\]

where \( y_1, \ldots, y_n \) are any elements of \( B_n \) whose residue classes form a basis for \( N/N \cdot N \) over \( F \).

In this paper we study the derivations of the algebras in \( K \). We are led to relationships between some of these algebras and recently discovered simple Lie algebras of characteristic \( p \). In particular, we obtain an intrinsic characterization of the simple Lie algebras \( V_r \) of A. A. Albert and M. S. Frank [1]. Also we display each of the simple Lie algebras \( L(G, \delta, f) \) of characteristic \( \neq 2 \) defined by Richard Block [2] as an ideal in the derivation algebra of a suitably chosen simple nodal noncommutative Jordan algebra \( A \). We assume characteristic \( p > 2 \) throughout.

1. Derivations. Since \( c_{ij} = -c_{ji} \) in (1), we have \( (fg+gf)/2 = f \cdot g \). That is, the commutative algebra \( A^+ \) attached to \( A \) is \( B_n \) itself.

Clearly any derivation of \( A \) is also a derivation of \( A^+ \). But the derivations of \( B_n \) are well-known [4, p. 107]:

\[
f \mapsto \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot a_k
\]

for arbitrary elements \( a_k \) in \( B_n \). Hence, if \( D \) is a derivation of \( A \), we have

\[
fD = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot a_k
\]

for all \( f \) in \( A \) and for certain elements \( a_k (=x_kD) \) in \( A \) \((k=1, \ldots, n)\). Occasionally we shall employ the notation \( D = (a_1, \ldots, a_n) \) used in [3] and [1].

Suppose that \( D \) is given by (4). Then \( D \) is a derivation of \( A^+ \), and also

\[
\frac{\partial h}{\partial x_i} D - \frac{\partial (hD)}{\partial x_i} = \sum_{k} \left( \frac{\partial^2 h}{\partial x_k \partial x_i} \cdot a_k - \frac{\partial^2 h}{\partial x_i \partial x_k} \cdot a_k - \frac{\partial h}{\partial x_k} \cdot \frac{\partial a_k}{\partial x_i} \right)
\]

for any \( h \) in \( A \). Hence
\[(fg) D - (fD)g - f(gD) = (f \cdot g) D - (fD) \cdot g - f \cdot (gD)\]

\[= \sum_{i,j} \left\{ \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} \right) D - \frac{\partial (fD)}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} + \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} D \right\} \]

\[= \sum_{i,j} \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial a_k}{\partial x_j} \cdot c_{ij} - \frac{\partial f}{\partial x_i} \cdot \frac{\partial a_k}{\partial x_j} \cdot c_{ij} \right) \]

\[= \sum_{i,j,k} \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial a_k}{\partial x_j} \cdot c_{ij} - \frac{\partial a_i}{\partial x_k} \cdot c_{ij} + \frac{\partial a_j}{\partial x_k} \cdot c_{ij} \right) \]

\[\text{Hence } D \text{ is a derivation of } A \text{ if and only if}\]

\[(5) \quad \sum_{k=1}^{n} \left( \frac{\partial^2 c_{ij}}{\partial x_k^2} \cdot a_k + \frac{\partial a_i}{\partial x_k} \cdot c_{jk} + \frac{\partial a_j}{\partial x_k} \cdot c_{ki} \right) = 0\]

for \(1 \leq i, j \leq n\). The equations (5) are redundant for \(j \geq i\), so we have proved

**Theorem 1.** Let \(A\) be a nodal noncommutative Jordan algebra in \(K\), multiplication being defined by (1). Then a mapping \(D\) on \(A\) is a derivation of \(A\) if and only if \(D\) has the form (4) for elements \(a_1, \ldots, a_n\) in \(A\) satisfying (5) for \(1 \leq i < j \leq n\).

Solution of the equations (5) in general seems a formidable task. Even a preliminary simplification of the problem by using (3) to normalize the \(c_{ij}\) seems exceedingly complex in the general situation. In the next two sections we treat two special cases: (i) \(n\) arbitrary, but all of the \(c_{ij}\) in \(F1\); (ii) \(n = 2\), but \(c_{ij}\) arbitrary in \(A\).

2. Algebras defined by skew-symmetric bilinear forms. Suppose that all of the \(c_{ij}\) are in \(F1\). That is, \(c_{ij} = \phi_{ij}1, \phi_{ij} = -\phi_{ji}\) in \(F\), not all \(\phi_{ij}\) zero, where \(x_i x_j = x_i x_j + \phi_{ij}1\). There is a unique skew-symmetric bilinear form \(\phi\) defined on the \(n\)-dimensional space \(M = \sum_{i=1}^{n} Fx_i\) (equivalently, defined on \(N/N-N\)) such that \(\phi(x_i, x_j) = \phi_{ij}\). Then

\[fg = f \cdot g + \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \phi(x_i, x_j)1.\]

If \(x_i (i = 1, \ldots, n)\) exist for which \(c_{ij} = \phi(x_i, x_j)1\), we shall say that \(A\) is defined by the skew-symmetric bilinear form \(\phi\). (Since \(N/N-N \subset N\) by [9, Theorem 5] we know that, if \(A\) is defined by different skew-symmetric bilinear forms, the forms are equivalent.) Let \(2r (2 \leq 2r \leq n)\) be the rank of \(\phi\).
A change of basis in $M$ gives $\phi(x_i, x_{r+i}) = 1 = -\phi(x_{r+i}, x_i)$ for $i = 1, \ldots, r$; $\phi(x_i, x_j) = 0$ otherwise. The form $\phi$ is nondegenerate if and only if $n = 2r$.

We remark that an algebra $A$ defined by a skew-symmetric bilinear form $\phi$ is simple if and only if $\phi$ is nondegenerate. For take the special basis above in $M$. If $n > 2r$, then $c_i,2r+1 = \phi_i,2r+1 = 0$ for $i = 1, \ldots, n$, so that $f x_{2r+1} = f \cdot x_{2r+1} = x_{2r+1} f$ for every $f$ in $A$ by (1). It follows that $C = x_{2r+1} \cdot A$ is an ideal of $A$, $C \neq 0$, $C \neq A$. If $n = 2r$, we shall see in §4 that $A$ is a simple nodal noncommutative Jordan algebra associated in a specific way with one of the algebras in a general class of simple Lie algebras of characteristic $p$. Of course the simplicity of $A$ may also be established directly in this particular case.

If we take the special basis above for $M$ in an algebra $A$ defined by a skew-symmetric form, equations (5) become

\begin{align*}
(6) \quad \frac{\partial a_i}{\partial x_{r+j}} &= \frac{\partial a_j}{\partial x_{r+i}}, \quad i, j = 1, \ldots, r;
(7) \quad \frac{\partial a_{i+j}}{\partial x_i} &= \frac{\partial a_{i+j}}{\partial x_j}, \quad i, j = 1, \ldots, r;
(8) \quad \frac{\partial a_i}{\partial x_j} + \frac{\partial a_{i+j}}{\partial x_{r+i}} = 0, \quad i, j = 1, \ldots, r;
\end{align*}

and

\begin{equation}
(9) \quad \frac{\partial a_j}{\partial x_i} = 0, \quad 1 \leq i \leq 2r; \quad 2r + 1 \leq j \leq n.
\end{equation}

Equations (6), (7), (8) involve only $a_1, \ldots, a_{2r}$. Hence the elements $a_{2r+k}$ ($k = 1, \ldots, n-2r$) are arbitrary in $F[x_{2r+1}, \ldots, x_n]$ by (9).

**Lemma 1.** Let $f$ be in $B_n = F[x_1, \ldots, x_n]$. For any $i$ ($1 \leq i \leq n$), write $R_i = F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ so that $B_n = R_i[x_i]$. Then there exists $g$ in $B_n$ satisfying

\begin{equation}
(10) \quad \frac{\partial g}{\partial x_i} = f
\end{equation}

if and only if $f$ is of the form

\begin{equation}
(11) \quad f = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_{p-2} x_i^{p-2}, \quad \beta_j \in R_i.
\end{equation}

If $f$ satisfies (11), there is always a solution $g$ of (10) having the form

\begin{equation}
(12) \quad g = x_i \cdot h, \quad h \in B_n.
\end{equation}

If $g$ is one solution of (10), then $g^*$ is a solution if and only if

\begin{equation}
(13) \quad g^* = g + h, \quad h \in R_i.
\end{equation}
The proof is straightforward.

Using the lemma, we give an inductive proof that the solutions of (6)-(9) are

\[ a_i = \frac{\partial g}{\partial x_{r+i}} + \sigma_i \cdot x_{r+i}^{p-1} \quad \text{for } i = 1, \ldots, r, \]

\[ a_{r+i} = -\frac{\partial g}{\partial x_i} + \sigma_{r+i} \cdot x_i^{p-1} \quad \text{for } i = 1, \ldots, r, \]

and

\[ a_{2r+k} = \sigma_{2r+k} \quad \text{for } k = 1, \ldots, n - 2r, \]

for arbitrary \( g \) in \( A \) and arbitrary \( \sigma_i, \sigma_{r+i}, \sigma_{2r+k} \) in \( F[x_{2r+1}, \ldots, x_n] \). Clearly any elements \( a_1, \ldots, a_n \) given by (14) and (15) satisfy equations (6)-(9). We are concerned only with establishing the fact that, if \( a_1, \ldots, a_{2r} \) satisfy (6)-(8), then \( a_1, \ldots, a_{2r} \) have the form given in (14).

Let \( a_1 = \beta_0 + \beta_1 \cdot x_{r+1} + \cdots + \beta_{p-2} \cdot x_{r+1}^{p-2} + \beta_{p-1} \cdot x_{r+1}^{p-1} \), \( \beta_j \in R_{r+1} \), so that, for any \( k \neq r+1 \),

\[ \frac{\partial a_1}{\partial x_k} = \frac{\partial \beta_0}{\partial x_k} + \cdots + \frac{\partial \beta_{p-1}}{\partial x_k} \cdot x_{r+1}^{p-1}. \]

Then, putting \( i = 1 \) in (6) and (8), we have

\[ \frac{\partial \beta_{p-1}}{\partial x_k} = 0 \quad \text{for } k = 1, \ldots, r, r + 2, \ldots, 2r \]

by (11). Also \( \partial \beta_{p-1}/\partial x_{r+1} = 0 \). By the lemma there exist \( g \in B_n \) and \( \sigma_1 \in F[x_{2r+1}, \ldots, x_n] \) such that \( a_i = \partial g/\partial x_{r+1} + \sigma_1 \cdot x_{r+1}^{p-1} \). For any \( t \) (\( 2 \leq t \leq 2r \)) we assume that (14) holds for \( i = 1, \ldots, t-1 \). There are two cases to be considered: \( t \leq r \), and \( t \geq r+1 \). If \( t \leq r \), we replace \( i \) in (6) by \( t \), and obtain

\[ \frac{\partial a_t}{\partial x_{r+j}} = \frac{\partial^2 g}{\partial x_{r+t} \partial x_{r+j}} \quad \text{for } j = 1, \ldots, t-1, \]

and

\[ \frac{\partial a_t}{\partial x_{r+j}} = \frac{\partial a_j}{\partial x_{r+t}} \quad \text{for } j = t+1, \ldots, r. \]

It follows that \( a_t = \partial g/\partial x_{r+t} + f \), where

\[ f = \frac{\partial h}{\partial x_{r+t}} + \sigma_t \cdot x_{r+t}^{p-1} \]

for some \( h \) satisfying \( \partial h/\partial x_{r+j} = 0 \) for \( j = 1, \ldots, t-1, \) and some \( \sigma_t \) satisfying...
for \( k = r + 1, \cdots, 2r \). Write \( g^* = g + h \). Then \( \partial g^*/\partial x_{r+i} = \partial g/\partial x_{r+i} \) for \( i = 1, \cdots, t-1 \), so that the inductive hypothesis remains valid with \( g^* \) replacing \( g \). Also \( a_i = \partial g^*/\partial x_{r+i} + \sigma_i \cdot x_{r+i}^{p-1} \) for \( \sigma_i \) satisfying (16) for \( k = r+1, \cdots, 2r \). Putting \( i = t \) in (8), we have

\[
\frac{\partial \sigma_i}{\partial x_j} = -\frac{\partial}{\partial x_{r+i}} \left( a_{r+j} + \frac{\partial g^*}{\partial x_j} \right),
\]

so (16) holds also for \( k = 1, \cdots, r \). That is, \( \sigma_i \in F[x_{2r+1}, \cdots, x_n] \), as desired. In the second case \( (t \geq r+1) \), we write \( t = r+s \). Then the assumption of the induction is that the first line of (14) holds for \( i = 1, \cdots, r \), and the second line for \( i = 1, \cdots, s-1 \). Putting \( j = s \) in (7), we have

\[
\frac{df}{dx_k} = 0
\]

for \( k = r+1, \cdots, 2r \). Also \( j = s \) in (7) yields (17) for \( k = 1, \cdots, s-1 \), and

\[
\frac{df}{dx_i} = \frac{\partial}{\partial x_i} \left( a_{r+i} + \frac{\partial g}{\partial x_i} \right)
\]

for \( i = s+1, \cdots, r \). Hence \( f = -\partial h/\partial x_s + \sigma_{r+s} \cdot x_s^{p-1} \) for some \( h \) satisfying \( \partial h/\partial x_k = 0 \) for \( k = 1, \cdots, s-1, r+1, \cdots, 2r \), and some \( \sigma_{r+s} \) satisfying

(18) \[
\frac{\partial \sigma_{r+s}}{\partial x_k} = 0 \quad \text{for } k = 1, \cdots, 2r.
\]

Write \( g^* = g + h \). Then \( \partial g^*/\partial x_k = \partial g/\partial x_k \) for \( k = 1, \cdots, s-1, r+1, \cdots, 2r \), so that the inductive hypothesis remains valid with \( g^* \) replacing \( g \). Also \( a_{r+s} = -\partial g^*/\partial x_s + \sigma_{r+s} \cdot x_s^{p-1} \) with \( \sigma_{r+s} \in F[x_{2r+1}, \cdots, x_n] \) by (18). We have established equations (14).

Now any \( D \) given by (4), (14), and (15) determines \( g \) modulo \( F[x_{2r+1}, \cdots, x_n] \) and the \( \sigma_i \) \((j = 1, \cdots, n)\) uniquely. Hence we have proved

**Theorem 2.** Let \( A \) be a nodal noncommutative Jordan algebra of dimension \( p^n \) which is defined by a skew-symmetric bilinear form \( \phi \) of rank \( 2r \). Then the derivation algebra \( D(A) \) of \( A \) has dimension

\[
p^n - p^{n-2r} + np^{n-2r} = p^{n-2r}(p^{2r} - 1 + n).
\]

The \( x_i \) in \( A \) may be chosen so that the derivations \( D \) of \( A \) have the form (4) with \( a_k \) as in (14) and (15).

Let \( D \) be given by (4) and \( E \) by
\[
(19) \quad fE = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot b_k
\]
for \( b_k \) in \( A \). Then it is well-known [4, p. 107] that \([D, E] = C = (c_1, \ldots, c_n)\) where
\[
(20) \quad c_i = \sum_{j=1}^{n} \left( \frac{\partial a_i}{\partial x_j} \cdot b_j - \frac{\partial b_i}{\partial x_j} \cdot a_j \right), \quad i = 1, \ldots, n.
\]
It is easily seen that, in case \( n > 2r \) above, there are many ideals in \( D(A) \).
If \( n = 2r \), however, (that is, if \( A \) is defined by a nondegenerate \( \phi \)) we have (14) with \( \sigma_k \in F1 \) (= F) for \( k = 1, \ldots, 2r \). Writing \( D(g; \sigma_1, \ldots, \sigma_{2r}) \) for \( D \) in (14), we obtain
\[
(21) \quad [D(f; \rho_1, \ldots, \rho_{2r}), D(g; \sigma_1, \ldots, \sigma_{2r})] = D(h; 0, \ldots, 0)
\]
from (20), where
\[
(22) \quad h = \sum_{j=1}^{r} \left\{ (\rho_j \sigma_{r+j} - \rho_{r+j} \sigma_j) x_j^{p-1} \cdot x_{r+j}^{p-1} + \left( \frac{\partial f}{\partial x_j} \cdot \frac{\partial g}{\partial x_{r+j}} - \frac{\partial f}{\partial x_{r+j}} \cdot \frac{\partial g}{\partial x_j} \right) \right\}.
\]
We recognize \( D(g) = D(g; 0, 0, \ldots, 0) \) as any element of the algebra \( V_{or} \) of Albert and Frank [1, p. 127]. Let \( \bar{A} \) be the subspace of \( B_n \) consisting of all elements of \( B_n \) for which the coefficient of \( x^{p-1} \cdot x_2^{p-1} \cdot \ldots \cdot x_n^{p-1} \) is zero. The \((p^{2r} - 2)\)-dimensional simple Lie algebra \( V_r \) of Albert and Frank consists of all \( D(g) \) with \( g \in \bar{A} \). It is known [1, pp. 127, 128] that \( V_r \) is the derived algebra \( V'_r \) of \( V_{or} \) and that \( \partial f/\partial x_j \cdot \partial g/\partial x_{r+j} - \partial f/\partial x_{r+j} \cdot \partial g/\partial x_j \) is in \( \bar{A} \). It follows from (11) that \( \partial f/\partial x_j, x_j^{p-1} \subseteq \bar{A} \), etc. Hence only the first term within the braces in (22) could fail to be in \( \bar{A} \). If \( r > 1 \), then \( h \) in (22) is in \( \bar{A} \), and \( D(h) \in V_r \). Hence \( V_r = V'_r \subseteq D(A)' \subseteq V_r \) if \( r > 1 \). If \( r = 1 \), there exists \( h \) in (22) which is not in \( \bar{A} \), so that \( D(A)' \subseteq V_{01} \) but \( D(A)' \neq V_1 \). However, \( V_1 = V'_1 \subseteq D(A)'' \subseteq V''_{01} = V_1 \). Of course \( D(A)'' \subseteq V_r \) for \( r > 1 \) also.

Thus we obtain the following intrinsic characterization of the \((p^{2r} - 2)\)-dimensional simple Lie algebras \( V_r \) of Albert and Frank:

**Theorem 3.** A Lie algebra \( L \) is a simple algebra \( V_r \) if and only if there is a \( p^{2r} \)-dimensional (simple) nodal noncommutative Jordan algebra \( A \) defined by a nondegenerate skew-symmetric bilinear form such that \( L \cong D(A)' \) in case \( r > 1 \), and \( L \cong D(A)'' \) in case \( r = 1 \) (actually \( L \cong D(A)'' \) in both cases).

A second characterization of the algebras \( V_r \) results from the following observation concerning Lie-admissible algebras.

In any nonassociative algebra \( A \), \( xy = x \cdot y + [x, y]/2 \). Therefore a linear transformation \( D \) on \( A \) is a derivation if and only if \( D \) is a derivation of both \( A^+ \) and \( A^- \), where \( A^- \) is the anticommutative algebra attached to \( A \) having the product \( [x, y] \). That is,
(23) \[ D(A) = D(A^+) \cap D(A^-). \]

Now in any nodal noncommutative Jordan algebra \( A \) in \( K \)

(24) \[ [f, g] = \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot 2c_{ij}, \]

and \( D \) in (4) is a derivation of \( A \) if and only if \( D \) is a derivation of \( A^- \).

If \( A^- \) is a Lie algebra, then the mappings \( \text{ad} g/2 \) defined by

(25) \[ f \to \frac{1}{2} [f, g] = \sum_i \frac{\partial f}{\partial x_i} \cdot \left( \sum_j \frac{\partial g}{\partial x_j} \cdot c_{ij} \right) \]

are inner derivations of \( A^- \), and therefore derivations of \( A \) since they are of the form (4) with

(26) \[ a_i = \sum_{j=1}^n \frac{\partial g}{\partial x_j} \cdot c_{ij}, \quad i = 1, \ldots, n. \]

The set \( \text{ad} A \) of all inner derivations (25) of \( A^- \), being an ideal of \( D(A^-) \), is an ideal of \( D(A) \) by (23).

We use the Jacobi identity

\[ 0 = [[f, g], h] + [[g, h], f] + [[h, f], g] \]

\[ = 2 \sum_{i,j} \left\{ \left[ \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij}, h \right] + \left[ \frac{\partial g}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} \cdot c_{ij}, f \right] + \left[ \frac{\partial h}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} \cdot c_{ij}, g \right] \right\} \]

\[ = 4 \sum_{i,j,k,t} \left\{ \frac{\partial}{\partial x_t} \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} \right) \cdot \frac{\partial h}{\partial x_k} \cdot c_{tk} + \frac{\partial}{\partial x_t} \left( \frac{\partial g}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} \cdot c_{ij} \right) \cdot \frac{\partial f}{\partial x_k} \cdot c_{tk} \right\} \]

\[ = 4 \sum_{i,j,k,t} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \frac{\partial g}{\partial x_t} \cdot \frac{\partial h}{\partial x_k} \cdot c_{ij} \cdot c_{tk} + \frac{\partial f}{\partial x_i} \cdot \frac{\partial^2 g}{\partial x_j \partial x_t} \cdot \frac{\partial h}{\partial x_k} \cdot c_{ij} \cdot c_{tk} + \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial^2 h}{\partial x_t \partial x_k} \cdot c_{ij} \cdot c_{tk} \right. \]

\[ + \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_t} \cdot c_{ij} \cdot c_{tk} + \frac{\partial^2 h}{\partial x_j \partial x_t} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_k} \cdot c_{ij} \cdot c_{tk} + \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_t} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} \cdot c_{tk} \]

\[ + \frac{\partial h}{\partial x_k} \cdot \frac{\partial g}{\partial x_t} \cdot \frac{\partial f}{\partial x_j} \cdot c_{ij} \cdot c_{tk} \bigg\} \]

\[ = 4 \sum_{i,j,k} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot \sum_i \left( \frac{\partial c_{ij}}{\partial x_i} \cdot c_{uk} + \frac{\partial c_{jk}}{\partial x_j} \cdot c_{ti} + \frac{\partial c_{ki}}{\partial x_k} \cdot c_{ij} \right) \]
to see that $A^-$ is a Lie algebra if and only if
\[ \sum_{i=1}^{n} \left( \frac{\partial c_{ij}}{\partial x_i} \cdot c_{ik} + \frac{\partial c_{jk}}{\partial x_i} \cdot c_{it} + \frac{\partial c_{ki}}{\partial x_i} \cdot c_{ij} \right) = 0 \]

for $i, j, k = 1, \ldots, n$. The equations (27) are redundant for $i \geq j$ and for $j \geq k$. Hence $A^-$ is a Lie algebra if and only if (27) holds for $1 \leq i < j < k \leq n$. (It follows that $A^-$ is a Lie algebra in case $n = 2$.)

The equations (27) are obviously satisfied in any algebra $A$ defined by a skew-symmetric bilinear form $\phi$. Using the basis employed before in $M = \sum_i Fx_i$, we have $ad g/2$ in the form (4) with $a_k$ given by (14) and (15) with $\sigma_1 = \cdots = \sigma_n = 0$. Therefore $ad A$ is an ideal of dimension $p^{n-2r}(p^{2r}-1)$ in $D(A)$ which is of dimension $p^{n-2r}(p^{2r}-1+n)$. If $\phi$ is nondegenerate, so that $n = 2r$, then $ad A$ is the (nonsimple) algebra $V_r$, of all $D(g)$ for $g \in A$.

**Theorem 4.** A Lie algebra $L$ is a simple Lie algebra $V_r$ if and only if there is a $p^2$-dimensional (simple) nodal noncommutative Jordan algebra $A$ defined by a nondegenerate skew-symmetric bilinear form such that $L = (ad A)'$.

3. The case $n = 2$. Let $A$ in $K$ have least possible dimension $p^2$. Then it is known [5, §3] that $A$ is simple. The vector space of $A$ coincides with $B_2 = F[x_1, x_2]$, $x_1^2 = x_2^2 = 0$, and multiplication in $A$ is defined by

\[ fg = f \cdot g + \left( \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1} \right) \cdot c \]

where $c$ has an inverse $c^{-1}$ in $B_2$. Also (5) reduces to the single equation

\[ \frac{\partial c}{\partial x_1} \cdot a_1 - \frac{\partial a_1}{\partial x_1} \cdot c + \frac{\partial c}{\partial x_2} \cdot a_2 - \frac{\partial a_2}{\partial x_2} \cdot c = 0, \]

which is equivalent to

\[ \frac{\partial}{\partial x_1} (c^{-1} \cdot a_1) + \frac{\partial}{\partial x_2} (c^{-1} \cdot a_2) = 0. \]

If, given $D$ in (4), we write $b \cdot D$ for the derivation

\[ f \rightarrow \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot (b \cdot a_k) \]

of $B_n$ (that is, $b \cdot D = (b \cdot a_1, \cdots, b \cdot a_n)$), and if we write $\delta(D)$ for the divergence [3, p. 715]

\[ \delta(D) = \sum_{k=1}^{n} \frac{\partial a_k}{\partial x_k}, \]

then condition (29) becomes
(30) \[ \delta(c^{-1} \cdot D) = 0. \]

Hence \( D \) is a derivation of \( A \) if and only if \( c^{-1} \cdot D \) is in the \((p^2 + 1)\)-dimensional Lie algebra \( M_2 \) of derivations of \( B_2 \) having divergence zero [3]. But \( D \leftrightarrow c^{-1} \cdot D \) is a vector space isomorphism, so the derivation algebra \( D(A) \) is a \((p^2 + 1)\)-dimensional algebra \( D(A) = c \cdot M_2 \). If \( c = 1 \), so that \( A \) is defined by a (non-degenerate) form \( \phi \), then \( D(A) = M_2 \) consists of the derivations (4) of \( B_2 \) given by (14) with \( r = 1 \), while \( D(A)' = V_{a_1} \) and \( D(A)'' = V_1 \) is simple. More generally two distinct situations arise, depending upon whether \( c^{-1} \) is in \( \bar{A} \) or not.

We digress momentarily to point out that (30) defines Lie algebras which generalize the algebras \( M_n \) of Frank [3], not only for \( n = 2 \), but for general \( n \). Let \( c \) be any invertible element of \( B_n \), and let \( D \) range over the derivations of \( B_n \) satisfying (30). The set \( c \cdot M_n \) of these derivations of \( B_n \) is a Lie algebra of dimension \((n - 1)p^n + 1\) since \( g \) in \( B_n \) implies

(31) \[ \delta(g \cdot [D, E]) = \delta(\delta(g \cdot D) \cdot E) - \delta(\delta(g \cdot E) \cdot D), \]

generalizing [3, Lemma 2]. For (20) gives

\[
\begin{align*}
\delta(\delta(g \cdot D) \cdot E) &- \delta(\delta(g \cdot E) \cdot D) \\
= &\sum_j \frac{\partial}{\partial x_j} \left\{ \sum_i \left( \frac{\partial}{\partial x_i} (g \cdot a_i) \cdot b_j - \frac{\partial}{\partial x_i} (g \cdot b_i) \cdot a_j \right) \right\} \\
= &\sum_{i,j} \left( \frac{\partial g}{\partial x_i} \cdot \frac{\partial a_i}{\partial x_j} \cdot b_j + g \cdot \frac{\partial^2 a_i}{\partial x_j \partial x_i} \cdot b_j - \frac{\partial g}{\partial x_i} \cdot \frac{\partial b_i}{\partial x_j} \cdot a_j - g \cdot \frac{\partial^2 b_i}{\partial x_j \partial x_i} \cdot a_j \right) \\
= &\sum_i \frac{\partial g}{\partial x_i} \cdot c_i + g \cdot \sum_i \frac{\partial c_i}{\partial x_i} \\
= &\delta(g \cdot [D, E]).
\end{align*}
\]

Putting \( g = c^{-1} \) in (31), we see that \([D, E] \) satisfies (30) in case \( D \) and \( E \) do, or \( c \cdot M_n \) is a Lie algebra. Also

(32) \[ (c \cdot M_n)' \subseteq c \cdot M_n'. \]

For suppose that \( D \) and \( E \) satisfy (30). Writing \( g = c^{-1} \), we have

\[
\sum_{j=1}^n \frac{\partial}{\partial x_j} (g \cdot a_i \cdot b_j - g \cdot b_i \cdot a_j)
= g \cdot \sum_{j=1}^n \left( \frac{\partial a_i}{\partial x_j} \cdot b_j - \frac{\partial b_i}{\partial x_j} \cdot a_j \right) + a_i \cdot \delta(g \cdot E) - b_i \cdot \delta(g \cdot D)
= g \cdot c_i
\]

for \( c_i \) in (20). Hence \( g \cdot [D, E] = (h_1, \cdots, h_n) \) where
\[ h_i = \sum_{j \neq i} \frac{\partial}{\partial x_j} (g \cdot a_i \cdot b_j - g \cdot b_i \cdot a_j). \]

Hence \( g \cdot [D, E] \) is in the algebra \( T_n = S_n \) defined by Frank [3, Lemma 3]. But \( S_n = M'_n \) (because \( S_n \) is simple in case \( n > 2 \) and by our earlier remarks in case \( n = 2 \)). Hence \( [D, E] \subseteq c \cdot M'_n \), establishing (32). It follows that \( c \cdot M'_n \) is a Lie algebra (of dimension \((n-1)(p^n-1)\), the known dimension of \( S_n \)), for \((c \cdot M'_n) \subseteq (c \cdot M'_n) \subseteq c \cdot M'_n \) by (32).

Returning to the case \( n = 2 \), we shall prove that \( D(A)' = (ad A)' \cong V_1 \) is a simple Lie algebra of dimension \( p^2 - 2 \) if \( c^{-1} \in A \), whereas \( D(A)' = ad A \) is a simple Lie algebra of dimension \( p^2 - 1 \) if \( c^{-1} \notin A \). We begin with a normalization of \( c \) by proper choice of \( x_i \) in \( A \).

**Theorem 5.** Let \( A \) be a (simple) nodal noncommutative Jordan algebra of dimension \( p^2 \) in \( K \) so that multiplication in \( A \) is defined by (28). Then \( x_i \) may be chosen in \( A \) so that \( c \) is in the form

\[ c = 1 + \alpha x_1 \cdot x_2, \quad \alpha \in F. \]

According as \( c^{-1} \) is or is not in \( A \) (for any choice of \( x_i \)), we have \( \alpha = 0 \) or \( \alpha \neq 0 \) in (33).

**Proof.** Write \( c^{-1} = \beta_0 + \beta_1 \cdot x_1 + \cdots + \beta_{p-2} \cdot x_1^{p-2} + \beta_{p-1} \cdot x_1^{p-1}, \) \( \beta_i \in F[x_1]. \)

Then \( \beta_0^{-1} \) exists, and \( c^{-1} \cdot (1 - \beta_0^{-1} \cdot \beta_{p-1} \cdot x_1^{p-1}) = \beta_0 + \beta_1 \cdot x_1 + \cdots + \beta_{p-2} \cdot x_1^{p-2} \). But then (11) and (12) imply that there exists \( y_2 = x_2 \cdot h \) such that

\[ \frac{\partial y_2}{\partial x_2} = c^{-1} \cdot (1 - \beta_0^{-1} \cdot \beta_{p-1} \cdot x_1^{p-1}). \]

Now \( \partial y_2 / \partial x_2 = x_2 \cdot \partial h / \partial x_2 + h \) implies that \( h \) and \( \beta_0 \) are congruent modulo \( N \). Hence \( h^{-1} \) exists, so that \( y_2 = \delta x_2 + n, \) \( n \in N \cdot N, \) \( \delta \neq 0. \) Let \( y_1 = x_1. \) Then \( A = F[y_1, y_2], \) \( y_1^2 = y_2 = 0. \) Now \( y_2^{p-1} = x_2^{p-1} \cdot h^{p-1}, \) so that \( x_2^{p-1} \in A \cdot y_2^{p-1}. \) But every element of \( A \cdot y_2^{p-1} \) has the form \( \rho \cdot y_2^{p-1} \) for \( \rho \in F[y_1]. \) Then (3) and (34) imply \( [y_1, y_2] / 2 = (\partial y_2 / \partial x_2) \cdot c = 1 - \beta_0^{-1} \cdot \beta_{p-1} \cdot x_1^{p-1} = 1 + \sigma \cdot y_2^{p-1} \) where

\[ \sigma = - \beta_0^{-1} \cdot \beta_{p-1} \cdot \rho \in F[y_1]. \]

That is, we may as well take \( c \) in (28) in the form

\[ c = 1 + \sigma \cdot x_2^{p-1}, \quad \sigma \in F[x_1]. \]

Now \( \sigma = \alpha_0 x_1 + \alpha_1 x_1 + \cdots + \alpha_{p-2} x_1^{p-2} + \alpha_{p-1} x_1^{p-1} \) for \( \alpha_j \in F. \) Then

\[ c^{-1} \cdot (1 + \alpha_{p-1} x_1^{p-1} \cdot x_2^{p-1}) = (1 - \sigma \cdot x_2^{p-1}) \cdot (1 + \alpha_{p-1} x_1^{p-1} \cdot x_2^{p-1}) \]

\[ = 1 - \alpha_0 x_2^{p-1} - \alpha_1 x_1 \cdot x_2^{p-1} - \cdots - \alpha_{p-2} x_1^{p-2} \cdot x_2^{p-1}. \]

By (11) and (12) there exists
\[ y_1 = x_1 \cdot (1 + \pi \cdot x_2^{p-1}), \quad \pi \in F[x_1], \]
such that
\[
\frac{\partial y_1}{\partial x_1} = c^{-1} \cdot (1 + \alpha x_1^{p-1} \cdot x_2^{p-1})
\]
where we have written \( \alpha \) for \( \alpha_{p-1} \in F \). Then \( y_1^{p-1} = x_1^{p-1} \cdot (1 - \pi \cdot x_2^{p-1}) \), or \( x_1^{p-1} = y_1^{p-1} \cdot (1 + \pi \cdot x_2^{p-1}) \). Let \( y_2 = x_2 \). Then \( A = F[y_1, y_2] \), while (3) and (35) imply that \([y_1, y_2]/2 = (\partial y_1/\partial x_1) \cdot c = 1 + \alpha x_1^{p-1} \cdot x_2^{p-1} = 1 + \alpha y_1^{p-1} \cdot (1 + \pi \cdot y_2^{p-1}) \cdot y_2^{p-1} = 1 + \alpha y_1^{p-1} \cdot y_2^{p-1} \). That is, we may take \( c \) in the form (33).

The final statement in the theorem could probably be established by a careful analysis of the argument above. Instead we note that, for any choice of \( x_i \), \( V_{01} \) consists of all \( D(g) \) with \( g \in A \), and \( V_1 = V_{01} \) of all \( D(g) \) for which \( g \in \tilde{A} \), while \([D(f), D(g)] = D(h) \) where
\[
D = \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1}.
\]
Since 1 is in \( \tilde{A} \), as well as being of the form (36), it follows that \( \tilde{A} \) consists of all linear combinations of elements of the form (36). By (24) we have \([f, g]/2 = h \cdot c \) for \( h \) in (36), or \([A, A] = \tilde{A} \cdot c \). Hence \( c^{-1} \in \tilde{A} \) if and only if \( 1 \in [A, A] \).

One point in the proof of the next theorem is deferred to the final section where we consider the simple Lie algebras \( L(G, \delta, f) \).

**Theorem 6.** Let \( A \) be a (simple) nodal noncommutative Jordan algebra of dimension \( p^2 \) in \( K \) so that multiplication in \( A \) is defined by (28). Then
\[
D(A) = c \cdot M_2
\]
where \( M_2 \) is the \((p^2 + 1)\)-dimensional algebra consisting of all derivations (4) of \( B_2 \) given by (14) with \( r = 1 \), and \( D(A)'' \) is simple. If \( c^{-1} \in \tilde{A} \), then \( D(A) \cong M_2 \), \( D(A)' \cong V_{01} \), and \( D(A)'' \cong V_1 \) is of dimension \( p^2 - 2 \). If \( c^{-1} \not\in \tilde{A} \), then \( D(A)' \cong D(A)'' = c \cdot M_2^* \) is of dimension \( p^2 - 1 \).

**Proof.** The case \( c^{-1} \in \tilde{A} \) has already been established since we may take \( c = 1 \) by Theorem 5. Suppose that \( c^{-1} \not\in \tilde{A} \) so we may take \( c \) in the form (33) with \( \alpha \neq 0 \). We have seen that (37) holds. Hence (32) implies that
\[
\dim D(A)' \leq \dim (c \cdot M_2^*) = p^2 - 1.
\]
We shall see in the next section that a class of central simple Lie algebras \( L_0 \) of dimension \( p^2 - 1 \) is obtained as follows: \( L_0 \cong \text{ad} A \) where \( A \) has multiplication defined by (28) with \( c = \gamma (1 + x_1) \cdot (1 + x_2) \) for \( \gamma \neq 0 \in F \). We trace through
the steps of the proof of Theorem 5 to see that $x_i$ may be chosen in this $A$ so that

$$\alpha = - \gamma^{p-1}$$

in (33): $c = -\gamma^{-1}(1+x_1)^{-1} \cdots (1+x_{p-1})^{-1} = -\gamma^{-1}(1+x_1)^{-1}(1-x_2+x_2^2-\cdots +x_2^{p-1})$

so that $\beta_0 = \beta_{p-1} = -\gamma^{-1}(1+x_1)^{-1}$. Then $y_1 = x_1$,

$$y_2 = \gamma^{-1}(1+x_1)^{-1} + x_2 \left( 1 - \frac{1}{2} x_2 + \cdots - \frac{1}{p-1} x_2^{p-2} \right),$$

$$y_2^{p-1} = \gamma^{-(p-1)}(1+x_1)^{-(p-1)} \cdot x_2^{p-1} \left( 1 - \frac{1}{2} x_2 + \cdots - \frac{1}{p-1} x_2^{p-2} \right)^{p-1}$$

$$= \gamma^{-(p-1)}(1+x_1)^{-(p-1)} \cdot x_2^{p-1},$$

so that $x_2^{p-1} = -\gamma^{-1}(1+x_1)^{-1} \cdot y_2^{p-1} = -\gamma^{-1}(1+y_1)^{p-1} \cdot y_2^{p-1}$. Then $\alpha$ in (33) is the coefficient of $y_2^{p-1}$, $y_2^{p-1}$ in $-\beta_0^{-1} \cdot \beta_{p-1} \cdot x_2^{p-1} = -x_2^{p-1}$; that is, we have (39). Let $H = F(\gamma)$ where $\gamma$ satisfies (39). Then $(\text{ad } A)_H = \text{ad}(A_H) \cong L_0$ (an algebra defined over $H$) is simple and of dimension $p^2-1$ over $H$. Hence $\text{ad } A$ is simple and of dimension $p^2-1$ over $F$. But $A \subseteq D(A)$ implies $p^2-1 = \dim(\text{ad } A) = \dim(\text{ad } A)' \leq \dim D(A)' \leq p^2-1$ by (38). Hence $\text{ad } A = D(A)' = D(A)'' = (c \cdot M_2)' = c \cdot M_2$.

We remark that equality holds in (32) for $n = 2$.

4. **The simple algebras** $L(G, \delta, f)$. The simple Lie algebras $L_0$ and $L_\delta$ of Albert and Frank [1] have been generalized by Block[3] in [2] to an extensive class of simple Lie algebras $L(G, \delta, f)$. Block has shown [2, Lemma 3] that each $V_r$ is an algebra $L(G, \delta, f)$. In this section we prove

**Theorem 7.** For any simple Lie algebra $L(G, \delta, f)$ (of characteristic $\neq 2$) there exists a simple nodal noncommutative Jordan algebra $A$ in $K$ such that $A^-$ is a Lie algebra and $L(G, \delta, f) \cong (\text{ad } A)'$, an ideal in $D(A)$. Actually $L_0 \cong \text{ad } A$.

If $L(G, \delta, f)$ is simple, then $G = G_0 + G_1 + \cdots + G_m$ is an elementary $p$-group [2, Theorem 2], so that each $G_k$ may be regarded as an $n_k$-dimensional vector space over the prime field $F_\rho$ of characteristic $p$. The order of $G$ is $p^n$

---

(1) I am indebted to Dr. Block for furnishing me with a copy of his excellent dissertation [2] before its publication. My Theorem 7 was suggested by his Lemma 3. The following remarks about [2] may be of interest: (i) each of the algebras $V_{m, \mu}$ is isomorphic to $V_m$, for $y_i = \mu_i x_i$, $y_{m+t} = \mu_i x_{m+t}$ ($i = 1, \ldots, m$) implies $\mu_i \partial \phi / \partial x_{m+t} = \partial \phi / \partial y_{m+t} = -\mu_i \partial \phi / \partial y_i$, and the coefficient of $(x_1 \cdots x_m)^{p-1}$ is zero if and only if the coefficient of $(y_1 \cdots y_m)^{p-1}$ is; (ii) if $F = F_\rho$, the prime field of characteristic $p$, then any simple $L(G, \delta, f)$ for which $G_0 = 0$ is isomorphic to $V_m$, for [2, Theorem 4] implies that in each $G_i \{ h(\delta), h(\beta_1), \ldots, h(\beta_r) \}$ and $(g(\beta_{k+1}), \ldots, g(\beta_r))$ are linearly independent sets (of elements in $F$) over $F_\rho$, so that $k = 0$ and $r = k+1 = 1$, requiring that each $G_i$ be $2$-dimensional over $F_\rho$ and that $L(G, \delta, f) \cong V_m$ by [2, Lemma 3] and (i) above.
where \( n = n_0 + n_1 + \cdots + n_m \), and we write \( q_{-1} = 0 \), \( q_k = n_0 + n_1 + \cdots + n_k \) \((k = 0, 1, \cdots, m)\). Let \( \sigma_1, \cdots, \sigma_{n_0} \) be any basis for \( G_0 \) over \( F_p \). Since \( \delta = \delta_0 + \delta_1 + \cdots + \delta_m \) where \( \delta_0 = 0 \) and \( \delta_k \neq 0 \) in \( G_k \) for \( k = 1, \cdots, m \), we may take a basis \( \sigma_{q_{k-1}+1}, \cdots, \sigma_{q_k-1}, \delta_k \) for \( G_k \) over \( F_p \) \((k = 1, \cdots, m)\). But then, defining \( \sigma_{q_k} \) by

\[
(40) \quad \delta_k = \sigma_{q_{k-1}+1} + \cdots + \sigma_{q_k-1} + \sigma_{q_k},
\]

we also have \( \sigma_{q_{k-1}+1}, \cdots, \sigma_{q_k} \) a basis for \( G_k \) over \( F_p \) \((k = 1, \cdots, m)\). Then \( \sigma_1, \cdots, \sigma_n \) is a basis for \( G \) over \( F_p \), and any \( \alpha \in G \) may be written uniquely in the form

\[
(41) \quad \alpha = \sum_{i=1}^{n} s_i \sigma_i, \quad s_i \in F_p.
\]

Now \( L(G, \delta, f) \) is the derived algebra of a Lie algebra \( L/Fu_0 \) of dimension \( p^n-1 \) over \( F \), where \( L \) has a basis consisting of \( p^n \) elements \( u_\alpha \) in \((1-1)\) correspondence with the elements \( \alpha \) of \( G \). By \((41)\) the \( u_\alpha \) are in \((1-1)\) correspondence with the \( n \)-tuples \((s_1, \cdots, s_n)\), \( s_i \in F_p \), and we shall represent the \( u_\alpha \) in this way. The skew-symmetric biadditive function \( f(\alpha, \beta) \) on \( G \) to \( F \) may be taken so that \( f(\alpha_k, \beta_l) = 0 \) for \( k \neq l \), \( \alpha_k \in G_k, \beta_l \in G_l \) \((k, l = 0, 1, \cdots, m)\). Writing \( f(\sigma_i, \sigma_j) = \alpha_{ij} \in F \), we see that

\[
(42) \quad \alpha_{ij} = 0 \text{ unless } q_{k-1} + 1 \leq i, j \leq q_k \text{ for some } k \quad (0 \leq k \leq m).
\]

Now \((41)\) and \( \beta = \sum_{j=1}^{n} t_j \sigma_j \) imply \( f(\alpha_k, \beta_l) = \sum_{j=1}^{q_k} s_{ij} \alpha_{ij} \). Since \( \delta = \sum_{k=n_0+1}^{n} \sigma_k \) by \((40)\), we see that \([2, (4)]\) defines multiplication in \( L \) by

\[
(43) \quad (s_1, \cdots, s_n)(t_1, \cdots, t_n) = \sum_{i,j=1}^{n} s_{ij} \alpha_{ij}(s_1 + t_1, \cdots, s_n + t_n)
\]

\[
+ \sum_{k=1}^{q_k} \left( \sum_{i,j=q_k-1+1}^{q_k} s_{ij} \alpha_{ij}(s_1 + t_1, \cdots, s_{q_k-1} + t_{q_k-1}, s_{q_k-1} + t_{q_k-1} + 1, \cdots, s_{q_k} + t_{q_k} - 1, s_{q_k+1} + t_{q_k+1}, \cdots, s_n + t_n) \right).
\]

Instead of the nilpotent generators \( x_i \) of \( B_n = F[x_1, \cdots, x_n] \) used in previous sections, we use at this point generators \( z_i = 1 + x_i \) \((i = 1, \cdots, n)\). We have \( z_i^p = 1 \), and every element of \( B_n \) may be written uniquely in the form

\[
(44) \quad f = \sum_{s_i \in F_p} \alpha_{s_1} \cdots z_1^{s_1} \cdots z_n^{s_n}, \quad \alpha_{s_1} \cdots \in F.
\]

Let \( A \) in \( K \) be of dimension \( p^n \) so that \( A^+ = B_n \). Then \((24)\) implies that multiplication in \( A^- \) is defined by

\[
[f, g] = \sum_{i,j} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} 2 \epsilon_{ij}
\]
since $\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial x_i}$. Equivalently, multiplication in $A^-$ is defined by

$$[s_1^t \cdots s_n^t, z_1 \cdots z_n]$$

(45)

$$= \sum_{i,j=1}^{n} s_{ij} s_1^{t_1} \cdots s_i^{t_i+1} \cdots s_j^{t_j-1} \cdots s_n^{t_n}.2c_{ij}.$$ 

Let

$$c_{ij} = 0$$

unless $q_{k-1} + 1 \leq i, j \leq q_k$ for some $k$ ($0 \leq k \leq m$),

(46) \hspace{1cm} 2c_{ij} = \alpha_{ij} s_i s_j$$

for $1 \leq i, j \leq n_0$,

$$2c_{ij} = \alpha_{ij} s_i s_j (z_{q_{k-1}+1} \cdots z_{q_k})^{-1}$$

for $q_{k-1} + 1 \leq i, j \leq q_k$ ($k = 1, \ldots, m$).

For typographical reasons we write $\{s_1^t \cdots s_n^t\}$ for $z_1^{t_1} \cdots z_n^{t_n}$. Then (45) and (46) imply

$$\{s_1, \ldots, s_n\}, \{t_1, \ldots, t_n\} = \sum_{i,j=1}^{n_0} s_{ij} \alpha_{ij} \{s_1 + t_1, \ldots, s_n + t_n\}$$

(47) \hspace{1cm} + \sum_{k=1}^{n} \left( \sum_{i,j=q_{k-1}+1}^{q_k} s_{ij} \alpha_{ij} \{s_1 + t_1, \ldots, s_{q_{k-1}} + t_{q_{k-1}}, s_{q_{k-1}+1} + t_{q_{k-1}+1} - 1, \ldots, s_{q_k} + t_{q_k} - 1, s_{q_k+1} + t_{q_k+1}, \ldots, s_n + t_n\} \right).$$

That is, $L \cong A^-$ by (43) and (47), and $L/Fu_0 \cong A^-/F1$.

In order to complete the proof of the theorem, we shall require the following

**Lemma 2.** Let $A$ be a noncommutative Jordan algebra such that $A^+$ is associative. If $[f, g] = 0$ for every $g \in A$, then $f \cdot A (= fA = Af)$ is an ideal of $A$. Thus, if $A$ is any simple nodal noncommutative Jordan algebra, $[f, g] = 0$ for every $g \in A$ if and only if $f \in F1$.

**Proof.** Clearly $fg = gf = f \cdot g$ for every $g \in A$. Then [6, (4)] implies $(f \cdot g)h = -(g \cdot h)f + (gh) \cdot f + (gf) \cdot h = -f \cdot g \cdot h + f \cdot (gh) + g \cdot f \cdot h \in f \cdot A$. If $A = F1 + N$ is simple, then either $f \cdot A = 0$, implying $f = 0$, or $f \cdot A = A$. In the latter case $f = \alpha 1 + z, \alpha \neq 0, z \in N$. But then $[s, g] = [f - \alpha 1, g] = 0$ for every $g$, implying $z = 0, f = \alpha 1 \in F1$.

We return to the proof of Theorem 7. $A$ is in $K$, since at least one of the $\alpha_{ij}$ in (46) is not zero [2, Theorem 2]. If $G = G_0$, then $L_0 = L(G, \delta, f) = L/Fu_0$ has dimension $p^n - 1$. If $G \neq G_0$, then $L(G, \delta, f) = (L/Fu_0)'$ has dimension $p^n - 2$. We shall show in both cases that $A$ is simple since $L(G, \delta, f)$ is. If $A$ is not simple, then $A$ has a nonzero ideal $B \subseteq N$. Since $N$ is not an ideal of $A$,

(48) \hspace{1cm} 1 \leq \dim B \leq p^n - 2.

$B^-$ is an ideal of $A^-$, and either $F1 \cap B^- = F1$, implying $1 \in B, B = A$, a con-
tradition, or \( F_1 \cap B^- = 0 \). Hence \( C^- = F_1 \oplus B^- \) is an ideal of \( A^- \). In case \( G = G_0 \), then \( A^-/F_1 \leq L(G, \delta, f) \) is simple, so the kernel \( C^-/F_1 \) of the natural homomorphism of \( A^-/F_1 \) onto \( A^-/C^- \) is either 0 or all of \( A^-/F_1 \). That is, either \( B^- = 0 \) or \( \dim B^- = p^n - 1 \), contradicting (48) in either event. In case \( G \neq G_0 \), \( L/F_{u_0} = L(G, \delta, f) + F_v \) contains an ideal \( M \) corresponding to \( C^-/F_1 \) in \( A^-/F_1 \). Clearly \( M \cong B^- \). Then \( M \cap L(G, \delta, f) \) is an ideal of the simple algebra \( L(G, \delta, f) \). In view of (48), it follows that either (i) \( M = L(G, \delta, f) \), or (ii) \( L/F_{u_0} = L(G, \delta, f) \oplus M \) where \( \dim B = \dim M = 1 \). In case (i), \( L/F_{u_0} = M + F_v \). Correspondingly, \( A^-/F_1 = C^-/F_1 + F_z \) where \( z \) may be taken to be in \( N \). Then \( A = C + F_z = F_1 + B + F_z, N = B + F_z \). Now \( B \) an ideal of \( A \) implies \( N \cap (B + F_z)(B + F_z) \subseteq B + F_z^2 \subseteq N \), a contradiction, since \( A \) is a nodal algebra. There remains the possibility (ii), \( L/F_{u_0} = (L/F_{u_0})' \oplus M \). Correspondingly, \( (A^-/F_1) = (A^-/F_1)' \oplus F_w \) where \( w \) may be taken in \( B \). Then \( B = F_w \). Since \( B^+ \) is a 1-dimensional ideal in \( A^+ = F[x_1, \ldots, x_n] \), we have \( w = \sigma x_1^{p_1-1} \cdots x_n^{p_1-1} \) for \( \sigma \neq 0 \in F \). Write \( c_{ij} = \alpha_{ij} + z_{ij} \) in (24), \( \alpha_{ij} \in F, z_{ij} \in N \). There exist \( i_0, j_0 \) such that \( \alpha_{i_0 j_0} \neq 0 \). Then \( [x_{i_0}, x_{j_0}^{p_1-1} \cdots x_n^{p_1-1}] \in B \) implies

\[
0 \equiv [x_{i_0}, x_1^{p_1-1} \cdots x_n^{p_1-1}] \equiv (\phi - 1) \sum_{i} x_1^{p_1-1} \cdots x_j^{p_1-2} \cdots x_n^{p_1-1} \cdot 2c_{i j} \\
= 2(\phi - 1) \sum_{i} \alpha_{i j} x_1^{p_1-1} \cdots x_j^{p_1-2} \cdots x_n^{p_1-1} \mod B = Fx_1^{p_1-1} \cdots x_n^{p_1-1},
\]

implying \( \alpha_{i_0 j_0} = 0 \), a contradiction. That is, \( A \) must be simple if \( L(G, \delta, f) \) is. That \( \text{ad} \ A \) is isomorphic to \( A^-/F_1 \) follows directly from Lemma 2.

In §2 we referred to the proof above for justification of the statement that any \( A \) defined by a nondegenerate form \( \phi \) is simple. This follows from the fact that \( V_r = L(G, \delta, f) \) where \( G_0 = 0, m = r, \) and \( G_k \) is of dimension 2 over \( F_p \) for \( k = 1, \ldots, r [2, \text{Lemma 3}] \). For then the \( c_{ij} \) defined by (46) are all in \( F_1 \). In the proof of Theorem 6 we relied on (46) for the case \( n = n_0 = 2 \). In that instance \( 2c = 2c_{12} = \alpha_{12} z_1 \cdot z_2 = \alpha_{12} (1 + x_1) \cdot (1 + x_2) \) with \( \alpha_{12} \neq 0 \).

We have not computed the derivations of the algebras \( A \) in Theorem 7. Instead we conclude with the following result which generalizes (26) in the direction of (14).

**Theorem 8.** Let \( A \) be in \( K \) so that multiplication is defined by (1). If \( A^- \) is a Lie algebra, then the mappings \( D \) defined by (4) with

\[
a_i = \sum_{j=1}^{n} \left( \frac{\partial g}{\partial x_j} + \alpha_j x_j^{p-1} \right) c_{i j}, \quad i = 1, \ldots, n,
\]

for any \( g \in A \) and any \( \alpha_j \in F \) (\( j = 1, \ldots, n \)), are derivations of \( A \).

Since \( \text{ad} \ g/2 \) in (25) is a derivation of \( A \), it is sufficient to verify that \( D \) in (4) is a derivation in case

\[
a_i = \sum_{k=1}^{n} \alpha_k x_k^{p-1} \cdot c_{ik}, \quad \alpha_k \in F.
\]
Now $D$ is a derivation of $A$ in case (5) with $k$ replaced by $t$ is satisfied. But (50) implies

$$\sum_{i=1}^{n} \left( \frac{\partial c_{ij}}{\partial x_t} \cdot a_t + \frac{\partial a_t}{\partial x_t} \cdot c_{it} + \frac{\partial a_j}{\partial x_t} \cdot c_{jt} \right)$$

$$= \sum_{k,t} \left( \frac{\partial c_{ij}}{\partial x_k} \cdot \alpha_k x_k^{p-1} \cdot c_{ik} + \alpha_k \frac{\partial (x_k^{p-1} \cdot c_{ik})}{\partial x_t} \cdot c_{jt} + \alpha_k \frac{\partial (x_k^{p-1} \cdot c_{jk})}{\partial x_t} \cdot c_{it} \right)$$

$$= \sum_{k} \alpha_k x_k^{p-1} \left\{ \sum_{i} \left( \frac{\partial c_{ij}}{\partial x_t} \cdot c_{ik} + \frac{\partial c_{ik}}{\partial x_t} \cdot c_{jt} + \frac{\partial c_{jk}}{\partial x_t} \cdot c_{it} \right) \right\}$$

$$+ \sum_{k} \alpha_k x_k^{p-2} \left( c_{ik} \cdot c_{jk} + c_{jk} \cdot c_{ki} \right)$$

$$= 0$$

by (27).

**References**


**Institute for Advanced Study,**

**Princeton, New Jersey**

**University of Connecticut,**

**Storrs, Connecticut**