A SEMIGROUP ASSOCIATED WITH A TRANSFORMATION GROUP(1)

BY

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Let (X, T, π) be a transformation group with compact Hausdorff phase space X, and let $G=[\pi^t/t\in T]$ be the transition group of (X, T, π). Then G is a group of homeomorphisms of X onto X and so may be regarded as a subset of X^X. The *enveloping semigroup* E of (X, T, π) is by definition the closure of G in X^X [2]. In the first half of this paper algebraic properties of E are studied and correlated with recursive properties of T. Here the main theorem states that proximal is an equivalence relation in X if and only if there is only one minimal right ideal in E. The latter half of the paper is concerned with the study of homomorphic images of transformation groups by means of their enveloping semigroups. For further reference see [2] and [3].

Topological semigroups occur in the literature; see [4]. However, the assumption is generally made that the semigroup multiplication is bilaterally continuous. This is a property which the multiplication in E does not enjoy.

Standing notation. Throughout this paper (X, T, π) will denote a transformation group with compact Hausdorff phase space, G its transition group, and E its enveloping semigroup. If Q is a concept defined in terms of (X, T, π) and (Y, T, ρ) is another transformation group with phase group T and compact Hausdorff phase space Y, then Q_Y or $Q(Y, T, \rho)$ will denote the same concept defined in terms of (Y, T, ρ). Thus G_Y denotes the set $[\rho^t/t\in T]$ and E_Y denotes the closure of G_Y in Y^Y.

REMARK 1. Since X^X may be regarded as the set of maps of X into X, a semigroup structure may be introduced into X^X. Thus if $p, q\in X^X$, then pq denotes the map of X into X such that $x(pq)=(xp)q$ $(x\in X)$. Provided with this structure and its cartesian product topology X^X becomes a compact semigroup. The maps $p\to qp$ $(p\in X^X)$ of X^X into X^X are continuous for all $q\in X^X$, and the maps $p\to pq$ $(p\in X^X)$ of X^X into X^X are continuous for all continuous maps $q\in X^X$. Moreover E is a closed hence compact sub-semigroup of X^X. For proofs see [1] and [2].

REMARK 2. The map σ: $E\times T\to E$ such that $(p, t)\sigma=p\pi^t$ $(p\in E, t\in T)$ defines a transformation group with phase space E and phase group T.

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DEFINITION 1. Let A be a non-null closed subset of X. Then A is *minimal under* T if $AT\subset A$ and if whenever B is a non-null closed subset of A with $BT\subset B$, then $B=A$.

DEFINITION 2. Let $x\in X$. Then x is an *almost periodic point under* T or T is *pointwise almost periodic at* x if $[xT]^-$ is minimal under T. The group T is said to be *pointwise almost periodic* if T is pointwise almost periodic at x for all $x\in X$. This is not the usual definition but a characterization; see [3, 4.05 and 4.07]. Note that $[xT]^-=xE$ for all $x\in X$.

DEFINITION 3. Let I be a nonvacuous subset of E. Then I is said to be a *right ideal in* E or simply an *ideal* if $IE\subset I$. An ideal I is said to be *minimal* if whenever K is a nonvacuous subset of I such that $KE\subset K$ then $K=I$. Notice that in the definition an ideal is not required to be a closed subset of E.

REMARK 3. Let I be a minimal ideal, and let $p\in E$. Then pI is a minimal ideal. Also, the restriction of σ to $I\times T$ defines a transformation group with phase space I. This transformation group will be denoted (I, T, σ).

LEMMA 1. *The following statements hold.*

(1) *Let $\emptyset\neq I\subset E$. Then I is a minimal right ideal if and only if I is minimal under T. Thus every minimal right ideal is closed.*

(2) *Let I be a minimal right ideal in E. Then xI is a minimal subset of X for all $x\in X$, where $xI=\bigcup[xp/p\in I]$.*

Proof. (1) Let I be a minimal right ideal, and let $p\in I$. Then $pE\subset IE\subset I$ and pE is a right ideal. Thus $I=pE$ which is closed by Remark 1. Now suppose K is a nonvacuous closed subset of I such that $KT=\bigcup[K\pi^t/t\in T]\subset K$. Then because K is closed and $E=\overline{G}$, $KE\subset K$, i.e. K is a right ideal; whence $K=I$. Thus I is a minimal subset of E.

Conversely, suppose I is a minimal subset of E. Then I is closed and $IG\subset I$, whence $IE\subset I$; i.e. I is a right ideal. Now let K be a nonvacuous right ideal contained in I and let $p\in K$. Then $pE\subset KE\subset K\subset I$, and pE is a closed subset of I such that $pET\subset pE$. Hence by the minimality of I, $pE=I$. Consequently $K=I$. The proof is completed.

(2) Let A be a non-null closed subset of xI such that $AT\subset A$. Let $K=[p/p\in I$ and $xp\in A]$. Then K is a non-null closed subset of I such that $KT\subset K$. Hence by (1) $K=I$ and so $xI=A$.

REMARK 4. Let K be a closed subset of E such that $K^2\subset K$. Then by [1, Lemma 1] there exists at least one idempotent (i.e. an element u of E such that $u^2=u$) in K. Thus every minimal right ideal I contains at least one idempotent.

Lemma (1) shows that there exists at least one minimal right ideal in E [3, 4.06].

LEMMA 2. *Let I be a minimal right ideal in E and let J be the set of idempotents of E contained in I. Then:*

(1) $pI=I$ *for all* $p\in I$.

(2) $up=p$ $(u\in J,\ p\in I)$.

(3) *If* $p\in I$, *then there exist* $s\in I$ *and* $u\in J$ *such that* $pu=p$ *and* $ps=sp=u$.

(4) *If* $aq=ar$ *with* $q,\ r\in I$ *and* $a\in E$, *then* $q=r$.

(5) *If* $u,\ v\in J$ *with* $u\neq v$, *then* $Iu\cap Iv=\emptyset$.

Proof. (1) Let $p\in I$. Then $pI\subset I^2\subset I$ and pI is a right ideal. Hence $pI=I$.

(2) Let $u\in J$ and $p\in I$. By (1) there exists $q\in I$ with $uq=p$. Hence $up=u^2q=uq=p$.

(3) Let $p\in I$. Set $H=[q/q\in I$ and $pq=p]$. Then by (1) and Remark 1 H is a nonempty closed subset of I. Furthermore $H^2\subset H$. Hence by Remark 4 there exists $u\in H\cap J$; i.e. $pu=p$. By (1) there exist $s,\ r\in I$ such that $ps=u$ and $sr=u$. Then $p=pu=psr=ur=r$ and so $ps=sp=u$.

(4) Let $p=qa$. Then $p\in I$ and $pq=pr$. By (3) there exist $s\in I$ and $u\in J$ such that $sp=u$. Then $q=uq=spq=spr=ur=r$ by (2).

(5) Let $pu=qv$ with $p,\ q\in I$ and $u,\ v\in J$. Then $pu=pu^2=qvu=qu$ by (2). Hence $qu=qv$ and so $u=v$.

Remark 5. Lemma 2 (4) shows that "left cancellation" holds to a limited extent in E. Also, given $p\in I$ (4) and (5) show that the s and u guaranteed in (3) are unique.

Definition 4. Let $u,\ v$ be two idempotents in E. Then u and v are said to be *equivalent* (symbolically $u\sim v$) if $uv=u$ and $vu=v$. Let $u\sim v$ and $v\sim w$. Then $uw=uvw=uv=u$ and $wu=wvu=wv=w$; i.e. $u\sim w$ and so $\sim$ is indeed an equivalence relation. Lemma 2 (2) shows that if $u\sim v$ and $u,\ v$ are in the same minimal right ideal then $u=v$.

Lemma 3. *Let* I_1 *and* I_2 *be minimal right ideals in* E, *and let* u_1 *be an idempotent in* I_1. *Then there exists a unique idempotent* $u_2\in I_2$ *such that* $u_1\sim u_2$.

Proof. The set u_1I_2 is a right ideal contained in I_1. Hence there exists $u_2\in I_2$ with $u_1u_2=u_1$, and u_2 is unique by Lemma 2 (4). Moreover $u_1u_2=u_1$ implies that $u_1u_2^2=u_1u_2$ whence $u_2^2=u_2$ by Lemma 2 (4). Similarly there exists $v_1\in I_1$ with $u_2v_1=u_2$ and $v_1^2=v_1$. Then $u_1=u_1u_2=u_1u_2v_1=u_1v_1=v_1$ by Lemma 2 (2). Thus $u_1=v_1$ and so $u_2u_1=u_2$. The proof is completed.

Proposition 1. *Let* I_1 *and* I_2 *be two minimal right ideals in* E, *and let* J_1 *and* J_2 *be the set of idempotents in* I_1 *and* I_2 *respectively. Then there exists a mapping* $\phi\colon I_1\to I_2$ *such that*:

(1) $J_1\phi=J_2$ *and* $u_1\phi\sim u_1$ *for all* $u_1\in J_1$.

(2) ϕ *is one-to-one and onto.*

(3) *The restriction of* ϕ *to* I_1u_1 *is a homeomorphism of* I_1u_1 *onto* $I_2(u_1\phi)$ *for all* $u_1\in I_1$.

Proof. For $u_1\in J_1$ let $u_1\psi$ be the unique element of J_2 such that $u_1\sim u_1\psi$, the existence of which is guaranteed by Lemma 3. Let $p_1\in I_1$. Then by Lemma

2 (2) and Remark 5 there exists a unique $u_1 \in J_1$ with $p_1 u_1 = p_1$. Let $p_1\phi = (u_1\phi)p_1$. If $p_1 \in J_1$ then $p_1 = u_1$ and $p_1\phi = u_1\phi = (u_1\psi)u_1 = u_1\psi$ since $u_1\psi \sim u_1$; i.e. ϕ agrees with ψ on J_1.

(1) The definition of ψ, and the fact that $\phi = \psi$ on J_1 follows from Lemma 3.

(2) Since I_1 and I_2 occur symmetrically in Proposition 1, $\eta: I_2 \to I_1$ such that $p_2\eta = (u_2\eta)p_2$ where $p_2 u_2 = p_2$ and $u_2\eta \in I_1$ with $u_2\eta \sim u_2$ is a well defined mapping of I_2 into I_1. Let $p_1 \in I_1$, $u_1 \in J_1$ and $p_1 u_1 = p_1$. Then $p_1\phi = u_2 p_1$ where $u_1 \sim u_2$. Since $p_1 u_2 = p_1$, $u_2 p_1 u_2 = u_2 p_1$ and $(u_2 p_1)\eta = (u_2\eta)u_2 p_1 = u_1 u_2 p_1 = u_1 p_1 = p_1$ since $u_1 \sim u_2$. Hence $p_1\phi\eta = p_1$ for all $p_1 \in I_1$. Similarly $p_2\eta\phi = p_2$ for all $p_2 \in I_2$. The proof of (2) is completed.

(3) Let $p_1 \in I_1$, $u_1 \in J_1$, $q_1 = p_1 u_1$, and $u_2 = u_1\phi$. Then $q_1 u_1 = q_1$ and so $q_1\phi = u_2 q_1$. Also $q_1 u_2 = p_1 u_1 u_2 = p_1 u_1 = q_1$ implies that $q_1\phi \in I_2 u_2$. Hence $(I_1 u_1)\phi \subset I_2 u_2$. Similarly $(I_2 u_2)\eta \subset I_1 u_1$, whence $(I_1 u_1)\phi = I_2 u_2$.

Now let $\{q_1^\alpha\}$ be a net of elements of $I_1 u_1$ such that $q_1^\alpha \to q_1 \in I_1 u_1$. Then $q_1^\alpha\phi = u_2 q_1^\alpha \to u_2 q_1 = q_1\phi$ by Remark 1. Hence the restriction of ϕ to $I_1 u_1$ is a continuous mapping of $I_1 u_1$ onto $I_2 u_2$. Similarly the restriction of η to $I_2 u_2$ is a continuous mapping of $I_2 u_2$ onto $I_1 u_1$. Since $\phi\eta =$ identity on I_1 and $\eta\phi =$ identity on I_2, the proof is completed.

Notation. Henceforth L will denote $\bigcup [I/I$ minimal right ideal in $E]$, M will denote $[u/u \in L$ and $u^2 = u]$, and A the set $[x/x \in X$ and T is almost periodic at $x]$.

Theorem 1. *Let $x \in X$. Then the following statements are equivalent.*

(1) *$x \in A$.*

(2) *$x \in xI$ for all minimal right ideals I.*

(3) *$x \in xJ(I)$ for all minimal right ideals I, where $J(I)$ denotes the set of idempotents in I.*

(4) *$x \in xM$.*

(5) *$x \in xL$.*

Proof. (1) implies (2). Let I be a minimal right ideal in E. Then by Lemma 1 xI is a minimal subset of X. Since $xI \subset [xT]^-$ and $x \in A$, $xI = [xT]^-$ whence $x \in xI$.

(2) implies (3). Let I be a minimal right ideal and $J = J(I)$. Let $H = [p/p \in I$ and $xp = x]$. By assumption $H \neq \emptyset$. Moreover H is a closed subset of I such that $H^2 \subset H$. Hence there exists an idempotent $u \in H$ by Remark 4, i.e. $u \in J$ and $xu = x$. Thus $x \in xJ$.

(3) implies (4) and (4) implies (5) are clear from the definition of M and L.

(5) implies (1). Assume (5). Then there exists a minimal right ideal I in E with $x \in xI$. Consequently $xt \in xIt = xI\pi^t \subset xI$ for all $t \in T$, and so $[xT]^- \subset xI$ since xI is closed. By Lemma 1, xI is a minimal subset of X. Hence $[xT]^- = xI$ since $[xT]^-$ is an invariant closed subset of xI. The proof is completed.

COROLLARY 1. *Let* $x \in X$. *Then* $[xT]^- \cap A = xL$.

Proof. Let $y \in xL$. Then $y = xp$ where $p \in I$ and I is a minimal right ideal. By Lemma 2 (3) there exists $u \in I$ with $pu = p$. Thus $y = xp = xpu = yu \in yI$, whence $y \in A$ by Theorem 1. Also $y = xp$ implies $y \in [xT]^-$.

Now let $y \in [xT]^- \cap A$. Then there exists $p \in E$ with $y = xp$. Moreover $y \in A$ implies $y \in yL = xpL = xL$ by Remark 3.

DEFINITION 5. Let $x, y \in X$. Then x and y are said to be *proximal* if given an index α of X, there exists $t \in T$ with $(xt, yt) \in \alpha$. The *proximal relation* P is defined to be that subset of $X \times X$ consisting of all proximal pairs (x, y) [2]. The relation P is reflexive and symmetric but in general not transitive. Theorem 2 is concerned with necessary and sufficient conditions on E to insure that P be transitive or in other words that P be an equivalence relation in X.

REMARK 6. In [2] it was remarked that if $x, y \in X$ then $(x, y) \in P$ if and only if there exists $p \in E$ with $xp = yp$. This condition may now be changed to read $(x, y) \in P$ if and only if $xr = yr$ for all r in some minimal right ideal I in E. For if $xp = xp$ for some $p \in E$, then $xr = yr$ for all $r \in I$ where $I = pK$ and K is an arbitrary minimal right ideal in E.

THEOREM 2. *The following statements are equivalent.*

(1) *P is an equivalence relation in X.*

(2) *E contains exactly one minimal right ideal.*

Proof. (1) implies (2). Assume (1). By Remark 4 there is at least one minimal right ideal in E. The proof will be completed by showing that there is at most one such ideal. Let I_1 and I_2 be distinct minimal right ideals, and let $u_1 \in I_1$ and $u_2 \in I_2$ be idempotents such that $u_1 \sim u_2$; such exist by Lemma 3. Let $x \in X$ and set $y = xu_1$ and $z = xu_2$. Then $yu_1 = xu_1^2 = xu_1$ and $zu_2 = xu_2^2 = xu_2$. Thus $(x, y) \in P$ and $(x, z) \in P$ by Remark 6. Hence $(y, z) \in P$ by assumption, and so by Remark 6 there exists a minimal right ideal I_3 such that $yr = zr$ for all $r \in I_3$. By Lemma 3 there exists an idempotent $u_3 \in I_3$ with $u_1 \sim u_2 \sim u_3$. Then $yu_3 = zu_3$ and $y = xu_1 = xu_1u_3 = yu_3 = zu_3 = xu_2u_3 = xu_2 = z$; i.e. $y = z$. Since x was arbitrary, $u_1 = u_2$, a contradiction.

(2) implies (1). Let $(x, y) \in P$ and $(y, z) \in P$, to show $(x, z) \in P$. By Remark 6 there exist minimal right ideals I_1 and I_2 such that $xr = yr$ $(r \in I_1)$ and $yp = zp$ $(p \in I_2)$. Since by assumption there is but one minimal right ideal I, $xr = yr = zr$ $(r \in I)$. The proof is completed.

REMARK 7. The transformation group (X, T, π) is said to be *distal* if $P = \Delta$ [1; 2]. Clearly if $P = \Delta$ then P is an equivalence relation in X and so by Theorem 2 there is one minimal right ideal I in E. In this case, however, E is a group [1] and so $I = E$.

COROLLARY 2. *Let P be an equivalence relation, and let $x \in X$. Then $A \cap [xT]^-$ is a closed subset of X.*

Proof. Use Corollary 1 and Theorem 2 and note that $L=I$ where I is the unique minimal right ideal.

COROLLARY 3. *Let T be pointwise almost periodic on X. Then the following statements hold.* (1) $(x, y)\in P$ *if and only if* $x\in yM$. (2) *P is an equivalence relation in X if and only if $M^2\subset M$.*

Proof. (1) If $x\in yM$, then there exists an idempotent u with $x=yu$. Hence $xu=yu$ and $(x, y)\in P$ by Remark 6. Conversely suppose $(x, y)\in P$. Then by Remark 6 there exists a minimal right ideal I with $xp=yp$ $(p\in I)$. Since $x\in A$, by Theorem 1 there exists $u\in J(I)\subset M$ with $x=xu$. Hence $x=xu=yu\in yM$.

(2) Let P be an equivalence relation, let I be the unique minimal ideal in E, and J its set of idempotents. Then $M=J$ and $J^2=J$ by Lemma 2 (2).

Now let $M^2\subset M$. Let $(x, y)\in P$ and let $(y, z)\in P$. Then by (1) $x\in ym$ and $y\in zM$. Hence $x\in zM^2\subset zM$. The proof is completed.

An interesting problem upon which Theorem 2 sheds some light is the following: Let H be a group of homeomorphisms of X onto X such that $xht=xth$ $(x\in X, h\in H, t\in T)$. Then what can be said about the almost periodic points under H? From general considerations it is known [3, 4.06] that there is at least one point $x_0\in X$ at which H is pointwise almost periodic. By the assumptions on H and T, this implies that H is pointwise almost periodic at all points of x_0T. Thus if $x_0T=X$, H is pointwise almost periodic. One might ask whether the condition that $x_0T=X$ could be relaxed to $[xT]^-=X$ $(x\in X)$ and still retain the conclusion that H is pointwise almost periodic. The answer is no in general as the following considerations show. Let T be abelian, let $[xT]^-=X$ for all $x\in X$, but suppose the transformation group $(X\times X, T, \eta)$, where $((x, y), t)\eta=(xt, yt)$ $(x, y\in X, t\in T)$, is not pointwise almost periodic. Such exist; see [2, Example 4]. Now consider the transformation group $(X\times X, T\times T, \xi)$ where

$$((x, y)\ (t, s))\xi = (xt, ys)\ (x, y\in X;\ t, s\in T).$$

Then $[(x, y)T\times T]^-=X\times X$ for all $x, y\in X$, $\eta^t\xi^{r,s}=\xi^{r,s}\eta^t$ $(r, s, t\in T)$ but T is not pointwise almost periodic on $X\times X$.

However, something may be said when $P(X, H, \sigma)$ is an equivalence relation, where $(x, h)\sigma=xh$ $(x\in X, h\in H)$.

COROLLARY 4. *Let $[xT]^-=X$ $(x\in X)$, let H be a group of homeomorphisms of X such that $xht=xth$ $(x\in X, h\in H, t\in T)$, let $P(X, H, \sigma)$ be an equivalence relation in X, and let there exist $y_0\in X$ with $[y_0H]^-=X$. Then $[xH]^-=X$ for all $x\in X$.*

Proof. Let $A(H)=[x/H$ is pointwise almost periodic at $x]$. Then as was remarked above there exists $x_0\in A(H)$ and so $x_0T\subset A(H)$. Consequently $[A(H)]^-=X$. Since $P(X, H, \sigma)$ is an equivalence relation in X, $[y_0H]^-\cap A(H)$

is closed by Corollary 2. Since $[y_0H]^- = X$, this means that $A(H) = [A(H)]^-$; whence $A(H) = X$. The proof is completed.

The last part of this paper is concerned with the study of homomorphic images of the transformation group (X, T, π).

DEFINITION 6. Let (Y, T, ρ) be a transformation group with compact Hausdorff space Y and phase group T, let $\phi: X \to Y$ be onto, then ϕ is a *homomorphism of* (X, T, π) *onto* (Y, T, ρ) provided that ϕ is continuous and $\pi^t\phi = \phi\rho^t$ for all $t \in T$. In [2] it is shown the ϕ induces a continuous semigroup homomorphism θ of E_X onto E_Y such that $xp\phi = (x\phi)(p\theta)$ $(x \in X, p \in E_X)$ and $\pi^t\theta = \rho^t$ $(t \in T)$. The mapping θ will be referred to as the *homomorphism induced* by ϕ [2].

LEMMA 4. *Let ϕ be a homomorphism of (X, T, π) onto (Y, T, ρ) and let θ be the homomorphism induced by ϕ. Then:*

(1) *θ is a homomorphism of (E_X, T, σ) onto (E_Y, T, σ) where $(p, t)\sigma = p\pi^t$ $(p \in E_X, t \in T)$ and $(q, t)\sigma = q\rho^t$ $(q \in E_Y, t \in T)$.*

(2) *If I is a right ideal in E_X, then $I\theta$ is a right ideal in E_Y.*

(3) *If K is a right ideal in E_Y, then $K\theta^{-1}$ is a right ideal in E_X.*

(4) *If I is a minimal right ideal in E_X, then $I\theta$ is a minimal right ideal in E_Y.*

(5) *If (Y, T, ρ) coincides with (X, T, π), i.e. ϕ is a homomorphism of (X, T, π) onto (X, T, π), then θ is the identity map of E_X onto E_X.*

(6) *Let I be a minimal right ideal in E_X, let $K = I\theta$, let N be the set of idempotents in K, and let $H = N\theta^{-1} \cap I$. Then $I/H = [pH/p \in I]$ and $K/N = [rN/r \in K]$ are semigroups where $(pH)(qH) = pqH$ $(p, q \in I)$ and $(rN)(sN) = rsN$ $(r, s \in K)$. If furthermore K/N is a Hausdorff space, then I/H and K/N are isomorphic.*

Proof. (1) By [2] θ is continuous onto, and $(pt)\theta = (p\pi^t)\theta = (p\theta)(\pi^t\theta) = (p\theta)\rho^t$ $(p \in E_X, t \in T)$.

(2) Let $r \in E_Y$ and $q = p\theta \in I\theta$ where $p \in I$. Pick $s \in E_X$ with $s\theta = r$. Then $ps \in I$ and $(ps)\theta = (p\theta)(s\theta) = qr \in I\theta$.

(3) Let $p \in K\theta^{-1}$ and $s \in E_X$. Then $p\theta \in K$ and $(ps)\theta = (p\theta)(s\theta) \in K(s\theta) = K$, whence $ps \in K\theta^{-1}$.

(4) Let K be a right ideal in E_Y such that $K \subset I\theta$. Then by (3) $K\theta^{-1}$ is a right ideal in E_X. Since $K\theta^{-1} \cap I \neq \emptyset$, $I \subset K\theta^{-1}$. Hence $I\theta \subset K$.

(5) By [2] $\pi^t\theta = \pi^t$ $(t \in T)$ whence θ is the identity on E_X since θ is continuous and $[\pi^t/t \in T]$ is dense in E_X.

(6) Let us first show that $p \in qH$ if and only if $p\theta \in (q\theta)N$ $(p, q \in I)$. Let $p \in qH$; then $p\theta \in (qH)\theta = (q\theta)N$. Conversely, let $p\theta = (q\theta)u$ with $u \in N$. By Lemma 2 (1) there exists $s \in I$ with $p = qs$. Thus $(q\theta)(s\theta) = p\theta = (q\theta)u$ whence $s\theta = u$ by Lemma 2 (4), since K is a minimal right ideal by (4).

Let $r, s \in K$ with $r = su$, $u \in N$. Then by Lemma 2 (3) there exists $v \in N$ with $sv = s$. Thus $s = sv = suv = rv \in rN$ by Lemma 2 (2). Consequently

$[rN/r \in K]$ is a partition of K and it makes sense to speak of the partition space K/N.

Let $p, q \in I$ with $p \in qH$. Then $p\theta \in (q\theta)N$ and so $q\theta \in (p\theta)N$ whence $q \in pH$. Thus $[pH/p \in I]$ is a partition of I.

Lemma 2 (2) shows that $Nr = r$ $(s \in K)$, whence $rNsN = rsN$ $(r, s \in K)$ and $pHqH = pqH$ $(p, q \in I)$.

The map $p \rightarrow (p\theta)N$ $(p \in I)$ of I onto K/N is continuous and so the map $\eta: I/H \rightarrow K/N$ such that $(pH)\eta = (p\theta)N$ $(p \in I)$ is a one-to-one continuous map of I/H onto K/N. If K/N is Hausdorff, this map is a homomorphism. Since $(pH \cdot qH)\eta = (pqH)\eta = (pq)\theta N = (p\theta)(q\theta)N = ((p\theta)N) \cdot ((q\theta)N) = (pH)\eta \cdot (qH)\eta$, in this case η is an isomorphism. The proof is completed.

A question which might be raised concerning homeomorphisms of transformation groups is the following. Suppose ϕ is a homomorphism of X onto Y and ψ is a homomorphism of Y onto X. Then are X and Y isomorphic? In general the answer is no. The following lemmas are concerned with this problem.

LEMMA 5. *Let I be a minimal right ideal in E and let η be a homomorphism of the transformation group (I, T, σ) onto itself. Then η is an isomorphism.*

Proof. Let $p \in I$ and $t \in T$. Then by assumption $(p\pi^t)\eta = (p\eta)\pi^t$. Hence $(pq)\eta = (p\eta)q$ $(p, q \in I)$ since $[\pi^t/t \in T]$ is dense in E, and left multiplication in E and η are continuous.

Now let u be an idempotent in I and set $r = u\eta$. Then $p\eta = (up)\eta = (u\eta)p = rp$ $(p \in I)$ by Lemma 2 (2). Thus if $p\eta = q\eta$, $rp = rq$, and so by Lemma 2 (4) $p = q$; i.e. η is one-to-one. The proof is completed.

LEMMA 6. *Let I be a minimal right ideal in E, let ϕ be a homomorphism of (I, T, σ) onto (Y, T, ρ), and let ψ be a homomorphism of (Y, T, ρ) onto (I, T, σ). Then ϕ is an isomorphism.*

Proof. The map $\phi\psi$ is a homomorphism of (I, T, σ) onto (I, T, σ). By Lemma 5 $\phi\psi$ is an isomorphism. Therefore ϕ is an isomorphism.

LEMMA 7. *Let ϕ be a homomorphism of (X, T, π) onto (Y, T, ρ) and let proximal be an equivalence relation in X. Then proximal is an equivalence relation in Y.*

Proof. Let I_1 and I_2 be minimal right ideals in E_Y. Then $I_1\theta^{-1}$ and $I_2\theta^{-1}$ are right ideals in E_X. Thus $I \subset I_1\theta^{-1} \cap I_2\theta^{-1}$ where I is the unique minimal ideal in E_X. Hence $I_1 \cap I_2 \neq \emptyset$ and so $I_1 = I_2$. The proof is completed.

DEFINITION 7. The transformation group (X, T, π) is said to be *locally almost periodic* provided that if $x \in X$ and V is a neighborhood of x, then there exist a neighborhood U of x and a syndetic subset A of T such that $(U \times A)\pi = UA \subset V$ [3].

REMARK 8. The following properties of the enveloping semigroup proved in [2] will be used in the sequel.

If P is a closed equivalence relation in X, then $E(X/P, T)$ is a group.

If (X, T, π) is a locally almost periodic, then P is a closed equivalence relation in X and $E(X/P, T)$ is a compact topological group referred to as the *structure group* of (X, T, π) and denoted Γ or Γ_X.

THEOREM 3. *Let T be pointwise almost periodic on X, let P be a closed equivalence relation in X, let I be the unique minimal right ideal in E and J the set of idempotents in I. Then:*

(1) I/J is a compact group.

(2) If T is locally almost periodic on X, then I/J is a compact topological group isomorphic with the structure group of (X, T, π) [2].

Proof. (1) Let $Y=[xP|x\in X]$ and (Y, T, ρ) the transformation group such that $(xP, t)\rho=xtP$ $(x\in X, t\in T)$. Let $\phi\colon X\to Y$ such that $x\phi=xP$ $(x\in X)$. Then ϕ is a homomorphism of (X, T, π) onto (Y, T, ρ). By [2] E_Y is a group. Let θ be the homomorphism of E onto E_Y induced by ϕ, and let $u\in J$. Then $u\theta=(u^2)\theta=(u\theta)(u\theta)$ and so $u\theta=e$, the identity element of E_Y. If $p\in E$, then pI is a minimal right ideal and so $pI=I$. Consequently $p\theta=(p\theta)u\theta=(pu)\theta\in I\theta$ and so $I\theta=E_Y$.

Now suppose $p\theta=e$ for some $p\in I$. Then $xp\phi=(x\phi)(p\theta)=x\phi$ $(x\in X)$. Consequently $(xp, x)\in P$. By Corollary 3 there exists $v=v_x\in J$ such that $xp=xv$, whence $xp^2=xvp=xp$ $(x\in X)$ by Lemma 2 (2). Thus $p^2=p\in J$; i.e. $J=e\theta^{-1}$. Therefore $I/J\cong E_Y$ by Lemma 4 (6).

(2) Statement (2) follows from (1) and the fact that in this case E_Y is a compact topological group.

COROLLARY 5. *Let (X, T, π) and (Y, T, ρ) be locally almost periodic transformation groups, let Γ_X and Γ_Y be the structure groups of (X, T, π) and (Y, T, ρ), and let ϕ be a homomorphism of (X, T, π) onto (Y, T, ρ). Then there exists a homomorphism of Γ_X onto Γ_Y.*

Proof. Let θ be the homomorphism of E_X onto E_Y induced by ϕ, let I_X and I_Y be the unique minimal right ideals of E_X and E_Y, and let J_X and J_Y be the idempotents in I_X and I_Y respectively. Then $I_X\theta=I_Y$ by Lemma 4 (4). Moreover $J_X\theta\subset J_Y$ implies that the map $pJ_X\to(p\theta)J_Y$ $(p\in I_X)$ is a homomorphism of I_X/J_X onto I_Y/J_Y. The proof is completed by Theorem 3.

COROLLARY 6. *If in addition to the hypotheses of Corollary 5, it is assumed that $\Gamma_X=\Gamma_Y$ and that θ maps J_X in a one-to-one manner onto J_Y, then θ is an isomorphism of I_X onto I_Y.*

Proof. Let $H=I_X\cap J_Y\theta^{-1}$. Then $J_X\subset H$. By Lemma 4 (6) the map $pH\to(p\theta)J_Y$ $(p\in I_X)$ is an isomorphism of I_X/H onto I_Y/J_Y. By Corollary 5 the map $pJ_X\to(p\theta)J_Y$ $(p\in I_X)$ is an isomorphism of I_X/J_X onto I_Y/J_Y since

$\Gamma_X = \Gamma_Y$. Hence the map $pH \to pJ_X$ is an isomorphism of I_X/H onto I_X/J_X. Consequently $H = J_X$.

Now let $p, q \in I_X$ with $p\theta = q\theta$. By Lemma 2 (1) there exists $s \in I_X$ such that $ps = q$. Then $p\theta = q\theta = (ps)\theta = (p\theta)(s\theta)$ and so $(p\theta)(s\theta) = (p\theta)(s\theta)^2$, whence $s\theta = (s\theta)^2$ by Lemma 2 (4). Thus $s\theta \in J_Y$ and so $s \in J_Y\theta^{-1} \cap I_X = H = J_X$; i.e. $s^2 = s$.

By Lemma 2 (3) there exists $u \in J_X$ with $pu = p$. Hence $p\theta = (p\theta)(u\theta) = (p\theta)(s\theta)$. Thus $u\theta = s\theta$ and so $u = s$ since $u, s \in J_X$. This means that $p = pu = ps = q$. The proof is completed.

Lemma 8. *Let P be a closed equivalence relation in X, let I be the unique minimal right ideal in E and let J be the idempotents in I, and let $\Gamma = E(X/P, T)$. Then there exists a natural one-to-one map ψ of I onto $\Gamma \times J$ induced by the projection ϕ of X onto X/P.*

Proof. Let θ be the homomorphism of E onto Γ induced by ϕ. Then as in Theorem 3 $I\theta = \Gamma$.

Let $p \in I$. Then by Lemma 2 (3) there exists a unique $u \in J$ with $pu = p$. Set $p\psi = (p\theta, u)$. Let $r \in \Gamma$ and $u \in J$. Then there exists $q \in I$ with $q\theta = r$. Let $p = qu$. Then $pu = p$ and $p\theta = q\theta u\theta = (q\theta)e = r$. Thus $p\psi = (r, u)$ and so ψ is onto.

Now suppose $p\psi = q\psi$ ($p, q \in I$). Let $p = pu$ and $q = qv$. Then $(p\theta, u) = (q\theta, v)$ implies that $u = v$ and $p\theta = q\theta$. By Lemma 2 (1) there exists $s \in I$ with $ps = q$. Thus $p\theta = q\theta = (pv)(sv)$ and so $s\theta = e$ whence $s \in J$ as in the proof of Theorem 3. Consequently $q = qv = qu = psu = pu = p$. The proof is completed.

Remark 9. The map ψ defined in Lemma 8 is in general not continuous.

Theorem 3 and its corollaries indicate that given a compact group Γ, a dense subgroup T, and a set J one should be able to construct the minimal ideal I of a locally almost periodic transformation group (X, T, π) with structure group Γ and such that J is the set of idempotents in I. Lemma 8 asserts that set theoretically at least I is $\Gamma \times J$. This idea was used in constructing Example 4 [2]. There Γ is the circle group and J is a two point set.

An interesting problem is to provide $\Gamma \times J$ with a compact Hausdorff topology and to determine the action of T on $\Gamma \times J$ such that $(\Gamma \times J, T)$ is locally almost periodic.

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