

ON ORDERED DIVISIBLE GROUPS⁽¹⁾

BY

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Introduction and remarks. In the theory of η_α -sets three main theorems stand out: that an η_α -set is universal for totally ordered sets of power not exceeding \aleph_α , that any two η_α -sets of power \aleph_α are isomorphic and that η_α -sets of power \aleph_α exist provided \aleph_α is a regular cardinal number and $\sum_{\delta < \alpha} 2^{\aleph_\delta} \leq \aleph_\alpha$. L. Gillman and M. Jerison [6] have shown that a real closed (totally ordered) field which is an η_α -set is universal for totally ordered fields of power not exceeding \aleph_α , provided $\alpha > 0$. Further, P. Erdős, L. Gillman and M. Henriksen [3] have shown that any two real closed fields which are η_α -sets of power \aleph_α are isomorphic, provided $\alpha > 0$. They also showed that real closed fields which are η_α -sets of power \aleph_α exist, assuming the continuum hypothesis, if $\alpha = 1$. The question of existence for $\alpha > 1$, even assuming \aleph_α to be regular and $\sum_{\delta < \alpha} 2^{\aleph_\delta} \leq \aleph_\alpha$, appears to be open.

In this paper it will be shown that given $\alpha > 0$ an η_α -group (i.e., a totally ordered Abelian divisible group which is an η_α -set) is universal for totally ordered Abelian groups of power not exceeding \aleph_α , that any two η_α -groups of power \aleph_α are isomorphic and finally that, given an η_α -set of power \aleph_α , η_α -groups of power \aleph_α exist.

BACKGROUND. Let α be an ordinal number. By $W(\alpha)$ is meant the set of all ordinal numbers δ such that $\delta < \alpha$. Let \aleph_α be a cardinal number. By ω_α is meant the least ordinal number such that $W(\omega_\alpha)$ is of power \aleph_α . A cardinal number \aleph_α is called regular if given an ordinal $\pi < \omega_\alpha$ and given a family of sets $(S_\delta)_{\delta < \pi}$, each of power less than \aleph_α , the power of $\bigcup_{\delta < \pi} S_\delta$ is less than \aleph_α .

Let T and T' be totally ordered sets and let f be a mapping of T into T' . We will call f order preserving (order reversing) if $x, y \in T$ and $x \leq y$ implies $f(x) \leq f(y)$, ($f(x) \geq f(y)$) and strictly order preserving (strictly order reversing) if f is order preserving (order reversing) and one-to-one. Let H and K be subsets of T . We will write $H < K$ if $h < k$ for every $h \in H$ and $k \in K$. Note, according to this definition, $\emptyset < K$ and $K < \emptyset$ for all subsets K of T including \emptyset . T is said to be an η_α -set if, given subsets H and K of T of power less than \aleph_α such that $H < K$ then there exists $t \in T$ such that $H < \{t\} < K$. Let T be an η_α -set and let H be a subset of T of power less than \aleph_α . By definition, $\emptyset < H$ and $H < \emptyset$. Thus there exist $u, v \in T$ such that $\emptyset < \{u\} < H$ and

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$H < \{v\} < \emptyset$: i.e., $\{u\} < H < \{v\}$. Hence, the definition of an η_α -set given above is equivalent to the classical definition. Let T be an η_α -set, let H and K be subsets of T of power less than \aleph_α and let $L = \{t \in T: H < \{t\} < K\}$. It is clear that $|L|$, the cardinal number of L , is not less than \aleph_α . The initial results on η_α -sets are to be found in [7]. For more recent results see [5] and [9].

Throughout this paper N will denote the set of positive integers, Z the integers and Q the rational numbers.

1. Let G be a group which is also a partially ordered set. G will be called a *right (left) partially ordered group* if given a, b and $c \in G$, $a < b$ implies $ac < bc$ ($ca < cb$). A group which is a right or a left partially ordered group may be referred to as a one-sided partially ordered group. A *partially ordered group* is a group which is both a right and a left partially ordered group. If G is a totally ordered set (i.e., a partially ordered set in which every pair of elements is order comparable), then "totally" may be substituted for "partially" in the definitions above.

Let T be a totally ordered set. Let $\Pi(T)$ denote the permutation group of T and let $A(T)$ denote the automorphism group of T (i.e., all strictly order preserving mappings of T onto T). The set $\Pi(T)$ may be partially ordered by defining $f < g$, $f, g \in \Pi(T)$, if $f(t) < g(t)$ for all $t \in T$. Under this order, the only order we will put on $\Pi(T)$, $\Pi(T)$ is a right partially ordered group. To show this let f, g and $h \in \Pi(T)$, $f < g$. Clearly $fh(t) = f(h(t)) < g(h(t)) = gh(t)$ for all $t \in T$; thus $fh < gh$. $A(T)$ is a subgroup of $\Pi(T)$ and therefore is a right partially ordered group. Furthermore, $A(T)$ is a left partially ordered group. To see this, let f, g and $h \in A(T)$, $f < g$. Remembering that h is necessarily strictly order preserving we see that $hf(t) = h(f(t)) < h(g(t)) = hg(t)$; thus $hf < hg$, proving the assertion above. Hence $A(T)$ is a partially ordered group.

Let G be a subgroup of a one-sided partially ordered group B . Let $C_B(G)$ denote the centralizer of G in B (i.e., $\{b \in B: bg = gb \text{ for all } g \in G\}$). Let $K_B(G) = \{b \in B: b \in C_B(G) \text{ and } b \text{ order comparable with each element of } G\}$.

PROPOSITION 1.1. *Let B be a one-sided partially ordered group and let G and H be Abelian subgroups of B such that $H \subset K_B(G)$. Then HG is a totally ordered Abelian subgroup of B .*

Proof. Clearly HG is Abelian; thus it is a partially ordered group. Let $t = hg$, $h \in H$ and $g \in G$. To show that HG is totally ordered it suffices to show that t is order comparable with 1. Since $h \in K_B(G)$, h is order comparable with g^{-1} ; hence hg is order comparable with 1, proving the proposition.

Let B be a group and let $S \subset B$. Let $n \in N$ and let $S^n = \{s^n: s \in S\}$. S will be called *divisible* if $S^n = S$ for all $n \in N$.

PROPOSITION 1.2. *Let B be a one-sided partially ordered group and let G be a totally ordered Abelian divisible subgroup of B . If $b \in K_B(G)$ then $b^Z \subset K_B(G)$.*

Proof. Let $n \in N$. Clearly $b^n \in C_B(G)$. Let $g \in G$ and let $h = g^{1/n}$. Since b is order comparable with h , b^n is order comparable with $h^n = g$; thus $b^n \in K_B(G)$. Clearly b^{-n} is also in $K_B(G)$. Hence $b^Z \subset K_B(G)$.

COROLLARY 1.1. *Let B be a one-sided partially ordered group and let G be a totally ordered Abelian divisible subgroup of B . If $b \in K_B(G)$ then Gb^Z is a totally ordered Abelian subgroup of B .*

That the condition of divisibility in Corollary 1.1 is crucial may be seen by considering the following example. Let

$$G = \{g_z: z \in Z \text{ and } g_z(x) = x + z, x \in R\}.$$

Clearly G is a totally ordered Abelian subgroup of $\Pi(R)$. It is also clear that G is not divisible. Let $b(x) = (1/8) \sin(2\pi x) + x + 1/2$. Clearly $b \in \Pi(R)$; in fact $b \in A(R)$. Furthermore $b \in K_{\Pi(R)}(G)$. However, $g_1(1/4) < b^2(1/4)$ and $b^2(3/4) < g_1(3/4)$. Thus b^2 is not order comparable with g_1 . Hence Gb^Z is not totally ordered.

PROPOSITION 1.3. *Let T be a totally ordered set and let H be a left totally ordered subgroup of $\Pi(T)$ which is transitive over T . Then $H \subset A(T)$.*

Proof. Let $x, y \in T, x < y$. Since H is transitive over $T, y = u(x)$ for some $u \in H$; thus $x < u(x)$. Since H is totally ordered, $1 < u$. Let $h \in H$. Since H is a left totally ordered group, $h1 < hu$; thus $h(x) < hu(x) = h(y)$. Hence $h \in A(T)$, proving the proposition.

COROLLARY 1.2. *Let T be a totally ordered set and let G be a totally ordered Abelian divisible subgroup of $\Pi(T)$. Let $b \in K_{\Pi(T)}(G)$ and assume that Gb^Z is transitive over T . Then Gb^Z is a totally ordered Abelian subgroup of $A(T)$.*

Proof. By Corollary 1.1, Gb^Z is a totally ordered Abelian subgroup of $\Pi(T)$. Since Gb^Z is transitive over T we may apply Proposition 1.3 and conclude that it is contained in $A(T)$.

PROPOSITION 1.4. *Let $(f_n)_{n \in N}$ be a sequence in a group G . Let $\phi(n) = l.c.m.(1, \dots, n)$ for each $n \in N$. Assume that $f_n^{\phi(n)/\phi(n-1)} = f_{n-1}$ for each $n \in N$ and let $H = \bigcup_{n \in N} f_n^Z$. H is Abelian and if it is torsion-free then it is divisible.*

Proof. Since $f_{n-1} \in f_n^Z, f_{n-1}^Z \subset f_n^Z$. Thus, H is the union of an ascending sequence of Abelian groups and is therefore Abelian. Clearly $f_1^{\phi(1)} = f_1$. Assume that $f_{n-1}^{\phi(n-1)} = f_1$. Clearly $f_n^{\phi(n)} = f_{n-1}^{\phi(n-1)} = f_1$. Let $k \in N$. We know that $\phi(k) = kt$ for some $t \in N$; thus $(f_k^t)^k = f_1$. Hence f_1 is divisible in H . Let $m, n \in N$. Thus there exists $x \in H$ such that $x^{m\phi(n)} = f_1$. We know that $f_1 = f_n^{\phi(n)}$; thus $x^{m\phi(n)} = f_n^{\phi(n)}$. Assume that H is torsion-free. Then $x^m = f_n$ and f_n is divisible by m ; thus H is divisible, proving the proposition.

Let α be an ordinal number. A one-sided partially ordered group B will

be called α -hyper-divisible if given a totally ordered Abelian divisible subgroup G of B of power less than \aleph_α then $K_B(G)$ is divisible.

PROPOSITION 1.5. *Let G be a totally ordered Abelian divisible subgroup of a one-sided partially ordered group B . Then $(K_B(G))^n \subset K_B(G)$ for all $n \in N$.*

Proof. Let $n \in N$ and let $f \in (K_B(G))^n$; thus $f = b^n$, $b \in K_B(G)$. By Proposition 1.2, $b^n \in K_B(G)$, proving the proposition.

COROLLARY 1.3. *For a one-sided partially ordered group B to be α -hyper-divisible it is necessary and sufficient that given a totally ordered Abelian divisible subgroup G of B , of power less than \aleph_α , $n \in N$ and $f \in K_B(G)$ there exists $t \in K_B(G)$ such that $f = t^n$.*

PROPOSITION 1.6. *Let G be a totally ordered Abelian divisible subgroup of power less than \aleph_α , of an α -hyper-divisible one-sided partially ordered group B and let $f \in K_B(G)$. There exists a totally ordered Abelian divisible subgroup G' of B such that $G \subset G'$, $f \in G'$ and $|G'| \leq \aleph_0 |G|$.*

Proof. Let $\phi(n) = \text{l.c.m.}(1, \dots, n)$, $n \in N$. Let $f_1 = f$ and assume f_{n-1} has been chosen in $K_B(G)$. Since B is α -hyper-divisible there exists $f_n \in K_B(G)$ such that $f_n^{\phi(n)/\phi(n-1)} = f_{n-1}$. Let $H = \bigcup_{n \in N} f_n^Z$.

Clearly H is Abelian. Since $f_n \in K_B(G)$, f_n is order comparable with 1; thus H is torsion-free. Hence, by Proposition 1.4, H is divisible. By Proposition 1.2, $f_n^Z \subset K_B(G)$; thus $H \subset K_B(G)$. Let $G' = HG$. By Proposition 1.1, G' is totally ordered and Abelian. Since both H and G are divisible, G' is divisible. Clearly $G \subset G'$. Since $f = f_1$, $f \in G'$. $|G'| \leq |H| \times |G| = \aleph_0 |G|$, proving the proposition.

Let H be an (additive) Abelian group and let U be a subgroup of H . If H is torsion-free then U is pure if and only if $u = ny$, $u \in U$, $n \in N$ and $y \in H$ implies $y \in U$. The following two propositions will be of use in §2.

PROPOSITION 1.7. *Let U be a subgroup of a torsion-free (additive) Abelian group H . There exists a subset U' of H such that:*

1. $|U'| \leq |U| \aleph_0$.
2. U' is the smallest pure subgroup of H containing U .
3. If f is an isomorphism of U into an (additive) Abelian divisible group G then f extends to a unique isomorphism F of U' into G .
4. If H and G are totally ordered, if G is Abelian and divisible and if f is a strictly order preserving isomorphism of U into G , then f can be extended to a strictly order preserving isomorphism F of U' into G . Further, F is unique.

Proof. Let $U' = \{h \in H : nh \in U \text{ for some } n \in N\}$. Clearly $|U'| \leq |U| \aleph_0$ and $U \subset U'$. Let $x, y \in U'$. Then there exists $n, m \in N$ such that $nx, my \in U$. Clearly $nm(x+y) = m(nx) + n(my) \in U$; thus $x+y \in U'$. Clearly $n(-x) \in U$; thus $-x \in U'$ and U' is a group. Let V be a pure subgroup of H which con-

tains U . We know that $nx \in U$; thus $nx \in V$. Since V is pure and H is torsion-free, $x \in V$; thus $U' \subset V$, proving 2. Proceeding to 3, let $F(x) = (1/n)f(nx)$. Since f is an isomorphism F is well defined. Clearly $F(x+y) = (1/nm)f(nm(x+y)) = (1/nm)f(nmx) + (1/nm)f(nmy) = F(x) + F(y)$. That F is unique may be seen by the following: let F' be an isomorphism of U' into G which extends f . Clearly $nF'(x) = F'(nx) = f(nx)$; thus $F'(x) = (1/n)f(nx) = F(x)$.

Given the conditions stated in 4, let F be the extension of f to U' . To show that F , an isomorphism, is a strictly order preserving mapping it suffices to show that $x > 0$ implies $F(x) > 0$. Since $x > 0$, $nx > 0$. Since f is strictly order preserving, $f(nx) > 0$. $F(x) = (1/n)f(nx) > 0$, proving the proposition.

The group U' will be referred to as the *purification of U in H* ⁽³⁾.

2. Let P and P' be partially ordered sets. A mapping h of P into P' will be called *l.u.b.-continuous* if given a nonempty totally ordered set $T \subset P$ such that l.u.b. T exists, then l.u.b. $h(T)$ exists and equals h (l.u.b. T). Clearly if h is l.u.b.-continuous then h is order preserving.

Let α be an ordinal number, let M be a set and let h be a mapping of a partially ordered set P into 2^M , the set of all subsets of M . Let $P_\alpha = \{p \in P: |h(p)| < \aleph_\alpha\}$. The pair (P, h) will be called α -*extendible* if given $p \in P_\alpha$ and $x \in M$ then there exists $p' \in P_\alpha$ such that $p \leq p'$ and $x \in h(p')$.

A partially ordered set in which every nonempty totally ordered subset has a least upper bound will be called *inductive*.

LEMMA 2.1. *Let P be an inductive partially ordered set and let h be a l.u.b.-continuous mapping of P into 2^M , where M is a set of power \aleph_δ . Let $\delta \leq \alpha$ and assume that \aleph_α is a regular cardinal. If $P_\alpha \neq \emptyset$ and if (P, h) is α -extendible, then there exists $f \in P$ such that $h(f) = M$.*

Proof. Since we are concerned exclusively with the cardinality of M we may let $M = W(\omega_\delta)$.

By an *admissible family* we shall mean a family $(f_\epsilon)_{\epsilon < \pi}$ of elements of P_α such that:

1. π is an ordinal number, $\pi \leq \omega_\delta$.
2. The mapping $\epsilon \rightarrow f_\epsilon$ is order preserving.
3. $W(\epsilon) \subset h(f_\epsilon)$ for all $\epsilon < \pi$.

By the *length* of an admissible family $(f_\epsilon)_{\epsilon < \pi}$ will be meant the ordinal π .

PROPOSITION 2.1. *If the length of an admissible family is π and if $\pi < \omega_\delta$ then the family can be extended to an admissible family of length $\pi + 1$.*

Proof. Let $(f_\epsilon)_{\epsilon < \pi}$ be an admissible family and assume that $\pi < \omega_\delta$.

Assume first that $\pi = 0$. Since $P_\alpha \neq \emptyset$ there exists $f_0 \in P_\alpha$. Clearly $W(0) = \emptyset \subset h(f_0)$.

⁽³⁾ The referee has observed that U' can be characterized as being the subgroup U' of H containing U such that U'/U is the torsion subgroup of H/U .

Assume now that $\pi > 0$ and that π is a limit ordinal. Let $f_\pi = \text{l.u.b. } (f_\epsilon)_{\epsilon < \pi}$. Since P is inductive f_π is a well defined element of P . Since h is l.u.b.-continuous, $h(f_\pi) = \bigcup_{\epsilon < \pi} h(f_\epsilon)$. Thus $|h(f_\pi)| \leq \sum_{\epsilon < \pi} |h(f_\epsilon)|$. For each $\epsilon < \pi$, $f_\epsilon \in P_\alpha$ thus $|h(f_\epsilon)| < \aleph_\alpha$. By assumption $\pi < \omega_\delta$ and $\omega_\delta \leq \omega_\alpha$; thus $\pi < \omega_\alpha$. Since \aleph_α is regular, $\sum_{\epsilon < \pi} |h(f_\epsilon)| < \aleph_\alpha$; thus $f_\pi \in P_\alpha$. Since π is a limit ordinal, $W(\pi) = \bigcup_{\epsilon < \pi} W(\epsilon)$. Further, $\bigcup_{\epsilon < \pi} W(\epsilon) \subset \bigcup_{\epsilon < \pi} h(f_\epsilon) = h(f_\pi)$; thus $W(\pi) \subset h(f_\pi)$.

Assume finally that $\pi > 0$ and that π is a nonlimit ordinal: i.e., $\pi = \epsilon + 1$, for some ordinal ϵ ; then $f_\epsilon \in P_\alpha$. Since (P, h) is α -extendible, there exists $f_\pi \in P_\alpha$ such that $f_\epsilon \leq f_\pi$ and $\epsilon \in h(f_\pi)$. Since h is l.u.b.-continuous, h is order preserving; thus $h(f_\epsilon) \subset h(f_\pi)$. Thus $W(\pi) = W(\epsilon) \cup \{\epsilon\} \subset h(f_\pi)$. Hence $W(\pi) \subset h(f_\pi)$.

Thus in each case $(f_\epsilon)_{\epsilon < \pi}$ can be extended to an admissible family of length $\pi + 1$, proving the proposition.

An admissible family of length π may be thought of as a mapping from $W(\pi)$ into P_α . Let Γ be the set of all admissible families. Let a, b be in Γ . We will write $a \subset b$ if b is an extension of a . Under inclusion Γ is partially ordered. Let T be a nonempty totally ordered subset of Γ and let $t = \bigcup_{a \in T} a$. Clearly $t \in \Gamma$; thus Γ is inductive. By Zorn's Lemma there exists a maximal element $m \in \Gamma$. Were the length of m less than ω_δ then, by Proposition 2.1, we could extend m to a larger admissible family, which is absurd. Hence m is of length ω_δ . Let $f = \bigcup_{\epsilon < \omega_\delta} m(\epsilon)$. Since P is inductive, f is in P . Since ω_δ is a limit ordinal, since h is l.u.b.-continuous and since condition 3 holds for m , $M = W(\omega_\delta) = \bigcup_{\epsilon < \omega_\delta} W(\epsilon) \subset \bigcup_{\epsilon < \omega_\delta} hm(\epsilon) = h(f)$. But $h(f) \subset M$; thus $h(f) = M$, proving the lemma.

A stronger version of Lemma 2.1 can be proved in which the condition that $\delta \leq \alpha$ and that \aleph_α is regular is replaced by the assumption that $\delta \leq cf(\alpha)$. (For a definition of $cf(\alpha)$ see, e.g., [10].)

COROLLARY 2.1. *Let M be a set of power \aleph_δ . Assume that \aleph_α is regular and that $\delta \leq \alpha$. Let h be an l.u.b.-continuous mapping of an inductive partially ordered set P into 2^M . If (P, h) is α -extendible then given $f_0 \in P_\alpha$ there exists $f \in P$ such that $f_0 \leq f$ and $h(f) = M$.*

Proof. Let $P' = \{p \in P; f_0 \leq p\}$. Clearly $P'_\alpha \neq \emptyset$. Further P' is an inductive partially ordered set. Restricting h to P' we see that it is l.u.b.-continuous. Clearly (P', h) is α -extendible. Thus, by Lemma 2.1, there exists $f \in P'$ such that $h(f) = M$. Since $f \in P'$, $f_0 \leq f$.

Let A and B be sets and let $A_0 \subset A$. A mapping f_0 of A_0 into B will be called a *partially defined mapping* of A into B . Let P be a set of partially defined mappings of A into B . Let Dp denote the domain of p and let Rp denote the range of p . Let $p, p' \in P$. If p' is an extension of p we will write $p \subset p'$. Under this order P is a partially ordered set. Further the mappings $p \rightarrow Dp$, $p \rightarrow Rp$ and $p \rightarrow (D \times R)p = (Dp) \times (Rp)$ of P into 2^A , 2^B and $2^A \times 2^B$,

respectively, are order preserving. Let T be a nonempty totally ordered subset of P . Let $t \in T$ and let $g(t)$ denote the graph of t . Under the order on P , given $p, p' \in P, p \subset p'$ if and only if $g(p) \subset g(p')$. Clearly $\bigcup_{t \in T} g(t)$ is the graph of a partially defined mapping of u of A into B ; u will be called the *union of T* . The set P will be called *union-inductive* if given a nonempty totally ordered subset T of P then the union of T is in P . The following is easily proved.

PROPOSITION 2.2. *Let P be a set of partially defined mappings of A into B , partially ordered by extension. If P is union-inductive then P is inductive and D, R and $D \times R$ are l.u.b.-continuous.*

THEOREM 2.1. *Let T and G be totally ordered (additive) Abelian groups, G being furthermore divisible and an η_α -set. Let P be the set of all partially defined strictly order preserving isomorphisms p of T into G for which Dp is a pure subgroup of T . Then P is union-inductive and (P, D) is α -extendible, provided $\alpha > 0$.*

Proof. Since T and G are totally ordered they are torsion-free. In particular 0 is a pure subgroup of T ; thus $P_\alpha \neq \emptyset$. It is well known (see, e.g., [8]) that the union of a nonempty totally ordered set of pure subgroups is itself pure; thus P is union-inductive.

Let $\alpha > 0$. It remains to show that (P, D) is α -extendible. Let $p \in P_\alpha$ and let $x \in T$. If $x \in Dp$ let $p' = p$. Assume that $x \notin Dp$. Let

$$\begin{aligned} H_n &= (1/n)p[] - \infty, nx[\cap Dp), \quad n \in N, \\ K_n &= (1/n)p[]nx, + \infty [\cap Dp), \quad n \in N, \end{aligned}$$

and let $H = \bigcup_{n \in N} H_n$ and let $K = \bigcup_{n \in N} K_n$.

We observe that $H < K$. To show this let $u \in H$ and let $u' \in K$. Clearly $u \in H_n$ and $u' \in K_{n'}$ for some $n, n' \in N$; thus $u = (1/n)p(t)$ and $u' = (1/n')p(t')$ for some t and t' in Dp such that $t < nx$ and $n'x < t'$. Clearly $n't < nn'x < nt'$. Thus $n'p(t) < np(t')$ and $u < u'$; hence we have proved that $H < K$.

Since $p \in P_\alpha, |Dp| < \aleph_\alpha$. Since $\alpha > 0, |H| + |K| \leq \aleph_0 |Dp| < \aleph_0 \aleph_\alpha = \aleph_\alpha$. Since G is an η_α -set there exists $y \in G$ such that $H < \{y\} < K$. Let $U = Dp + Zx$. Since Dp is pure the sum is direct, thus there exists a homomorphism p' of U into G which extends p and such that $p'(x) = y$.

Let $u \in U$ such that $u > 0$. Since Dp is pure u is uniquely expressible as $t + zx, t \in Dp$ and $z \in Z$. If $z = 0, p'(u) = p(t) > 0$, since p is strictly order preserving. Since $u > 0$, we know that $zx > -t$ and that $t > -zx$. If $z > 0$, using the first of these two inequalities, we see that $(1/z)p(-t) \in H_z \subset H < \{y\}$; thus $p(t) + zy > 0$. If $z < 0$, using the second inequality, we see that $(1/-z)p(t) \in K_{-z} \subset K > \{y\}$; thus $p(t) + zy > 0$. Hence $p'(u) > 0$, proving that p' is a strictly order preserving isomorphism of U into G .

The group U , which is Dp' , may not be a pure subgroup of T . Let V be the purification of U in T . By Proposition 1.7, p' extends to s , a strictly order

preserving isomorphism of V into G . Since V is pure in T , $s \in P$. Since $\alpha > 0$ and $p \in P_\alpha$, $|V| \leq |U| \aleph_0 = (|Dp| + \aleph_0) \aleph_0 < \aleph_\alpha$; thus $s \in P_\alpha$. Clearly $x \in U \subset V$ and $p \subset s$. Hence (P, D) is α -extendible, proving the theorem.

THEOREM A. *Let T be a totally ordered (additive) Abelian group of power not exceeding \aleph_α and let G be a totally ordered (additive) Abelian divisible group which is an η_α -set. Let f_0 be a strictly order preserving isomorphism of a subgroup U , of power less than \aleph_α , of T into G . If $\alpha > 0$ then f_0 extends to a strictly order preserving isomorphism of T into G .*

Proof. Assume $\alpha > 0$. From the existence of G we may assume that \aleph_α is regular⁽⁴⁾. Let T_0 be the purification of U in T and let p_0 be the extension of f_0 to T_0 . We know that $|T_0| \leq \aleph_0 |U| < \aleph_\alpha$, since $\alpha > 0$. Let P be the set of all partially defined strictly order preserving isomorphisms p of T into G for which Dp is a pure subgroup of T . Clearly $p_0 \in P_\alpha$. By Theorem 2.1, P is union-inductive and (P, D) is α -extendible. By Proposition 2.2, P is inductive and D is l.u.b.-continuous. By Corollary 2.1, there exists $f \in P$ such that $p_0 \subset f$ and $Df = T$: that is, f is an order preserving isomorphism of T into G which extends f_0 , and the theorem is proved.

Theorem A is not true if we allow α to be 0. To see this, consider the following example. Let $T = \{a + b2^{1/2} : a, b \in Q\}$. Clearly T is a countable set and yet it can not be mapped isomorphically into Q , an η_0 -set.

Let A and B be sets and let P be a set of partially defined mappings of A into B .

PROPOSITION 2.3. *If P is a set of partially defined one-to-one mappings of A into B such that (P, D) and (P^{-1}, D) are α -extendible then $(P, D \times R)$ is α -extendible.*

Proof. In the abstract case P_α is determined by P , α and h . Since the elements of P are one-to-one mappings, the sets P_α as defined by D , R and $D \times R$ are identical.

Let $p \in P_\alpha$ and let $(x, y) \in A \times B$. Since (P, D) is α -extendible, there exists $t \in P_\alpha$ such that $p \subset t$ and $x \in Dt$. Clearly $t^{-1} \in (P^{-1})_\alpha$. By hypothesis (P^{-1}, D) is α -extendible. Thus there exists $w \in (P^{-1})_\alpha$ such that $t^{-1} \subset w$ and such that $y \in Dw$. Since $(P^{-1})_\alpha = (P_\alpha)^{-1}$, $w^{-1} = s \in P_\alpha$. Thus $p \subset s$ and $(x, y) \in (D \times R)s$; hence the proposition is proved.

We can now prove the second of our major results.

THEOREM B. *Let G and G' be totally ordered (additive) Abelian divisible groups which are η_α -sets of power \aleph_α . Any order preserving isomorphism f_0 of a subgroup U of G of power less than \aleph_α into G' extends to an order preserving isomorphism of G onto G' , provided $\alpha > 0$.*

⁽⁴⁾ Hausdorff [7] has shown that if \aleph_α is singular then G is an $\eta_{\alpha+1}$ -set. $\aleph_{\alpha+1}$ is necessarily regular.

Proof. Assume $\alpha > 0$ and let P be the set of all partially defined order preserving isomorphisms p of G into G' for which Dp is a pure subgroup of G . Clearly $P_\alpha \neq \emptyset$. By Theorem 2.1, (P, D) is α -extendible. Let $p \in P$. Since Dp is a pure subgroup of G (a divisible group), Dp is divisible. Since p is an isomorphism of Dp onto Rp , Rp is divisible and is therefore pure. By Theorem 2.1, (P^{-1}, D) is α -extendible. By Proposition 2.3, $(P, D \times R)$ is α -extendible. Theorem 2.1 tells us that P is union-inductive. By Proposition 2.2, P is inductive and $D \times R$ is l.u.b.-continuous.

Let V be the purification of U in G and let p_0 be the extension of f_0 to V , which exists and is unique by Proposition 1.7. Clearly $p_0 \in P$. By Proposition 1.7, $|V| \leq |U| \aleph_\alpha$. By hypothesis $|U| < \aleph_\alpha$ and $\alpha > 0$; thus $p_0 \in P_\alpha$.

Since G is an η_α -set of power \aleph_α , \aleph_α is a regular cardinal. By Corollary 2.1, there exists $f \in P$ such that $p_0 \subset f$ and such that $(D \times R)f = G \times G'$: i.e., f extends f_0 and is an order preserving isomorphism of G onto G' , proving the theorem.

The remainder of this paper will be devoted to a proof of the existence of totally ordered (additive) Abelian divisible groups which are η_α -sets of power \aleph_α .

Let E be an η_α -set of power \aleph_α . As remarked earlier, \aleph_α is necessarily regular.

THEOREM 2.2. *Let G be a totally ordered Abelian subgroup of $A(E)$ of power less than \aleph_α and let $H, K \subset G$ such that $H \cup K = G$ and $H < K$. Let $\{H, K\}$ be the set of all partially defined strictly order preserving mappings p of E into E such that the following conditions hold:*

(2.1) *If $t \in Dp$ and $g \in G$ then $g(t) \in Dp$ and $gp(t) = pg(t)$. (Hence $GDp = Dp$.)*

(2.2) *If $t \in Dp$ then $H(t) < p(t) < K(t)$.*

Then $(\{H, K\}, D)$ is α -extendible.

Proof. Let $p \in \{H, K\}_\alpha$ and let $e \in E$. If $e \in Dp$ let $p' = p$. Assume that $e \notin Dp$. Since $H < K$, $H(e) < K(e)$. Let $U =]-\infty, e[\cap Dp$ and let $V =]e, +\infty[\cap Dp$. Since p is strictly order preserving, $p(U) < p(V)$. Let $g \in H$ and let $t \in V$. Since $e < t$, $g(e) < g(t)$. By condition (2.2), $g(t) < p(t)$; thus $g(e) < p(t)$ and therefore $H(e) < p(V)$. Similarly, $p(U) < K(e)$. Hence $H(e) \cup p(U) < K(e) \cup p(V)$. The sets on either side of this inequality are of power less than \aleph_α . Since E is an η_α -set, there exists a point $s(e)$ in E such that $H(e) \cup p(U) < \{s(e)\} < K(e) \cup p(V)$. By condition (2.1), since $e \notin Dp$, $G(e) \cap Dp = \emptyset$. Let s extend p and be defined on $G(e)$ as follows: $s(g(e)) = g(s(e))$ for all $g \in G$.

We wish to show that s is strictly order preserving. To show this let $x, x' \in Ds$, $x < x'$. Assume first that $x = g(e)$ and $x' = g'(e)$, $g, g' \in G$. Since G is totally ordered, g and g' are uniquely determined by x and x' . Since $x < x'$, $g < g'$; thus $s(x) = s(g(e)) = gs(e) < g's(e) = s(g'(e)) = s(x')$. Now let $x = g(e)$, $g \in G$, and $x' \in Dp$. Since $x < x'$, $e < g^{-1}(x')$. By condition (2.1), $g^{-1}(x') \in Dp$. Furthermore $g^{-1}(x') \in V$. We know that $s(e) < p(V)$; thus $s(e) < pg^{-1}(x')$. By

condition (2.1), $pg^{-1}(x') = g^{-1}p(x')$. Thus $s(x) = sg(e) = gs(e) < p(x') = s(x')$. If $x \in Dp$ and $x' = g(e)$, $g \in G$, then $g^{-1}(x) < e$. Thus $g^{-1}(x) \in U$ and $pg^{-1}(x) < s(e)$. Hence $s(x) < s(x')$. Since p is strictly order preserving and since s is an extension of p , $x, x' \in Dp$ implies $s(x) < s(x')$. Hence s is strictly order preserving.

Clearly s satisfies condition (2.1). As for condition (2.2), let $h \in H$ and $k \in K$ and let $t \in Ds$. We know that p satisfies condition (2.2); thus $t \in Dp$ implies $H(t) < \{s(t)\} < K(t)$. Let $t = g(e)$, $g \in G$. We know that $H(e) < \{s(e)\} < K(e)$; thus $h(e) < s(e) < k(e)$. Clearly $h(t) = hg(e) = gh(e) < gs(e) = s(t) < gk(e) = kg(e) = k(t)$. Hence s satisfies condition (2.2), and therefore $s \in \{H, K\}$.

We know that $Ds = Dp \cup G(e)$; thus $|Ds| \leq |Dp| + |G|$. Since $p \in P_\alpha$, $|Dp| < \aleph_\alpha$. By hypothesis $|G| < \aleph_\alpha$; thus $|Ds| < \aleph_\alpha$ and $s \in \{H, K\}_\alpha$. Clearly $e \in Ds$, thus the theorem is proved.

Continuing in the notation of Theorem 2.2, we have the following:

PROPOSITION 2.4. *The set $\{H, K\}$ of partially defined mappings of E into E is union-inductive.*

Proof. Let T be a nonempty totally ordered subset of $\{H, K\}$ and let s be the union of T . Let $x \in Ds$ and let $g \in G$. Clearly there exists $p \in T$ such that $x \in Dp$. Thus $g(x) \in Dp \subset Ds$ and $gs(x) = sg(x)$. Further $H(x) < s(x) < K(x)$; thus $s \in \{H, K\}$, proving the proposition.

PROPOSITION 2.5. $\{H, K\}^{-1} = \{K^{-1}, H^{-1}\}$.

Proof. The mappings in $\{H, K\}$ are strictly order preserving, therefore they are one-to-one mappings; thus $\{H, K\}^{-1}$ is well defined. Since G is a totally ordered group $K^{-1} \cup H^{-1} = G$ and $K^{-1} < H^{-1}$; thus $\{K^{-1}, H^{-1}\}$ is well defined and is, indeed, the set of all partially defined strictly order preserving mappings s of E into E such that (2.1) holds and such that the following holds:

(2.3) If $w \in Ds$ then $K^{-1}(w) < s(w) < H^{-1}(w)$.

STATEMENT 2.1. Let p be a partially defined one-to-one mapping of E into E which satisfies condition (2.1). Then p^{-1} satisfies condition (2.1).

Proof. Let $t \in Dp$, $g \in G$ and let $w = p(t)$. Clearly $w \in Rp = Dp^{-1}$. By condition (2.1), $g(t) \in Dp$ and $gp(t) = pg(t)$. Thus $g(w) = gp(t) = pg(t) \in Rp = Dp^{-1}$. Further, $g(w) = g(p^{-1}(g(w))) = pgp^{-1}(w)$; thus $p^{-1}g(w) = gp^{-1}(w)$. Hence p^{-1} satisfies condition (2.1), proving the statement.

STATEMENT 2.2. Let p be a partially defined one-to-one mapping of E into E . p satisfies condition (2.2) if and only if p^{-1} satisfies condition (2.3).

Proof. p satisfies condition (2.2), if and only if $H(t) < \{p(t)\} < K(t)$ for all $t \in Dp$: i.e., $K^{-1}(p(t)) < \{t\} < H^{-1}(p(t))$. Let $w = p(t)$. We have shown that p satisfies condition (2.2) if and only if p^{-1} satisfies condition (2.3), proving the statement.

Returning now to the thesis of Proposition 2.5, let $p \in \{H, K\}$. Clearly $p^{-1} \in \{H, K\}^{-1}$. According to Statement 2.1 and Statement 2.2, p^{-1} satisfies

conditions (2.1) and (2.3); thus $p^{-1} \in \{K^{-1}, H^{-1}\}$. Now let $s \in \{K^{-1}, H^{-1}\}$. Thus s satisfies conditions (2.1) and (2.3). By Statement 2.2, s^{-1} satisfies condition (2.2); thus $s^{-1} \in \{H, K\}$ and hence $s \in \{H, K\}^{-1}$, proving the proposition.

COROLLARY 2.2. *The pair $(\{H, K\}, D \times R)$ is α -extendible.*

Proof. According to Theorem 2.2, $(\{H, K\}, D)$ and $(\{K^{-1}, H^{-1}\}, D)$ are α -extendible. By Proposition 2.5, $\{K^{-1}, H^{-1}\} = \{H, K\}^{-1}$. Applying Proposition 2.3 we see that $(\{H, K\}, D \times R)$ is α -extendible, proving the corollary.

We now can prove a lemma which will be used explicitly in the proof of the existence theorem.

LEMMA 2.2. *Let E be an η_α -set of power \aleph_α , $\alpha > 0$, and let G be a totally ordered Abelian subgroup of $A(E)$ of power less than \aleph_α . Let x and y be in E . There exists $f \in K_{A(E)}(G)$ such that $y = f(x)$.*

Proof. If there exists $g \in G$ such that $y = g(x)$ then let $f = g$. Clearly $g \in K_{A(E)}(G)$. If no such g exists let $H = \{g \in G: g(x) < y\}$ and let $K = \{g \in G: g(x) > y\}$. Clearly $H \cup K = G$ and $H < K$. By Proposition 2.4, $\{H, K\}$ is union-inductive. By Proposition 2.2, $\{H, K\}$ is inductive and $D \times R$ is l.u.b.-continuous. By Corollary 2.2, $(\{H, K\}, D \times R)$ is α -extendible. Let $u(g(x)) = g(y)$, for each $g \in G$. Since G is a totally ordered group, u is strictly order preserving. Let $t \in Du$. Since $Du = G(x)$, $t = g'(x)$ for some $g' \in G$. Let $g \in G$. Clearly $g(t) = gg'(x) \in G(x) = Du$ and $gu(t) = gug'(x) = gg'(y) = ugg'(x) = ug(t)$; thus u satisfies condition (2.1). To show that u satisfies condition (2.2), it suffices to show that $H(g'(x)) < \{g'(y)\} < K(g'(x))$. Since G is Abelian it suffices to show that $H(x) < \{y\} < K(x)$, which is true; thus $u \in \{H, K\}$. By hypothesis $|G| < \aleph_\alpha$; thus $|Du| < \aleph_\alpha$ and $u \in \{H, K\}_\alpha$. By Corollary 2.1, there exists $f \in \{H, K\}$ such that $(D \times R)f = E \times E$ and such that $u \subset f$. Thus $f \in A(E)$ and, by conditions (2.1) and (2.2), $f \in K_{A(E)}(G)$. Further $y = u(x) = f(x)$, proving the lemma.

LEMMA 2.3. *Let E be an η_α -set of power \aleph_α and assume that $\alpha > 0$. The group $A(E)$ is an α -hyper-divisible partially ordered group.*

A proof of this lemma will be given in §3.

THEOREM C. *Let E be an η_α -set of power \aleph_α ; $\alpha > 0$. Given a totally ordered Abelian divisible subgroup G_0 of $A(E)$, of power less than \aleph_α , it is contained in a totally ordered Abelian divisible subgroup G of $A(E)$ which is simply transitive over E ; thus G is an η_α -set of power \aleph_α .*

Proof. Let P be the set of all totally ordered Abelian divisible subgroups of $A(E)$. Let P be ordered by inclusion. Let x be a fixed element in E . Let $v(G) = G(x)$ for each $G \in P$. Thus v maps P into 2^E . Clearly v is union-inductive; thus P is inductive and v is l.u.b.-continuous.

Let $G \in P_\alpha$ and let $y \in E$. By Lemma 2.2, there exists $f \in K_{A(E)}(G)$ such that $y = f(x)$. By Lemma 2.3, $A(E)$ is α -hyper-divisible. According to Proposition 1.6, there exists a totally ordered Abelian divisible subgroup G' of $A(E)$ such that $G \subset G'$, $f \in G'$ and $|G'| \leq |G| \aleph_0$. Thus $G' \in P_\alpha$ and $y \in G'(x) = v(G')$. Hence (P, v) is α -extendible.

By hypothesis $G_0 \in P_\alpha$. By Corollary 2.1, there exists $G \in P$ such that $v(G) = E$: i.e., $G(x) = E$. Let $e, e' \in E$. There exist $g, g' \in G$ such that $e = g(x)$ and $e' = g'(x)$; thus $g'g^{-1}(e) = e'$ proving that G is transitive over E . Since G is totally ordered, G is simply transitive over E . Finally, since G is totally ordered, G is order isomorphic with $G(x) = E$; thus G is a totally ordered Abelian divisible group which is an η_α -set of power \aleph_α , proving the theorem.

Having assumed the existence of an η_α -set of power \aleph_α it has been shown that totally ordered Abelian divisible groups which are η_α -sets of power \aleph_α exist.

3. It remains to prove Lemma 2.3. Let E be an η_α -set of power \aleph_α , $\alpha > 0$. We must prove that $A(E)$ is α -hyper-divisible. By Corollary 1.3, it suffices to prove that given a totally ordered Abelian divisible subgroup G of $A(E)$, of power less than \aleph_α , $n \in N$ and $f \in K_{A(E)}(G)$ then there exists $t \in K_{A(E)}(G)$ such that $f = t^n$.

If $n = 1$ then there is nothing to prove. Assume that $n > 1$. If $f \in G$ let $t = f^{1/n}$. Since G is divisible, such an element exists and is in G . Assume now that $f \notin G$. Since $f \in K_{A(E)}(G)$, f is order comparable with 1. Since $f \notin G, f \neq 1$; hence either f or f^{-1} is greater than 1. We may assume, without loss of generality, that $f > 1$.

Let P be the set of all partially defined strictly order preserving mappings p of E into E such that the following conditions hold:

- (3.1) p, f and the elements of G restricted to Dp are permutations of Dp .
- (3.2) $p \in K_{\Pi(Dp)}(G|Dp)^{(\ast)}$.
- (3.3) $f = p^n$ over Dp .

Clearly P is union-inductive. Thus P is inductive and D is l.u.b.-continuous. Our main task is to show that (P, D) is α -extendible. Having shown this, Lemma 2.1 will be invoked to prove the lemma.

Let $H = \{g \in G: g < f\}$ and let $K = \{g \in G: g > f\}$. Since $f \in K_{A(E)}(G)$ and $f \notin G, H \cup K = G$. Clearly $H < K$. Thus $H < \{f\} < K$. It is clear that (3.2) and (3.3) combine to give us:

- (3.3') $H^{1/n} < \{p\} < K^{1/n}$ over Dp .
- (By $S^{1/n}$ is meant $\{s^{1/n}: s \in S\}$, where $S \subset G$.)

Let $p \in P_\alpha$ and let $u \in E$. If $u \in Dp$ let $p' = p$. Assume that $u \notin Dp$. An *admissible sequence* is a sequence $(e_j)_{0 \leq j < m}$ of points in E such that:

- (3.4) $u = e_0$.
- (3.5) $H^{1/n}(e_{j-1}) < \{e_j\} < K^{1/n}(e_{j-1})$, for $0 < j < m$.
- (3.6) $x < e_{j-1} < x', x, x' \in Dp$ implies $p(x) < e_j < p(x')$, for $0 < j < m$.

(*) $G|Dp = \{g|Dp: g \in G\}$, where by $g|Dp$ is meant the restriction of g to Dp .

$$(3.7) \quad K^{(i-n)/n}(f(e_0)) < \{e_j\} < H^{(i-n)/n}(f(e_0)), \text{ for } 0 \leq j < m.$$

$$(3.8) \quad e_j \neq gf^r(e_i), \text{ for every } g \in G, r \in Z, \text{ for } 0 \leq i < j < m.$$

Let $(e_j)_{0 \leq j < m}$ be an admissible sequence. By the length of this sequence will be meant the number m . Clearly any initial segment of an admissible sequence is an admissible sequence.

PROPOSITION 3.1. *If $(e_j)_{0 \leq j < m}$ is an admissible sequence, then $e_j \notin Dp$, $0 \leq j < m$.*

Proof. By condition (3.4), $e_0 = u$ and $u \notin Dp$. Assume $e_{j-1} \notin Dp$, $0 < j < m$. Thus $\{x \in Dp: x < e_{j-1}\} \cup \{x \in Dp: x > e_{j-1}\} = Dp$. Since p is a permutation of Dp , $p\{x \in Dp: x < e_{j-1}\} \cup p\{x \in Dp: x > e_{j-1}\} = Dp$. By (3.6) e_j lies between these two sets whose union is Dp . Thus $e_j \notin Dp$, proving the proposition.

PROPOSITION 3.2. *Given an admissible sequence of length m , then $x < e_j < x'$, $x, x' \in Dp$, implies $p^{-i}(x) < e_0 < p^{-i}(x')$, $0 \leq j < m$.*

Proof. Clearly the statement is true for $j=0$. Let $0 < j < m$ and assume that the statement is true for $j-1$. Let $x \in Dp$ such that $x < e_j$ and assume for a moment that $p^{-1}(x) \geq e_{j-1}$. By Proposition 3.1, $e_{j-1} \notin Dp$. By (3.1), $p^{-1}(x) \in Dp$; thus $p^{-1}(x) > e_{j-1}$. By (3.6), $pp^{-1}(x) > e_j$; i.e., $x > e_j$, which is absurd. Thus $p^{-1}(x) < e_{j-1}$. We have assumed that this implies $p^{-(j-1)}p^{-1}(x) < e_0$; i.e., $p^{-i}(x) < e_0$. By a similar argument it can be shown that $e_j < x'$, $x' \in Dp$, implies $e_0 < p^{-i}(x')$. Thus the proposition is proved.

LEMMA 3.1. *There exists an admissible sequence of length n .*

A proof of Lemma 3.1 will be given in §4.

Henceforth let $(e_j)_{0 \leq j < n}$ be an admissible sequence of length n . Let m be an integer. Clearly m can be written as $rn+i$, $0 \leq i < n$, $r, i \in Z$. Further, r and i are uniquely determined by m . If $r \neq 0$ let e_m be defined as $f^r(e_i)$; thus for all integers m , $e_m = f^r(e_i)$.

PROPOSITION 3.3. *Let $y = g(e_m)$ and $y' = g'(e_{m'})$, $g, g' \in G$ and $m, m' \in Z$. $y = y'$ if and only if $g = g'$ and $m = m'$.*

Proof. Assume that $y = y'$, $m = rn+i$, $m' = r'n+i'$ and $0 \leq i, i' < n$. If $i \neq i'$ we may assume, without loss of generality, that $i > i'$. Clearly $e_i = g^{-1}g'f^{r-r'}(e_{i'})$, which is untenable by condition (3.8). Thus $i = i'$ and $f^{r-r'}(e_i) = g^{-1}g'(e_i)$. We know that $f \in K_{A(B)}(G)$. By Proposition 1.2, $f^Z \subset K_{A(B)}(G)$; thus $f^{r-r'} = g^{-1}g'$. Since $f \notin G$, $r-r' = 0$ and $g = g'$. If, on the other hand, $g = g'$ and $m = m'$ then $r = r'$ and $i = i'$; thus $y = y'$, proving the proposition.

Let $T = \{g(e_m): g \in G \text{ and } m \in Z\}$. Since T is a subset of E , a totally ordered set, it is a totally ordered set. Clearly f and the elements of G act as permutations on T as they do on Dp . By Proposition 3.1, $(e_j)_{0 \leq j < n} \cap Dp = \emptyset$;

thus $T \cap Dp = \emptyset$. Let $y \in T$. By Proposition 3.3, y can be written as $g(e_m)$ for one and only one pair (g, m) , where $g \in G$ and $m \in \mathbb{Z}$. Let p' extend p and let $p'(y) = g(e_{m+1})$ for each $y \in T$.

PROPOSITION 3.4. *p' is a partially defined mapping of E into E which satisfies conditions (3.1)–(3.3).*

Proof. Both Dp and T are subsets of E . Since p' acts as a permutation on both these sets, p' is a partially defined mapping of E into E . Further p', f and the elements of G , restricted to Dp' , are permutations of Dp' : i.e., p' satisfies condition (3.1).

Let $g, g' \in G$ and let $y = g(e_m), m \in \mathbb{Z}$. Clearly $p'g'(y) = p'g'g(e_m) = g'g(e_{m+1}) = g'p'g(e_m) = g'p'(y)$; thus $p'g' = g'p'$ over T . Hence $p' \in C_{\Pi(Dp')}(G|Dp')$. Let $h \in H^{1/n}$ and let $k \in K^{1/n}$. Let $m = rn + i, 0 \leq i < n$. If $i + 1 < n$ we may invoke condition (3.5) and conclude that $h(e_i) < e_{i+1} < k(e_i)$. If $i + 1 = n$ we may invoke condition (3.7) and conclude that $k^{-1}(f(e_0)) < e_{n-1} < h^{-1}(f(e_0))$; thus $h(e_{n-1}) < f(e_0) < k(e_{n-1})$. Remembering that $f(e_0) = e_n$ and that, in this case, $i = n - 1$, we see that $h(e_i) < e_{i+1} < k(e_i)$. Thus this holds for all $i, 0 \leq i < n$. Since $f \in K_{A(E)}(G)$ we may apply Proposition 1.2 and conclude that $f^r \in K_{A(E)}(G)$. Thus $h(f^r(e_i)) < f^r(e_{i+1}) < k(f^r(e_i))$. Since G is Abelian and totally ordered, $h(y) < p'(y) < k(y)$; hence $H^{1/n} < \{p'\} < K^{1/n}$ over T . By (3.3'), this also holds over Dp ; thus it holds over Dp' . We know that $H \cup K = G$. Since G is divisible, $H^{1/n} \cup K^{1/n} = G$. We conclude that p' is order comparable with the elements of G : i.e., $p' \in K_{\Pi(Dp')}(G|Dp')$ and p' satisfies condition (3.2). Clearly $(p')^n(y) = g(e_{m+n}) = gf(e_m) = fg(e_m) = f(y)$; thus p' satisfies condition (3.3), proving the proposition.

PROPOSITION 3.5. *The group $G' = G((p')^2)$ is transitive over T .*

Proof. It has been shown that $p' \in K_{\Pi(Dp')}(G|Dp')$. Let $x \in T$. Clearly x may be written as $g(e_m), g \in G$ and $m \in \mathbb{Z}$. Let $g(p')^m = a$. Clearly $a \in G'$ and $x = a(e_0)$. Let $y \in T$. By an argument similar to the preceding, there exists $b \in G'$ such that $y = b(e_0)$. Thus $y = ba^{-1}(x)$ and $ba^{-1} \in G'$, proving the proposition.

COROLLARY 3.1. *p' restricted to T is strictly order preserving.*

Proof. By Proposition 3.5 and Corollary 1.2, $G'|T \subset A(T)$; thus in particular, $p'|T$ is in $A(T)$. Hence p' is strictly order preserving over T , proving the corollary.

PROPOSITION 3.6. *Let $x, y \in Dp$ and $t \in T$. If $x < t < y$ then $p'(x) < p'(t) < p'(y)$.*

Proof. Let $t = g(e_m), g \in G$ and $m \in \mathbb{Z}$. Let $m = rn + i, 0 \leq i < n$; thus $t = f^r g(e_i)$. Let $v = g^{-1}f^{-r}(x)$ and $w = g^{-1}f^{-r}(y)$. Since $g^{-1}f^{-r}$ is strictly order preserving,

$v < e_i < w$. If $i = n - 1$ we may apply Proposition 3.2 and conclude that $p^{-i}(v) < e_0 < p^{-i}(w)$. Thus $p'(v) = p^{i+1}p^{-i}(v) = fp^{-i}(v) < f(e_0) < fp^{-i}(w) = p'(w)$. Since $i = n - 1$, $p'(e_i) = e_n = f(e_0)$; thus $p'(v) < p'(e_i) < p'(w)$. If $i < n - 1$ we may invoke condition (3.6) and conclude that $p'(x) < p'(t) < p'(y)$, proving the proposition.

COROLLARY 3.2. *p' is strictly order preserving.*

Proof. p' extends p , which is strictly order preserving; thus p' is strictly order preserving over Dp . By Corollary 3.1, p' is strictly order preserving over T . Combining these results with Proposition 3.6, we see that p' is strictly order preserving over $Dp \cup T = Dp'$, proving the corollary.

Hence $p' \in P$. Clearly $|Dp'| = |Dp| + |T|$. Since $p \in P_\alpha$, $|Dp| < \aleph_\alpha$. By hypothesis $|G| < \aleph_\alpha$ and $\alpha > 0$; thus $|T| = \aleph_0 |G| < \aleph_\alpha$. Hence $|Dp'| < \aleph_\alpha$ and $p' \in P_\alpha$. Furthermore p' extends p and $u = e_0 \in T \subset Dp'$; thus (P, D) is α -extendible. Clearly the empty mapping is in P ; thus $P_\alpha \neq \emptyset$.

By Lemma 2.1, there exists $t \in P$ such that $Dt = E$. By condition (3.1), t is a permutation of Dt : i.e., t is a permutation of E . Since $t \in P$, t is strictly order preserving; thus, by condition (3.2), $t \in K_{A(B)}(G)$. By condition (3.3), $f = t^n$ over E ; thus Lemma 2.3 is proved.

4. The proof of Lemma 3.1 will be given in this section.

Let $e_0 = u$ and consider the sequence $\{e_0\}$. Clearly this sequence satisfies condition (3.4). In this case conditions (3.5), (3.6) and (3.8) are vacuous. In this case condition (3.7) states that $K^{-1}(f(e_0)) < \{e_0\} < H^{-1}(f(e_0))$; this is true since $H < \{f\} < K$. Thus $\{e_0\}$ is an admissible sequence of length 1.

LEMMA 4.1. *Any admissible sequence of length m can be extended to an admissible sequence of length $m + 1$, provided $1 \leq m < n$.*

Granting Lemma 4.1 for a moment, we see that Lemma 3.1 is proved.

LEMMA 4.2. *Let $(e_j)_{0 \leq j < m}$ be an admissible sequence of length m , $1 \leq m < n$. Then*

$$L_0(m) = H^{1/n}(e_{m-1}) \cup p\{x \in Dp : x < e_{m-1}\} \cup K^{(m-n)/n}(f(e_0)) \\ < K^{1/n}(e_{m-1}) \cup p\{x \in Dp : x > e_{m-1}\} \cup H^{(m-n)/n}(f(e_0)) = L_1(m).$$

This lemma will be proved shortly. Clearly $|L_i(m)| \leq |G| + |Dp| + |G| < \aleph_\alpha$ for $i = 0, 1$. Since E is an η_α -set, there exist at least \aleph_α points $x \in E$ such that $L_0(m) < \{x\} < L_1(m)$. The set $S(m) = \{gf^r(e_i) : g \in G, r \in Z \text{ and } 0 \leq i < m\}$ is of power $|G| \aleph_0$. Since $\alpha > 0$, $|S(m)| < \aleph_\alpha$. Thus there exists e_m in E such that $L_0(m) < \{e_m\} < L_1(m)$ and such that $e_m \notin S(m)$: i.e., $(e_j)_{0 \leq j < m+1}$ is an admissible sequence of length $m + 1$, proving Lemma 4.1.

It remains to prove Lemma 4.2. $L_0(m)$ is the union of three sets, as is $L_1(m)$. It will be shown, in Propositions 4.1 through 4.9, that each of the three sets, considered in Lemma 4.2, whose union is $L_0(m)$ is less than each of the

three sets whose union is $L_1(m)$. Having shown this, Lemma 4.2 will be proved.

PROPOSITION 4.1. $H^{1/n}(e_{m-1}) < K^{1/n}(e_{m-1})$.

This is true since $H < K$ and $n > 0$.

PROPOSITION 4.2. *Given $h \in H^{1/n}$ and $x \in Dp$ such that $e_{m-1} < x$ then $h(e_{m-1}) < p(x)$.*

Proof. By (3.3'), $H^{1/n} < \{p\}$; thus $h(e_{m-1}) < h(x) < p(x)$, proving the proposition.

By symmetry we also have:

PROPOSITION 4.3. *Given $k \in K^{1/n}$ and $x \in Dp$ such that $e_{m-1} > x$, then $k(e_{m-1}) > p(x)$.*

PROPOSITION 4.4. $H^{1/n}(e_{m-1}) < H^{(m-n)/n}(f(e_0))$.

Before proving this it will be useful to establish the following:

STATEMENT 4.1. If $q, s \in Q$ such that $qs > 0$ then $H^q H^s = H^{q+s}$ and $K^q K^s = K^{q+s}$.

Proof. $H^q H^s = \{x^q y^s : x, y \in H\}$; thus $H^{q+s} \subset H^q H^s$. Let $x, y \in H$, $x \leq y$. Since $qs > 0$, q and s have the same sign; thus $x^q y^s$ lies between x^{q+s} and y^{q+s} , both of which are in H^{q+s} . Since G is divisible, H^{q+s} is an interval of G ; thus $x^q y^s \in H^{q+s}$. Hence $H^q H^s \subset H^{q+s}$. A similar argument can be used to prove the second assertion, proving the statement.

Proof of Proposition 4.4. Let $h \in H^{1/n}$ and let $h' \in H^{(m-n)/n}$. By Statement 4.1, $h^{-1}h' \in H^{(m-1-n)/n}$. By condition (3.7), $\{e_{m-1}\} < H^{(m-1-n)/n}(f(e_0))$: i.e., $e_{m-1} < h^{-1}h'f(e_0)$. Hence $h(e_{m-1}) < h'f(e_0)$, proving the proposition.

By symmetry we also have:

PROPOSITION 4.5. $K^{(m-n)/n}(f(e_0)) < K^{1/n}(e_{m-1})$.

PROPOSITION 4.6. *Given $x, x' \in Dp$ such that $x < e_{m-1} < x'$, then $p(x) < p(x')$.*

This is true since p is strictly order preserving.

PROPOSITION 4.7. *Given $h \in H^{(m-n)/n}$ and $x \in Dp$ such that $x < e_{m-1}$, then $p(x) < hf(e_0)$.*

Proof. Applying Proposition 3.2 we see that $p^{1-m}(x) < e_0$. By condition (3.3'), $H^{1/n} < \{p\}$. Further $m < n$; thus $\{p^{m-n}\} < H^{(m-n)/n}$. Hence $p^{m-n} = p^{mf-1}$ implies $p^m < hf$. Clearly $p(x) = p^m p^{1-m}(x) < hf p^{1-m}(x) < hf(e_0)$, proving the proposition.

By symmetry we have:

PROPOSITION 4.8. *Given $x \in Dp$ and $k \in K^{(m-n)/n}$ such that $x > e_{m-1}$, then $p(x) > kf(e_0)$.*

Since $H < K$ and $m < n$ we have:

PROPOSITION 4.9. $K^{(m-n)/n}(f(e_0)) < H^{(m-n)/n}(f(e_0))$.

Thus Lemma 4.2 is proved.

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