THE DUAL SPACES OF C^*-ALGEBRAS(1)

BY

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Introduction. The idea of the structure space (or dual space) $\hat{A}$ of an associative algebra A was introduced by Jacobson in [8]. The space $\hat{A}$ consists of all kernels of irreducible representations of A, with the hull-kernel topology: An ideal I in $\hat{A}$ is in the closure of a subset B of $\hat{A}$ if I contains the intersection of the ideals in B. For unrestricted infinite-dimensional A, the dual space need not be Hausdorff or even T_1; and in many situations it is not very useful. However, Gelfand and others have shown that for commutative Banach algebras the dual space is a powerful tool. For noncommutative Banach algebras, too, the study of the dual space has been found fruitful. Kaplansky [12] has analyzed the dual spaces of C^*-algebras whose irreducible *-representations all consist of completely continuous operators. The importance of this study is emphasized by the fact that the group algebras of connected semi-simple Lie groups having faithful matrix representations all fall into this category (see [7]).

This paper deals with some questions concerning the dual spaces of noncommutative C^*-algebras, especially the group C^*-algebras of certain groups. The contents of its three chapters are as follows:

Chapter I centers around the equivalence theorem (Theorem 1.2)(2). This is a theorem specifically about C^*-algebras. It states that, if $\mathcal{S}$ is a family of *-representations of a C^*-algebra A, and T is a *-representation of A which vanishes for those elements for which all S in $\mathcal{S}$ vanish, then positive functionals associated with T are weakly* approximated by sums of positive functionals associated with $\mathcal{S}$. In another form, it states a one-to-one correspondence between closed two-sided ideals of a C^*-algebra and certain subsets of the positive cone of its conjugate space. In the latter form, the theorem was communicated to this author by R. Prosser, who also suggested the short proof of Theorem 1.1 given here. An interesting corollary of this theorem is the following: If G is a locally compact group, the hull-kernel topology of the dual space of its group C^*-algebra is equivalent to the topology which Godement defined in [5] for the space $\hat{G}$ of irreducible unitary representations of G, using functions of positive type. Let us refer to this simply as the topology of $\hat{G}$.

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(2) A theorem essentially identical with our equivalence theorem has been proved by Takenouchi (Theorem 1 of [20]).

The equivalence theorem leads naturally to the ideas of weak containment and weak equivalence. Theorem 1.6 shows that every set of representations of a C^*-algebra A is weakly equivalent to a unique closed set of irreducible representations (compare the definition of the spectrum of a positive functional, in [5, p. 43]). Theorem 1.7 relates weak equivalence to the construction of continuous direct integrals of representations. Theorem 1.8 is a digression, and relates the topology of $\hat{A}$ to the condition that a discrete direct sum of completely continuous representations be completely continuous.

It should be noted that in this paper the elements of $\hat{A}$ are the (topologically) irreducible *-representations of A, rather than the kernels of these.

Chapter II arose from Kaplansky's observation [12, Theorem 4.1] that the Hausdorff property of the dual space $\hat{A}$ of a C^*-algebra A is related to the continuity of the real-valued functions $T \rightarrow \|T_x\|$ ($T \in \hat{A}$, x fixed). We ask what is the relation between the topology of $\hat{A}$ and the functions $T \rightarrow \|T_x\|$ for an arbitrary C^*-algebra A. The answer is Theorem 2.1. We also ask how the topology of $\hat{A}$ is related to the functions $T \rightarrow \text{Trace}\ (T_x)$ (supposing that Trace (T_x) exists for many x and T). Theorems 2.2 and 2.3 are partial answers to this question. In §10, we generalize Kaplansky's result [12, Theorem 4.2] that a C^*-algebra A all of whose irreducible representations have the same finite dimension has a Hausdorff dual space. In fact, we show that if $\{T^i\}$ is a net of elements of $\hat{A}$, all of which are of dimension $\leqq n$, and if $T^i \rightarrow S^m$ ($m = 1, \cdots, r$), where $S^1, \cdots, S^r$ are distinct elements of $\hat{A}$, then $\sum_{m=1}^{r} \dim S^m \leqq n$.

Chapter III applies the results of Chapter II to calculate explicitly the topologies of the duals of the $n \times n$ complex unimodular groups G (all of whose irreducible representations together with their characters, are listed in [4]). The result is Theorem 3.1. The topologies are not Hausdorff, though their deviations from this property are rather weak (see Corollaries 2 and 6 of Theorem 3.1). To illustrate, we recall that in the 2×2 case the elements of $\hat{G}$ fall into three classes: (i) the principal series of representations $T^{m,r}$ (m an integer, r real), (ii) the supplementary series of representations T^s $(0 < s < 1)$, and (iii) the identity representation I. Now the topology of $\hat{G}$ is the natural topology of the parameters with one exception: as $s \rightarrow 1-$, T^s converges both to I and to $T^{2,0}$. This failure of the Hausdorff property stems from the behavior of the characters. If $\gamma^{m,r}$, γ^s, γ^I are the characters of $T^{m,r}$, T^s, and I respectively, it arises from the fact that

$$\lim_{s \rightarrow 1-} \gamma^s = \gamma^I + \gamma^{2,0}.$$

A further fact about $\hat{G}$, true for all n, is that each principal series is closed in $\hat{G}$. This has the interesting consequence (Theorem 3.2) that the regular representation of G weakly contains the representations of the principal nondegenerate series, and no others.

Chapter I. Weak containment and the equivalence theorem

1. The equivalence theorem.

Theorem 1.1[3]. *Let A be any norm-closed self-adjoint algebra of operators on a Hilbert space H. Then any continuous positive linear functional ϕ on A can be approximated in the weak* topology (i.e., pointwise on A) by natural positive functionals on A, that is, positive functionals ψ of the form*

$$\psi(a) = \sum_{i=1}^{k} (ax_i, x_i) \qquad (x_i \in H).$$

In fact, the approximating functionals ψ may be assumed to have norm equal to or less than $\|\phi\|$.

Proof. Let N be the family of all natural positive functionals ψ on A for which $\|\psi\| \leqq 1$, considered as a subset of the conjugate space of the real Banach space B of all Hermitian elements of A. Then N is a convex set containing 0. We verify that the polar set $N_\pi = \{a \in B \mid \psi(a) \geqq -1 \text{ for } \psi \text{ in } N\}$ (see [2, p. 17]) consists of those a in B whose negative part a_- satisfies $\|a_-\| \leqq 1$; and hence that the "bipolar" $(N_\pi)^\pi = \{\psi \in B^* \mid \psi(a) \geqq -1 \text{ for } a \text{ in } N_\pi\}$ consists of *all* positive functionals ψ with $\|\psi\| \leqq 1$. Applying the theorem on "bipolars," [2, (8), p. 20], which says that $(N_\pi)^\pi$ is the weak* closure of N, we conclude that every positive functional ϕ with $\|\phi\| \leqq 1$ is a weak* limit of natural positive functionals ψ with $\|\psi\| \leqq 1$. Q.E.D.

If A is a C^*-algebra, a **-representation* T of A is a homomorphism of A into the bounded operators on some Hilbert space $H = H(T)$, involution in A going into the adjoint operation. A positive functional ϕ on A is *associated with* a *-representation T if there is an x in $H(T)$ for which $\phi(a) = (T_a x, x)$ $(a \in A)$; ϕ is *associated with* a family $\mathcal{S}$ of *-representations if it is associated with some S in $\mathcal{S}$.

Theorem 1.2. *Let A be any C^*-algebra, T a *-representation of A, and $\mathcal{S}$ a family of *-representations of A. The following four conditions are equivalent:*

(i) *The kernel J of T contains the intersection I of the kernels of the representations in $\mathcal{S}$;*

(ii) *Every positive functional on A associated with T is a weak* limit of finite linear combinations of positive functionals associated with $\mathcal{S}$;*

(iii) *Every positive functional on A associated with T is a weak* limit of finite sums of positive functionals associated with $\mathcal{S}$;*

(iv) *Every positive functional ϕ on A associated with T is a weak* limit of finite sums ψ of positive functionals associated with $\mathcal{S}$ for which $\|\psi\| \leqq \|\phi\|$.*

Proof. It is trivial that (iv)→(iii)→(ii). Assume (ii), and let a belong to

[3] This theorem is the same as Lemma 2.1 of [20]. The proof given here was communicated to the author by R. Prosser.

$I-J$. Then there is a positive functional ϕ associated with T for which $\phi(a)\neq 0$. But, by (ii), ϕ is a weak* limit of linear combinations ψ of functionals associated with $\mathcal{S}$; for such ψ, $\psi(a)=0$. Hence $\phi(a)=0$; and we have a contradiction. Thus $I\subset J$; and we have shown that (ii) implies (i).

Now assume (i), and let $S^0=\sum_{S\in\mathcal{S}}\oplus S$. Since $J\supset I=\text{Kernel }(S^0)$, a positive functional ϕ associated with T vanishes on I, and hence induces a continuous positive functional ϕ' on A/I, which may be identified with the range $S^0(A)$ of S^0. Since the latter is norm-closed, apply Theorem 1.1 and approximate ϕ' weakly* by sums ψ' of natural positive functionals on $S^0(A)$ for which $\|\psi'\|\leq\|\phi'\|$. Thus, passing back to A, we approximate ϕ weakly* by sums ψ of positive functionals associated with S^0, for which $\|\psi\|\leq\|\phi\|$. But each positive functional associated with S^0 is itself a norm-limit of sums of positive functionals associated with $\mathcal{S}$. The last two statements combine to give (iv). The proof of the Theorem is now complete.

This theorem will be referred to as the equivalence theorem. If $\mathcal{S}$ and T are such as to satisfy conditions (i)–(iv) of the theorem, we shall say that T is *weakly contained* in $\mathcal{S}$.

The equivalence of (i) and (ii) is evidently valid for any Banach *-algebra. However, the equivalence of (i) and (iii) or (iv) depends on the special properties of C^*-algebras; it fails, for example, for the L_1 group algebras of certain groups (see remark following Theorem 3.2).

In case T is a cyclic representation, with cyclic vector ξ (that is, the $T_a\xi$, $a\in A$, are dense in $H(T)$), each condition of the equivalence theorem is equivalent to:

(ii′) *Merely the positive functional $\phi_0(a)=(T_a\xi,\xi)$ is a weak* limit of finite linear combinations of positive functionals associated with* $\mathcal{S}$.

Indeed: Assume (ii′); and let $\eta\in H(T)$, $\phi(a)=(T_a\eta,\eta)$. Choose y in A so that $\|T_y\xi-\eta\|$ is small. Then ϕ', defined by $\phi'(a)=(T_aT_y\xi, T_y\xi)=\phi_0(y^*ay)$, approximates ϕ in the norm, hence weakly*. Pick a net $\{\psi_i\}$ of linear combinations of positive functionals associated with $\mathcal{S}$ so that $\psi_i\rightarrow\phi_0$ weakly*. If $\psi_i'(a)=\psi_i(y^*ay)$, then $\{\psi_i'\}$ is again a net of linear combinations of positive functionals associated with $\mathcal{S}$, and converges weakly* to ϕ', i.e., to a functional approximating ϕ weakly*. Thus, by the arbitrariness of ϕ, (ii) holds.

2. **Adjunction of a unit.** Let A be a C^*-algebra without unit, and A_1 the C^*-algebra obtained by adjoining a unit 1 to A (see [1, p. 275] or [18, p. 207]). A *-representation T of A is *nowhere trivial* if $\xi=0$ whenever $T_a\xi=0$ for all a in A, or, equivalently, if the linear span of the $T_a\xi$ ($a\in A$, $\xi\in H(T)$) is dense in $H(T)$. To each *-representation T of A, let T^1 be the *-representation of A_1 coinciding with T on A, and for which $T^1(1)$ is the identity operator in $H(T)$. If $\mathcal{S}$ is a family of *-representations of A, $\mathcal{S}^1$ will mean $\{T^1 \mid T\in\mathcal{S}\}$.

If I is a closed two-sided ideal of A, let $I^1=\{a+\lambda\cdot 1 \mid a\in A,\ \lambda \text{ complex};\ ay+\lambda y\in I \text{ for all } y \text{ in } A\}$.

LEMMA 1.1. *I^1 is a closed two-sided ideal of A_1 with $A\cap I^1=I$. In fact, if*

*T is a nowhere trivial *-representation of A with kernel I, then I^1 is the kernel of T^1.*

Proof. There exists a nowhere trivial *-representation T of A with kernel I. We have $T^1_{a+\lambda\cdot 1}=0$ if and only if $0=T^1_{a+\lambda\cdot 1}T_y\xi=T_{ay+\lambda y}\xi$ for all y in A and ξ in $H(T)$, i.e., if and only if $ay+\lambda y\in I$ for all y in A. Thus $I^1=\text{Kernel }(T^1)$.

LEMMA 1.2. *If $\mathcal{J}$ is a family of closed two-sided ideals of A, and J is a closed two-sided ideal of A, then $J\supset\bigcap_{I\in\mathcal{J}} I$ if and only if $J^1\supset\bigcap_{I\in\mathcal{J}} I^1$.*

Proof. If the second condition holds, intersect it with A to get the first (using Lemma 1.1). Let the first condition hold; and suppose $a+\lambda\cdot 1\in\bigcap_{I\in\mathcal{J}} I^1$. Then $ay+\lambda y\in I$ for all I in $\mathcal{J}$ and y in A; so that by the first condition $a+\lambda\cdot 1\in J^1$.

Combining Lemmas 1.1 and 1.2, one obtains:

LEMMA 1.3. *If T is a nowhere trivial *-representation of A, and $\mathcal{S}$ is a family of nowhere trivial *-representations of A, then T is weakly contained in $\mathcal{S}$ if and only if T^1 is weakly contained in $\mathcal{S}^1$.*

3. **Application to groups.** Let G be a locally compact topological group with unit element e. Its group algebra $L_1(G)$ with respect to left-invariant Haar measure is a Banach *-algebra, and there is a natural one-to-one correspondence between the unitary equivalence classes of unitary representations of G and those of the nowhere trivial *-representations of $L_1(G)$ (see [14]). In this correspondence irreducible representations of G correspond to irreducible representations of $L_1(G)$, and vice versa.

Now introduce into $L_1(G)$ a new norm $\|\ \|_c$ defined by

$$\|x\|_c = \sup_T \|T_x\|,$$

where T runs over all *-representations of $L_1(G)$. (This is the minimal regular norm; see [16], or [18, p. 235].) The completion of $L_1(G)$ under $\|\ \|_c$ is a C^*-algebra called $C^*(G)$, the *group C^*-algebra* of G. The correspondence between representations of G and of $L_1(G)$ carries over into an exactly similar correspondence between unitary representations of G and *-representations of $C^*(G)$, irreducible representations of one corresponding to irreducible representations of the other.

If T is a unitary representation of G, and $\mathcal{S}$ is a family of unitary representations of G, we say that T is *weakly contained* in $\mathcal{S}$ if this is the case when T and $\mathcal{S}$ are considered as representations of $C^*(G)$.

We shall now show that, in the case of groups, the weak containment relation can be defined in terms of the uniform convergence on compacta of functions of positive type. The essential argument for this is given in [5].

We observe first that the continuous positive functionals on $C^*(G)$ and on $L_1(G)$ are essentially the same.

LEMMA 1.4. *The restriction map is a one-to-one norm-preserving map of the set of all continuous positive functionals on $C^*(G)$ onto the set of all continuous positive functionals on $L_1(G)$.*

Proof. It follows almost immediately from the definition of $C^*(G)$ that the restriction map is one-to-one and onto (see [16] or [18]). We need only prove that it preserves norm.

If x is a non-negative function in $L_1(G)$, and I is the one-dimensional identity representation of G, we have $\|x\|_c \geqq \|I_x\| = \int_G x(g)dg = \|x\|_{L_1(G)} \geqq \|x\|_c$. Hence

$$\|x\|_{L_1(G)} = \|x\|_c \qquad \text{for } x \in L_1(G),\ x \geqq 0. \tag{1}$$

Now let $\{U_i\}$ be a net of compact neighborhoods of e converging to e; and let x_i be a continuous non-negative function on G, vanishing outside U_i, with $\int_G x_i(g)dg = 1$. By (1), $\|x_i\|_{L_1(G)} = \|x_i\|_c = 1$; hence $\{x_i\}$ is an approximate identity satisfying $\|x_i\| = 1$ in both $L_1(G)$ and $C^*(G)$. If f is a continuous positive linear functional on $C^*(G)$, and f' is its restriction to $L_1(G)$, we have (see [18, p. 172])

$$\begin{aligned}\|f'\| &= \sup_i f'(x_i^* * x_i) = \sup_i f(x_i^* * x_i)\\ &= \|f\|.\end{aligned}$$

Thus the restriction mapping preserves the norm.

By Lemma 1.4, the norm of a continuous positive functional f is the same whether f acts on $L_1(G)$ or on $C^*(G)$. If $\{f_i\}$ is a net of such functionals, with uniformly bounded norm $\|f_i\|$, then weak* convergence of $\{f_i\}$ to f means the same with respect to $L_1(G)$ as it does with respect to $C^*(G)$.

If T is a unitary representation of G, and $0 \neq \xi \in H(T)$, the function F on G defined by $F(g) = (T_g\xi, \xi)$ is a *function of positive type associated with* T. If $\mathcal{S}$ is a family of unitary representations of G, a function of positive type is *associated with* $\mathcal{S}$ if it is associated with some T in $\mathcal{S}$. Functions F of positive type are extensively investigated in [5]. They are bounded and continuous, with $F(e) = \sup_{g \in G} |F(g)|$. Considered as elements of $L_\infty(G)$, or the dual of $L_1(G)$, they are precisely the positive continuous linear functionals on $L_1(G)$.

A family Φ of functions of positive type on G will be said to be *closed invariant* if:

(i) Φ is closed in the topology of uniform convergence on compacta;

(ii) if $\phi \in \Phi$, n is a positive integer, $r_1, \cdots, r_n$ are complex numbers, $h_1, \cdots, h_n$ are elements of G, and ψ is defined on G by

$$\psi(g) = \sum_{i,j=1}^{n} \bar{r}_j r_i \phi(h_j^{-1} g h_i),$$

then $\psi \in \Phi$.

Now by an argument based on Gelfand's lemma on the weak* conver-

gence of functionals, and similar to that used for the proof of Lemma C, [5, p. 43], we derive the following:

LEMMA 1.5. *If Φ is a closed invariant family of functions of positive type, and if $\{\phi_i\}$ is a net of elements of Φ such that:*
(i) *$\|\phi_i\|$ is bounded uniformly in i;*
(ii) *$\phi_i \to \phi$ weakly* (as elements of $(L_1(G))^*$), then $\phi \in \Phi$.*

THEOREM 1.3. *If G is a locally compact group, T is a unitary representation of G, and $\mathcal{S}$ is a family of unitary representations of G, then T is weakly contained in $\mathcal{S}$ if and only if every function of positive type on G associated with T can be approximated uniformly on compact sets by sums of functions of positive type associated with $\mathcal{S}$.*

Proof. The "if" part of the theorem follows easily from Lemma 1.4 and the equivalence theorem. To prove the converse, suppose that T is weakly contained in $\mathcal{S}$; and let F be a function of positive type associated with T, corresponding to the positive functional ϕ on $C^*(G)$. By the equivalence theorem

$$\phi_i \to \phi \text{ weakly*}, \tag{2}$$

where each ϕ_i is a sum of positive functionals on $C^*(G)$ associated with $\mathcal{S}$, and the $\|\phi_i\|$ are uniformly bounded in i. Let F_i be the function of positive type corresonding to ϕ_i.

We define Φ to be the set of all uniform-on-compacta limits of sums of functions of positive type associated with $\mathcal{S}$. It is easy to verify that Φ is closed invariant. Now $F_i \in \Phi$, and, by (2) and Lemma 1.4,

$$F_i \to F \text{ weakly* } (\text{in}(L_1(G))^*).$$

Also, by Lemma 1.4, the $\|F_i\|$ are uniformly bounded in i. Applying Lemma 1.5, we conclude that $F \in \Phi$, which completes the proof of the theorem.

COROLLARY. *If G, T, $\mathcal{S}$ are as in the theorem, T is weakly contained in $\mathcal{S}$, and H is a closed subgroup of G, then the restriction of T to H is weakly contained in the family of all restrictions to H of members of $\mathcal{S}$.*

4. **The dual space.** The relation of weak containment, applied to irreducible representations, gives the closure operation in the dual space.

For any C^*-algebra, the *dual space* $\hat{A}$ will be the set of all unitary equivalence classes of irreducible *-representations of A. If $\mathcal{S} \subset \hat{A}$, the *closure* $\bar{\mathcal{S}}$ of $\mathcal{S}$ will be defined as the set of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$, i.e., for which $\bigcap_{S \in \mathcal{S}}$ Kernel $(S) \subset$ Kernel (T).

This definition of closure in a set of ideals is essentially given in [19, p. 349].

Our $\hat{A}$ differs from the Jacobson structure space (see [8]) in two minor respects: First, its elements are representations, not ideals (note that two different irreducible representations might have the same kernel). Secondly,

the representations in $\hat{A}$ are required to be only topologically, not algebraically, irreducible. Kadison in [9] has shown that all irreducible *-representations of a C^*-algebra are algebraically irreducible; so that the importance of the second difference is much reduced. However, in proving that the closure defined above generates a topology, we need not use Kadison's rather abstruse result to make Jacobson's classical proof directly applicable; a slight modification of the latter will suffice.

LEMMA 1.6. *The above closure operation in $\hat{A}$ generates a topology.*

The topology defined by this closure is called the *hull-kernel* topology of $\hat{A}$. Unless the contrary is stated, we assume $\hat{A}$ equipped with this topology.

If G is a locally compact group, the dual space $\hat{G}$ will be the set of all unitary equivalence classes of irreducible unitary representations of G, equipped with the *hull-kernel* topology, i.e., the topology of $(C^*(G))^\wedge$ transferred to $\hat{G}$ by the natural correspondence between $\hat{G}$ and $(C^*(G))^\wedge$.

Let A be any C^*-algebra, T any element of $\hat{A}$ belonging to the closure of a subset $\mathcal{S}$ of $\hat{A}$. By the equivalence theorem, each positive functional ϕ associated with T is a weak* limit of sums of positive functionals associated with $\mathcal{S}$. Since, however, we are dealing now with irreducible representations, it is now possible to make a stronger statement: Each such ϕ is a weak* limit of positive functionals associated with $\mathcal{S}$; sums are unnecessary. We next prove this.

First let the C^*-algebra A have a unit 1. By P we denote the set of all normed (i.e., $\phi(1)=1$) positive functionals on A, and by N the set of all indecomposable positive functionals. Let Q be a weakly* closed subset of P, and L the weak* closure of the set of all convex linear combinations of elements of Q.

LEMMA 1.7. *Every extreme point of L lies in Q.*

Proof. Let $C(Q)$ be the space of all continuous complex functions on the compact Hausdorff space Q, and $M(Q)$ the set of all positive Baire measures on Q of total mass 1. Each element μ of $M(Q)$ corresponds naturally to an element of $(C(Q))^*$; and the weak* topology of $(C(Q))^*$ transferred to $M(Q)$ will be called the weak* topology of $M(Q)$. Evidently $M(Q)$ is weakly* compact.

To each μ in $M(Q)$ and x in A, let

$$\phi_\mu(x) = \int_Q \phi(x)d\mu\phi.$$

Evidently $\phi_\mu \in P$, and the map $\mu \rightarrow \phi_\mu$ is continuous in the weak* topologies of $M(Q)$ and P. So its range $\phi(M(Q))$ is compact, hence weakly* closed in P. On the other hand, $\phi(M(Q))$ contains all convex linear combinations of elements of Q, and the latter are dense in $\phi(M(Q))$. It follows that

$$L = \phi(M(Q)). \tag{3}$$

Now let ψ be an extreme point of L. By (3) $\psi = \phi_\mu$, $\mu \in M(Q)$. The lemma will be proved if we show that μ is a point mass, i.e., that its closed hull contains only one point.

Let f_0 be a point in the closed hull of μ; and assume that the closed hull contains another point distinct from f_0. Then, for all sufficiently small open Baire neighborhoods U of f_0,

$$0 < \mu(U) < 1. \tag{4}$$

Fix such a U. For each Baire set R let

$$\mu_1(R) = \frac{\mu(R \cap U)}{\mu(U)}, \qquad \mu_2(R) = \frac{\mu(R - U)}{\mu(Q - U)}.$$

Then the μ_i belong to $M(Q)$ and, if $\psi_i = \phi_{\mu_i}$, we have

$$\psi = \mu(U)\psi_1 + (1 - \mu(U))\psi_2, \tag{5}$$

and by (3)

$$\psi_i \in L. \tag{6}$$

Since ψ is an extreme point of L, (4), (5), and (6) give

$$\psi_1 = \psi. \tag{7}$$

But

$$\psi_1(x) = \int \phi(x) d\mu_1 \phi = \frac{1}{\mu(U)} \int_U \phi(x) d\mu\phi \to f_0(x)$$

as U closes down on f_0. This combined with (7) shows that $\psi = f_0$ for each f_0 in the closed hull of μ. Hence there can only be one point in the closed hull of μ.

THEOREM 1.4. *Let A be an arbitrary C^*-algebra, $\mathcal{S}$ a subset of $\hat{A}$, and T an element of $\hat{A}$. Then the following three conditions are equivalent:*

(i) $T \in \bar{\mathcal{S}}$;

(ii) *some nonzero positive functional associated with T is a weak* limit of finite linear combinations of positive functionals associated with $\mathcal{S}$;*

(iii) *every nonzero positive functional ϕ associated with T is the weak* limit of some net $\{\psi_i\}$ of positive functionals associated with $\mathcal{S}$ such that $\|\psi_i\| \leqq \|\phi\|$.*

Proof. It is trivial that (iii) implies (ii); the equivalence theorem gives that (ii) implies (i). To prove that (i) implies (iii), we assume that $T \in \bar{\mathcal{S}}$.

Suppose first that A has a unit 1. Let ϕ be a normed positive functional associated with T, Q the weak* closure of the set of all normed positive functionals associated with $\mathcal{S}$, and L the weak* closed convex hull of Q. The

equivalence theorem assures us that $\phi \in L$. But now ϕ is indecomposable, hence is certainly an extreme point of L. By Lemma 1.7, $\phi \in Q$. This proves (iii) in case A has a unit.

If A has no unit, adjoin one to get A_1. Define T^1, $\mathcal{S}^1$ as in §2. By Lemma 1.3, $T^1 \in (\mathcal{S}^1)^-$. If ϕ is a positive functional of norm 1 associated with T, extend it to a normed positive functional ϕ' on A_1 associated with T^1. So by the last paragraph ϕ' is the weak* limit of a net $\{\phi_i'\}$, where each ϕ_i' is a normed positive functional associated with $\mathcal{S}^1$. Restricting ϕ_i' to ϕ_i on A, we have

$$\|\phi_i\| \leqq 1, \qquad \phi = \lim \phi_i \text{ (in the weak* topology).}$$

The following theorem for groups bears the same relation to Theorem 1.4 as Theorem 1.3 does to the equivalence theorem. Its proof is omitted, since it is obtained by applying Lemma 1.5 to Theorem 1.4 in the same way that Lemma 1.5 was applied to the equivalence theorem in the proof of Theorem 1.3.

THEOREM 1.5. *Let G be a locally compact group, $\mathcal{S}$ a subset of $\hat{G}$, and T an element of $\hat{G}$. The following three conditions are equivalent*:

(i) $T \in \bar{\mathcal{S}}$;

(ii) *some function of positive type associated with T is a uniform-on-compacta limit of sums of functions of positive type associated with $\mathcal{S}$*;

(iii) *every function of positive type associated with T is a uniform-on-compacta limit of functions of positive type associated with $\mathcal{S}$.*

It might be conjectured, by analogy with Theorem 1.4, that Theorem 1.5 would remain true on replacing "sums of" by "finite linear combinations of" in condition (ii). This however is not so. A counter-example is provided by Theorem 3.2 of this paper, together with the observation that any bounded continuous complex function on G is a uniform-on-compacta limit of finite linear combinations of functions of positive type associated with the regular representation of G.

As we have already remarked, the essential difference between Theorems 1.2 and 1.4 (also between Theorems 1.3 and 1.5) is the replacement of convergence of sums of positive functionals by convergence of single positive functionals, in the case that the representations are irreducible. There are other cases in which this replacement is possible. For example, Takenouchi has pointed out [20, p. 154] that (in our terminology) a representation V of a locally compact group G is weakly contained in the regular representation R if each function of positive type associated with V is a uniform-on-compacta limit of functions of positive type associated with the regular representation.

Let A be a C^*-algebra without unit, and A_1 the C^*-algebra obtained by adjoining a unit 1 to A. For each T in $\hat{A}$, T^1 (for definition, see §2) is in $\hat{A}_1$.

Besides the T^1, there is only one other element of $\hat{A}_1$, namely, the one-dimensional representation τ sending $a+\lambda\cdot 1$ into λ ($a\in A$). Thus $\hat{A}$ may be identified (as a set) with $\hat{A}_1-\{\tau\}$. Lemma 1.3 now gives:

LEMMA 1.8. *The topology of $\hat{A}$ is that of $\hat{A}_1$ relativized to $\hat{A}_1-\{\tau\}$.*

Observe that τ belongs to the closure of $\hat{A}_1-\{\tau\}$. Otherwise A_1 would contain an element $a+\lambda\cdot 1$ not in A, belonging to the kernels of all T^1 ($T\in\hat{A}$). This means that $T_a+\lambda I=0$ for all T in $\hat{A}$ (I is the identity operator in $H(T)$). But $\lambda\neq 0$; hence $T_{(-a/\lambda)}=I$ for all T in $\hat{A}$. It follows that $-a/\lambda$ is a unit element of A.

5. **Weak equivalence.** Let $\mathcal{S}$ and $\mathcal{G}$ be any two families of *-representations of a C^*-algebra A. If each S in $\mathcal{S}$ is weakly contained in $\mathcal{G}$, we say that $\mathcal{S}$ is *weakly contained* in $\mathcal{G}$. If $\mathcal{S}$ and $\mathcal{G}$ are each weakly contained in the other, they are *weakly equivalent.*

The following remarks are trivial: (i) If $\mathcal{S}\subset\mathcal{G}$, $\mathcal{S}$ is weakly contained in $\mathcal{G}$; (ii) the relation of weak containment is reflexive and transitive; (iii) the relation of weak equivalence is an equivalence relation; (iv) if $\mathcal{S}\subset\hat{A}$, $\mathcal{G}\subset\hat{A}$, then $\mathcal{S}$ is weakly contained in $\mathcal{G}$ if and only if $\mathcal{S}\subset\overline{\mathcal{G}}$; $\mathcal{S}$ is weakly equivalent to $\mathcal{G}$ if and only if $\overline{\mathcal{S}}=\overline{\mathcal{G}}$; (v) any *-representation T of A is weakly equivalent to any direct sum of copies of T.

THEOREM 1.6. *If $\mathcal{S}$ is any family of *-representations of a C^*-algebra A, there exists a (unique) closed subset $\mathcal{G}$ of $\hat{A}$ which is weakly equivalent to $\mathcal{S}$. It consists of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$.*

Proof. (A) First assume A has a unit 1; and define P as the set of all positive linear functionals ϕ on A with $\phi(1)=1$; and Q as the smallest convex weakly *closed subset of P containing all positive functionals associated with $\mathcal{S}$. I claim that an extreme point of Q is an extreme point of P (compare [5, Proposition 6, p. 40]).

Let ϕ be an extreme point of Q; and assume

$$\phi = r\psi_1 + (1-r)\psi_2 \qquad (0 < r < 1, \psi_i \in P).$$

If ϕ is associated with a representation T, $\phi(a)=(T_a\xi, \xi)$ (ξ cyclic in $H(T)$), then a well-known majorization theorem [18, Theorem 1, p. 237] supplies us with an η in $H(T)$ for which $\psi_1(a)=(T_a\eta, \eta)$. Since ξ is cyclic, there is a sequence $\{y_n\}$ of elements of A with $\|T_{y_n}\xi\|=1$, $T_{y_n}\xi\to\eta$. Put $\phi_n(a)=\phi(y_r{}^*ay_n)$; then $\phi_n\in P$, and $\phi_n\to\psi_1$ weakly*. Since $\phi\in Q$, also $\phi_n\in Q$; and hence $\psi_1\in Q$. Similarly $\psi_2\in Q$. Since ϕ is an extreme point of Q, this gives $\psi_1=\psi_2=\phi$. Hence ϕ is an extreme point of P; and the claim is justified.

Now let $\mathcal{G}$ be the (closed) set of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$. It suffices to show that $\mathcal{S}$ is weakly contained in $\mathcal{G}$. If ϕ is in P and associated with $\mathcal{S}$, and if E is the set of all extreme points of Q, then by the Krein-Milman theorem ϕ is a weak* limit of convex linear combinations of elements

of E. On the other hand, elements of E are extreme points of P, hence are associated with representations in $\mathcal{G}$. Thus $\mathcal{S}$ is weakly contained in $\mathcal{G}$.

(B) If A has no unit, adjoin a unit 1 to get A_1. Defining $\mathcal{S}^1$ as in §2, we obtain by (A) a closed subset $\mathcal{G}^1$ of $\hat{A}_1$ which is weakly equivalent to $\mathcal{S}^1$. Let $\mathcal{G}$ be the subset of $\hat{A}$ corresponding to $\mathcal{G}^1 - \{\tau\}$ (see Lemma 1.8). $\mathcal{G}$ is obviously weakly contained in $\mathcal{S}$. That $\mathcal{S}$ is weakly contained in $\mathcal{G}$ follows from the two facts that $\mathcal{S}^1$ is weakly contained in $\mathcal{G}^1$ and that τ vanishes on A.

We will call the $\mathcal{G}$ of Theorem 1.6 the *spectrum* of $\mathcal{S}$. This definition is a generalization of [5, Definition 2, p. 43].

6. **Weak equivalence and direct integrals.** Several authors (for example, [6]; for other references, see [18]) have studied direct integrals of *-representations of C^*-algebras. If this notion is defined topologically, rather than purely measure-theoretically, one can conclude the weak equivalence of the direct integral representation with the set of component representations.

For details concerning direct integrals, we refer the reader to [6].

Fix a locally compact Hausdorff space T; with each t in T associate a Hilbert space H_t. A *vector field* will be a function ξ on T such that $\xi(t) \in H_t$ for each t. An *operator field* will be a function B on T such that, for each t, $B(t)$ is a bounded linear operator on H_t.

A *continuity basis* is a family F of vector fields such that: (i) if $\xi, \eta \in F$ and r, s are complex, then $r\xi + s\eta \in F$; (ii) if $\xi \in F$, $\|\xi(t)\|$ is continuous on T; (iii) for each t_0 in T, $\{\xi(t_0) \mid \xi \in F\}$ is dense in H_{t_0}.

For the rest of this section, we fix a continuity basis F. A vector field ξ is *continuous* if for each t_0 in T, and each $\epsilon > 0$, there is a neighborhood U of t_0 and an η in F such that $\|\xi(t) - \eta(t)\| < \epsilon$ for all t in U. An operator field B is *continuous* if, for all continuous vector fields ξ, the map $t \to B(t)\xi(t)$ is a continuous vector field.

Now let m be a fixed regular Borel measure on T whose closed hull is T; and denote by H' the inner product space of all continuous vector fields ξ for which

$$\int_T \|\xi(t)\|^2 dmt < \infty,$$

equipped with the inner product

$$(\xi, \eta) = \int_T (\xi(t), \eta(t)) dmt.$$

Following [18], the Hilbert space H obtained by completing H' will be denoted by

$$H = \int_T H_t (dmt)^{1/2},$$

the *direct integral* of the H_t.

A bounded linear operator b on H is a *direct integral* operator if there is a continuous operator field B such that, for $\xi \in H'$, $b\xi \in H'$ and $(b\xi)(t) = B(t)\xi(t)$. We describe this by writing $b = \int \oplus B(t)$. The operator field B is uniquely determined by b.

LEMMA 1.9. (i) *If* $b = \int \oplus B(t)$, *then* $\|b\| = \sup_t \|B(t)\|$.
(ii) *If* $b = \int \oplus B(t)$, $c = \int \oplus C(t)$, *then*

$$rb + sc = \int \oplus (rB(t) + sC(t)),$$

$$bc = \int \oplus B(t)C(t).$$

(iii) *If* $b = \int \oplus B(t)$, $b^* = \int \oplus C(t)$, *then* $C(t) = B(t)^*$.

(Note that b might be a direct integral operator, without b^* being one.)

Now suppose A is a C^*-algebra, and S a $*$-representation of A in $H = \int_T H_t (dmt)^{1/2}$ such that $S_x = \int \oplus S_x^{(t)}$ is a direct integral operator for each x in A. It follows from Lemma 1.9 that, for each t in T, the map $S^{(t)}: x \to S_x^{(t)}$ is a $*$-representation of A in H_t. We then say that S is a *direct integral* of the $S^{(t)}$, and write $S = \int \oplus S^{(t)}$.

THEOREM 1.7. *If* $S = \int \oplus S^{(t)}$ *is a direct integral representation of a* C^*-*algebra* A, *then* S *is weakly equivalent to* $\{S^{(t)} \mid t \in T\}$.

Proof. (A) Let $t_0 \in T$. We shall show that $S^{(t_0)}$ is weakly contained in S. Letting $\xi \in F$, it is sufficient to approximate

$$\phi(x) = (S_x^{(t_0)} \xi(t_0), \xi(t_0)) \tag{8}$$

in the weak* topology by positive functionals associated with S. For each compact neighborhood U of t_0, f_U will be a continuous non-negative function on T vanishing outside U, for which

$$\int_T (f_U(t))^2 dmt = 1. \tag{9}$$

Putting $\xi_U(t) = f_U(t)\xi(t)$, $\phi_U(x) = (S_x \xi_U, \xi_U)$, we get (note $\xi_U \in H'$):

$$\begin{aligned} \phi_U(x) &= \int_T (S_x^{(t)} \xi_U(t), \xi_U(t)) dmt \\ &= \int_U (f_U(t))^2 (S_x^{(t)} \xi(t), \xi(t)) dmt. \end{aligned} \tag{10}$$

Now ϕ_U is a positive functional associated with S. By (8), (9) and (10), $\lim_{U \to t_0} \phi_U(x) = \phi(x)$ for $x \in A$.

(B) To prove that S is weakly contained in the set of all $S^{(t)}$, pick $\xi \in H'$ with compact support D, and put

$$\phi(x) = (S_x\xi, \xi) \tag{11}$$
$$= \int_D \phi^t(x) dmt,$$

where $\phi^t(x) = (S_x^{(t)}\xi(t), \xi(t))$. It is enough to show that ϕ is weakly* approximated by finite sums of the ϕ^t. But this is evident from the integral (11).

7. **The regular representation of a group.** It is of interest to inquire which locally compact groups G have the property—which we shall refer to as property (R)—that their regular representation weakly contains all irreducible representations. Godement [5, p. 77] has observed that, if the regular representation of G weakly contains merely the one-dimensional identity representation of G, then it has property (R). It is well known that this is the case for compact groups and locally compact Abelian groups. But it is not true for all locally compact groups. Takenouchi has shown in [20] that, if G is a locally compact group whose factor group modulo the connected component of the identity is compact, then property (R) holds if and only if G is of type (C) in the sense of Iwasawa.

In Theorem 3.2 we shall determine exactly which irreducible representations of the $n \times n$ complex unimodular group are weakly contained in the regular representation.

The following remark is mildly interesting:

LEMMA 1.10. *Let L be the left-regular representation of a locally compact group G ($L_y(x) = y * x$ for $x \in L_2(G)$, $y \in L_1(G)$). Then G has property* (R) *if and only if $\|y\|_c = \|L_y\|$ for all y in $L_1(G)$ (see* §3 *for $\| \ \|_c$).*

Proof. Property (R) holds if and only if the kernel of L on $C^*(G)$ is $\{0\}$ i.e., L is an isometry on $C^*(G)$. For this it suffices to know that L is an isometry on $L_1(G)$ with respect to $\| \ \|_c$.

8. **Completely continuous representations.** A *-representation T of a C^*-algebra A is *completely continuous* if T_a is completely continuous for all a in A.

Fix a C^*-algebra A.

LEMMA 1.11. *If T is a completely continuous element of $\hat{A}$, then $\{T\}$ is closed.*

Proof. If $I = \text{Kernel } (T)$, A/I is the algebra of *all* completely continuous operators on $H(T)$. If $S \in \{\overline{T}\}$, Kernel $(S) \supset I$, so that S induces an irreducible representation of A/I. But the latter is well-known to have no irreducible representation other than the identity map. Hence $S = T$.

THEOREM 1.8. *Let $T^i \in \hat{A}$ for each i in an index set N. Form the direct sum $T = \sum_{i \in N} \oplus T^i$. Then T is completely continuous if and only if the following*

three conditions hold:

(i) *Each* T^i *is completely continuous*;

(ii) *For each* i, *there are only finitely many distinct* j *in* N *with* $T^i \cong T^j$;

(iii) *The set* $\mathcal{G} = \{T^i \mid i \in N\}$ *has no limit points in* $\hat{A}$.

Proof. (A) Assume T completely continuous. Obviously (i) and (ii) hold. To show that $\mathcal{G}$ has no limit points, it is enough to show that it is closed in $\hat{A}$. Indeed, if this has been done, it will follow, since every subrepresentation of T is completely continuous, that every subset of $\mathcal{G}$ is closed in $\hat{A}$. But a set all of whose subsets are closed has no limit points.

Let I be the intersection of the kernels of the T^i; then $I = \text{Kernel } (T)$. Pick an irreducible representation S of A whose kernel contains I. It suffices to show $S \in \mathcal{G}$.

We will consider T as a faithful representation of $B = A/I$, and S as an irreducible representation of B. Now B is (via T) a C^*-algebra of completely continuous operators. By the structure theorem for such (see [10, Theorem 8.3]), B is isomorphic to the $C^{(\infty)}$-sum

$$B \cong \sum_{r \in W} \oplus B^{(r)},$$

where each $B^{(r)}$ is the algebra of all completely continuous operators on some Hilbert space. To each s in W corresponds an irreducible representation S^s of B:

$$S^s \colon \sum_{r \in W} \oplus a_r \to a_s;$$

and all irreducible representations of B are of this form. In particular $S \cong$ some S^s. On the other hand, each T^i ($i \in N$) gives rise to a representation of B, and these distinguish points of B. It follows that all S^s occur among the T^i. In particular, S occurs among the T^i, i.e., $S \in \mathcal{G}$.

(B) Now assume (i), (ii), and (iii); let I be the closed two-sided ideal of all x in A for which T_x is completely continuous. It suffices to show that $I = A$.

Assume then that $I \neq A$. Then A has an irreducible representation S whose kernel contains I. Now S is not a limit point of $\mathcal{G}$, so it is not in the closure of $\mathcal{G}' = \mathcal{G} - \{S\}$. Pick an x_0 belonging to the kernels of all T^i in $\mathcal{G}'$, but not to the kernel of S.

Now either $S \in \mathcal{G}$ or $S \notin \mathcal{G}$. If $S \notin \mathcal{G}$, then $\mathcal{G} = \mathcal{G}'$, so that $x_0 \in \text{Kernel } (T)$, i.e., $T_{x_0} = 0$. If $S \in \mathcal{G}$, then T_{x_0} is 0 on the subspaces of $H(T)$ corresponding to all $T^i \not\cong S$, while, on the subspace of $H(T)$ corresponding to the $T^i \cong S$, T_{x_0} is completely continuous by (i) and (ii). Thus in either case T_{x_0} is completely continuous; and $x_0 \in I \subset \text{Kernel } (S)$. This contradicts $x_0 \notin \text{Kernel } (S)$.

Chapter II. Norm, trace, and topology in the dual

9. **Norm and topology.** Throughout this chapter, A will be an arbitrary fixed C^*-algebra.

For each x in A, $T\rightarrow\|T_x\|$ is a numerical function on $\hat{A}$. In general, this function is not continuous. It is, however, lower semi-continuous, as we now show.

LEMMA 2.1. *If* $\mathcal{S}\subset\hat{A}$ *and* $T\in\hat{A}$, *then* $T\in\bar{\mathcal{S}}$ *if and only if for each* x *in* A,

$$\|T_x\| \leqq \sup_{S\in\mathcal{S}}\|S_x\|. \tag{1}$$

Proof. If $T\notin\bar{\mathcal{S}}$, by the definition of closure there is an x in A for which $\sup_{S\in\mathcal{S}}\|S_x\|=0$, $\|T_x\|>0$. So (1) fails.

Assume $T\in\bar{\mathcal{S}}$. By Lemma 1.8 it is sufficient to assume that A has a unit 1. Again, since $\|T_{x*x}\|=\|T_x\|^2$, we may assume without loss of generality that x is positive.

Choose ξ in $H(T)$ so that $\|\xi\|=1$ and

$$(T_x\xi, \xi) \geqq \|T_x\| - \frac{\epsilon}{2}\cdot \tag{2}$$

By Theorem 1.4, there exist $S^1, \cdots, S^n$ in $\mathcal{S}$, $\xi_i\in H(S^i)$ $(i=1, \cdots, n)$, and non-negative $\lambda_1, \cdots, \lambda_n$, such that $\|\xi_i\|=1$, $\sum_{i=1}^n\lambda_i=1$, and

$$\left|\left\{\sum_{i=1}^n\lambda_i(S_x^i\xi_i, \xi_i)\right\} - (T_x\xi, \xi)\right| < \frac{\epsilon}{2}\cdot \tag{3}$$

But $(S_x^i\xi_i, \xi_i)\leqq\|S_x^i\|$, so that

$$\sum_{i=1}^n\lambda_i(S_x^i\xi_i, \xi_i) \leqq \max_{i=1}^n\|S_x^i\| \leqq \sup_{S\in\mathcal{S}}\|S_x\|. \tag{4}$$

Combining (3) and (4), we get

$$(T_x\xi, \xi) \leqq \sup_{S\in\mathcal{S}}\|S_x\| + \frac{\epsilon}{2}\cdot \tag{5}$$

Now (1) follows from (2), (5), and the arbitrariness of ϵ.

From this we immediately obtain:

LEMMA 2.2. *For each* x *in* A, *the function* $T\rightarrow\|T_x\|$ *is lower semi-continuous, i.e., if* $T^i\rightarrow T$ *in* $\hat{A}$, *then* $\liminf_i\|T_x^i\|\geqq\|T_x\|$.

LEMMA 2.3. *For every net* $\{T^i\}$ *of elements of* $\hat{A}$, *and every element* x *of* A,

$$\sup_{S\in\mathcal{G}}\|S_x\| \geqq \limsup_i\|T_x^i\|,$$

where $\mathcal{G}$ *is the set of all cluster points of* $\{T^i\}$.

Proof. For each index j, $\mathcal{S}_j$ will be the closure in $\hat{A}$ of $\{T^i|i>j\}$. The $\mathcal{S}_j$ form a decreasing net of closed sets, and

$$\mathcal{G} = \bigcap_j \mathcal{S}_j. \tag{6}$$

Let $I_j = \bigcap_{S \in \mathcal{S}_j}$ Kernel (S), $K = \bigcap_{S \in \mathcal{G}}$ Kernel (S). I claim that

$$K = \left(\bigcup_j I_j \right)^{-}. \tag{7}$$

Indeed, let $x \notin (\bigcup_j I_j)^-$. Since $(\bigcup_j I_j)^-$ is a closed two-sided ideal, there is a T in $\hat{A}$ such that $T_x \neq 0$, and

$$\left(\bigcup_j I_j \right)^{-} \subset \text{Kernel } (T). \tag{8}$$

Since each $\mathcal{S}_j$ is closed, (8) implies that T belongs to all $\mathcal{S}_j$, i.e., $T \in \mathcal{G}$. Therefore $T_x \neq 0$ gives $x \notin K$. Thus $K \subset (\bigcup_j I_j)^-$. The opposite inclusion is obvious. This proves (7).

Now $\{I_j\}$ is an increasing net of closed two-sided ideals of A. Denote as usual by x/I_j the element of the C^*-algebra A/I_j corresponding to x. Applying to (7) an elementary argument valid in all Banach spaces, we have for all x in A

$$\lim_j \|x/I_j\| = \|x/K\|. \tag{9}$$

Now, since $\|x\| = \sup_{T \in \hat{A}} \|T_x\|$ in any C^*-algebra (see p. 411 of [10]),

$$\|x/I_j\| = \sup_{S \in \mathcal{S}_j} \|S_x\|, \tag{10}$$

$$\|x/K\| = \sup_{S \in \mathcal{G}} \|S_x\|. \tag{11}$$

Combining (9), (10), and (11), we get for all x,

$$\lim_j \sup_{S \in \mathcal{S}_j} \|S_x\| = \sup_{S \in \mathcal{G}} \|S_x\|. \tag{12}$$

But, by the definition of $\mathcal{S}_j$,

$$\lim_j \sup_{S \in \mathcal{S}_j} \|S_x\| \geqq \lim_j \sup_{i > j} \|T^i_x\| = \lim_i \sup \|T^i_x\|.$$

This and (12) complete the proof.

THEOREM 2.1. *Let A be an arbitrary C^*-algebra, $\{T^i\}$ a net of elements of $\hat{A}$, and $\mathcal{S}$ a closed subset of $\hat{A}$. The following two conditions are equivalent*:

(i) *For all x in A, $\lim_i \|T^i_x\| = \sup_{S \in \mathcal{S}} \|S_x\|$.*

(ii) *For all subnets $\{T'^j\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have*

$$T'^j \to S \textit{ if and only if } S \in \mathcal{S}.$$

Proof. (A) Assume (ii). Then $\mathcal{S}$ is the set of all cluster points of $\{T^i\}$. By Lemma 2.3,

$$\limsup_i \|T^i_x\| \leqq \sup_{S\in\mathcal{S}} \|S_x\|. \tag{13}$$

On the other hand, if $S\in\mathcal{S}$, we have $T^i\rightarrow S$, so that by Lemma 2.2

$$\|S_x\| \leqq \liminf_i \|T^i_x\|.$$

Combining this with (13) we get (i).

(B) Assume (i). Let $S^0\notin\mathcal{S}$; then there exists x in A such that $S^0_x\neq 0$, $S_x=0$ for all S in $\mathcal{S}$. By (i), $\lim_i\|T^i_x\|=0$. If at the same time S^0 is a cluster point of $\{T^i\}$, we pick a subnet $\{T'^j\}$ of $\{T^i\}$ for which $T'^j\rightarrow S^0$. By Lemma 2.2

$$\|S^0_x\| \leqq \liminf_j \|T'^j_x\| = 0.$$

This contradicts $S^0_x\neq 0$. Thus all cluster points of $\{T^i\}$ are in $\mathcal{S}$.

Let $S^0\in\mathcal{S}$. I claim that $T^i\rightarrow S^0$. Indeed, if this were not so, there would be a subnet $\{T'^j\}$ of $\{T^i\}$ and a neighborhood U of S^0 such that all T'^j are outside U. Hence there would be an x such that $S^0_x\neq 0$, $T'^j_x=0$ for all j. Hence by (i)

$$\sup_{S\in\mathcal{S}}\|S_x\| = \lim_i \|T^i_x\| = \lim_j \|T'^j_x\| = 0,$$

which contradicts $S^0_x\neq 0$. Thus every element of $\mathcal{S}$ is a limit of $\{T^i\}$.

We have proved that $\mathcal{S}$ coincides with the set of all limits, and also with the set of all cluster points, of $\{T^i\}$. But this is exactly (ii). The proof is complete.

It may be worth mentioning the status of Lemmas 2.2 and 2.3 and Theorem 2.1 for an arbitrary Banach *-algebra A. For such an A, one defines the topological space $\hat{A}$ just as in §4, as the set of all irreducible *-representations with the hull-kernel topology. Now Lemma 2.3 is still valid in the general case, but Lemma 2.2 is in general false; consider, for example, the Banach *-algebra of complex functions continuous on the closed unit disc and analytic in its interior. As for Theorem 2.1, it holds whenever Lemma 2.2 does.

Corollary 1. *If A is a C^*-algebra, and $\{T^i\}$ a net of elements of $\hat{A}$, the following are equivalent*:

(i) $\lim_i\|T^i_x\|=0$ *for all x in A*;

(ii) *no subnet of $\{T^i\}$ converges to any limit*;

(iii) *A has no unit, $(T^i)^1 \to \tau$ in $\hat{A}_1$, and no subnet of $\{(T^i)^1\}$ converges to any other limit* (*see* §2 *for* A_1, T^1, τ).

Proof. Theorem 2.1, with $\mathcal{S}$ taken as the void set, shows that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from Lemmas 1.7 and 1.8.

Theorem 2.1 has as a simple corollary a connection between the Hausdorff property and the continuity of the functions $T \to \|T_x\|$.

Let the *ideal structure space* X of A be defined as the set of all kernels of irreducible *-representations, topologized with the hull-kernel topology. X is obtained by identifying elements of $\hat{A}$ with the same kernel. Since, for $T \in \hat{A}$, $\|T_x\|$ depends only on the kernel of T, we may define $N_x(I)$ as $\|T_x\|$ whenever $T \in \hat{A}$, Kernel $(T) = I$.

COROLLARY 2 (*see Theorem* 4.1 *of* [12]). *X is Hausdorff if and only if each N_x is continuous on X.*

10. **Trace and topology.** We fix a C^*-algebra A, with dual $\hat{A}$. The dimension of a representation S of A will be called dim S, and the trace of an operator B $\mathrm{Tr}(B)$.

LEMMA 2.4. *If $\{T^i\}$ is a net of elements of $\hat{A}$, n is an integer, and* dim $T^i \leqq n$ *for each i, then $\{T^i\}$ can converge to no more than a finite number of distinct limits, and, for each such limit S,* dim $S \leqq n$.

Proof. (A) Let $T^i \to S$. To prove dim $S \leqq n$, we give a well-known argument using polynomial identities (see §2, [11]). Let

$$\mathrm{B} = A \Big/ \bigcap_i \text{Kernel } (T^i).$$

Since the elements of B are separated by the T^i, B must satisfy the standard polynomial identity for the $n \times n$ matrix algebra. Hence all its irreducible representations are of dimension $\leqq n$. But, since S is in the closure of the $\{T^i\}$, its kernel contains $\bigcap_i$ Kernel (T^i); hence it induces an irreducible representation of B. Combining these facts, we get dim $S \leqq n$.

(B) Pick such a positive integer p that there do not exist as many as p linear operators $A_1, \cdots, A_p$ on an n-dimensional Hilbert space for which $\|A_k\| \leqq 1$ $(k=1, \cdots, p)$, $\|A_k - A_j\| \geqq 1/2$ $(k, j=1, \cdots, p;\ k \neq j)$. We complete the proof by contradicting the assumption that $\{T^i\}$ converges to p distinct limits $S^1, \cdots, S^p$.

By (A) and Lemma 1.11, each one-point set $\{S^k\}$ is closed in $\hat{A}$. Thus, for $k=1, \cdots, p$, there is a Hermitian element x_k of A such that

$$\|S^j_{x_k}\| = \delta_{kj}. \tag{19}$$

Let F be the following real-valued continuous function on the reals:

$$F(t) = \begin{cases} 1 & \text{if} \quad t \geqq 1, \\ t & \text{if} \quad -1 \leqq t \leqq 1, \\ -1 & \text{if} \quad t \leqq -1. \end{cases}$$

Applying F to x_k (see [12, p. 227]), we have by (19)

$$\|F(x_i)\| \leqq 1, \qquad S^j_{F(x_k)} = F(S^j_{x_k}) = S^j_{x_k}.$$

Replacing the x_k by the $F(x_k)$, we may assume $\|x_k\| \leqq 1$. Therefore

$$\|T^i_{x_k}\| \leqq 1 \text{ for all } i, \text{ and all } k = 1, \cdots, p. \tag{20}$$

Fix $k \neq j$. Since S^k belongs to the closure of all subnets of $\{T^i\}$, Lemma 2.1 gives

$$1 = \|S^k_{x_k - x_j}\| \leqq \liminf \|T^i_{x_k - x_j}\|.$$

Hence there is an i_0 such that $\|T^i_{x_k-x_j}\| \geqq 1/2$ for all $i \succ i_0$. We may therefore pick an i such that

$$\|T^i_{x_k} - T^i_{x_j}\| \geqq 1/2 \qquad \text{for all } k, j = 1, \cdots, p;\ k \neq j.$$

This combined with (20) contradicts the definition of p.

LEMMA 2.5. *Let $\{T^i\}$ be a net of n-dimensional representations in $\hat{A}$, and suppose $S^1, S^2, \cdots, S^r$ are distinct elements of $\hat{A}$ such that*

(i) *for all subnets $\{T'^i\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have $T'^i \to S$ if and only if $S =$ some S^k;*

(ii) *for each x in A, $\mathrm{Tr}(T^i_x)$ approaches some limit $\sigma(x)$.*

Then there exist positive integers $m_1, \cdots, m_r$ such that

$$\sum_{k=1}^{r} m_k \dim S^k \leqq n, \tag{21}$$

$$\sigma(x) = \sum_{k=1}^{r} m_k \mathrm{Tr}(S^k_x) \qquad \textit{for all } x \textit{ in } A. \tag{22}$$

Proof. By Lemma 2.4, r is finite, and $\dim S^k \leqq n$. By (i) and Theorem 2.1,

$$\lim \|T^i_x\| = \max_{k=1} \|S^k_x\| \qquad (x \in A). \tag{23}$$

Denote $\dim S^k$ by d_k.

(A) Let $P_{k,j}$ $(j = 1, \cdots, d_k)$ be orthogonal one-dimensional projections in $H(S^k)$. We shall first show that there are positive elements x_{kj} in A $(k = 1, \cdots, r;\ j = 1, \cdots, d_k)$ for which

(a) $S^q_{x_{kj}} = \delta_{qk} P_{kj}$;

(b) There exists i_0 such that, for $i > i_0$, the $T^i_{x_{kj}}$ $(k=1, \cdots r; j=1, \cdots, d_k)$ are orthogonal nonzero projections in $H(T^i)$, with dim $T^i_{x_{kj}}$ independent of i and j.

To prove this, we select $\sum_{k=1}^r d_k$ distinct positive integers

$$w_{kj} \ (k = 1, \cdots, r; j = 1, \cdots, d_k),$$

and put $B_k = \sum_{j=1}^{d_k} w_{kj} P_{kj}$. Choose a positive element z of A such that

$$S^k_z = B_k \qquad \text{for each } k. \tag{24}$$

Pick $1/4 \geqq \epsilon > 0$. I claim that i_0 can be found to satisfy the following, which we will call property (P):

For $i > i_0$, T^i_z has at least one eigenvalue in each interval $[w_{kj} - \epsilon, w_{kj} + \epsilon]$, and no eigenvalue lying outside $[0, \epsilon]$ and also outside all $[w_{kj} - \epsilon, w_{kj} + \epsilon]$.

Indeed: Fix k, j. Let F be a continuous non-negative function on the reals which is 0 outside $[w_{kj} - \epsilon, w_{kj} + \epsilon]$, and 1 at w_{kj}. By (24)

$$S^q_{F(z)} = F(S^q_z) = F(B_q) = \delta_{qk} P_{kj}.$$

Hence $S^k_{F(z)} \neq 0$. So, by (23), $\lim_i \|T^i_{F(z)}\|$ exists and is not 0. Therefore we may choose i_0 so that, for $i > i_0$, $\|T^i_{F(z)}\| > 0$, i.e. $F(T^i_z) \neq 0$. But the latter implies that T^i_z has an eigenvalue in $[w_{kj} - \epsilon, w_{kj} + \epsilon]$.

Thus i_0 can be chosen to satisfy the first half of property (P).

Now pick a non-negative continuous function G on the reals which is 0 at 0 and at each w_{kj} $(k=1, \cdots, r; j=1, \cdots, d_k)$, and is 1 at all points which lie outside all the intervals $[-\epsilon, \epsilon]$ and

$$[w_{kj} - \epsilon, w_{kj} + \epsilon] \ (k = 1, \cdots, r; j = 1, \cdots, d_k).$$

Then

$$S^q_{G(z)} = G(S^q_z) = G(B_q) = 0,$$

so that by (23)

$$\lim_i \|T^i_{G(z)}\| = 0.$$

Now choose i_0 so that not only does the first half of property (P) hold, but also, for all $i > i_0$, $\|T^i_{G(z)}\| = \|G(T^i_z)\| < 1$, i.e., T^i_z has no eigenvalues at places where G is 1. Then i_0 satisfies property (P). Fix this i_0.

Now, for $k=1, \cdots, r; j=1, \cdots, d_k$, select a non-negative continuous function K_{kj} on the reals which is 1 on $[w_{kj} - \epsilon, w_{kj} + \epsilon]$ and 0 outside $[w_{kj} - 2\epsilon, w_{kj} + 2\epsilon]$. If $x_{kj} = K_{kj}(z)$, we have by (24)

$$S^q_{x_{kj}} = K_{kj}(S^q_z) = K_{kj}(B_q)$$
$$= \delta_{qk}P_{kj},$$

which is (a). If $i \succ i_0$, then by property (P), $T^i_{x_{kj}} = K_{kj}(T^i_z)$ is a nonzero projection on $H(T^i)$, and all the $T^i_{x_{kj}}$ ($k=1, \cdots, r; j=1, \cdots, d_k$) are orthogonal.

Let d^i_{kj} be the dimension of the range of $T^i_{x_{kj}}$. Now $d^i_{kj} = \mathrm{Tr}(T^i_{x_{kj}})$, which by hypothesis approaches $\sigma(x_{kj})$. But a convergent set of integers is eventually constant. Denoting the eventually constant value of d^i_{kj} by m_{kj}, I claim that m_{kj} depends only on k.

Indeed, fix k, and let j_1 and j_2 be two of the integers $1, \cdots, d_k$. Select a partial isometry C on $H(S^k)$ so that $CC^* = P_{k,j_1}$, $C^*C = P_{k,j_2}$; and let $u \in A$, $S^q_u = \delta_{qk}C$. Then

$$S^q_{(uu^* - x_{k,j_1})} = 0 \qquad \text{for all } q.$$

Hence by (23)

$$\left\| T^i_{(uu^* - x_{k,j_1})} \right\| \underset{i}{\rightarrow} 0;$$

so that

$$Tr(T^i_{uu^*}) - \mathrm{Tr}(T^i_{x_{k,j_1}}) \underset{i}{\rightarrow} 0.$$

Similarly

$$\mathrm{Tr}(T^i_{u^*u}) - \mathrm{Tr}(T^i_{x_{k,j_2}}) \underset{i}{\rightarrow} 0.$$

But $\mathrm{Tr}(T^i_{uu^*}) = \mathrm{Tr}(T^i_{u^*u})$, so that

$$\mathrm{Tr}(T^i_{x_{k,j_1}}) - \mathrm{Tr}(T^i_{x_{k,j_2}}) \underset{i}{\rightarrow} 0.$$

But this means that $m_{k,j_1} = m_{k,j_2}$, which proves that m_{kj} depends only on k. Write m_k for m_{kj}.

(B) I claim that the m_k thus defined have the properties (21) and (22).

Indeed, for large enough i, the $T^i_{x_{kj}}$ are orthogonal projections in $H(T^i)$ of dimension m_k. Hence

$$n \geqq \sum_{k,j} m_k = \sum_{k=1}^{r} m_k d_k,$$

which is (21).

Consider now any Hermitian element x of A such that

$$P_{kj}S_x^k = S_x^k P_{kj} \qquad (k = 1, \cdots, r; j = 1, \cdots, d_k). \tag{25}$$

Then for suitable real λ_{kj}, we have $S_x^k = \sum_j \lambda_{kj} P_{kj}$. Now look at $u = x - \sum_{k,j} \lambda_{kj} x_{kj}$. We have

$$S_u^q = \sum_j \lambda_{qj} P_{qj} - \sum_{k,j} \lambda_{kj} S_{x_{kj}}^q = 0$$

by (a). Hence by (23),

$$\lim_i \mathrm{Tr}(T_u^i) = 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_i \left\{ \mathrm{Tr}(T_x^i) - \sum_{k,j} \lambda_{kj} \mathrm{Tr}(T_{x_{kj}}^i) \right\} \\ &= \lim_i \left\{ \mathrm{Tr}(T_x^i) - \sum_{k=1}^r m_k \sum_{j=1}^{d_k} \lambda_{kj} \right\} \\ &= \lim_i \left\{ \mathrm{Tr}(T_x^i) - \sum_{k=1}^r m_k \,\mathrm{Tr}(S_x^k) \right\}, \end{aligned}$$

from which follows (22).

Now, for any preassigned Hermitian x in A, there is a set of P_{kj} for which (25) holds. Thus (22) will be established if we show that the m_k are independent of the P_{kj} with which we start. For this, choose a Hermitian x so that S_x^k is the identity operator on $H(S^k)$, while $S_x^q = 0$ for $q \neq k$. This x satisfies (25) for any set of P_{kj}. Hence using this x in (22), we have

$$\sigma(x) = m_k d_k.$$

Thus m_k is independent of the choice of the P_{kj}. We have therefore shown that (22) holds for all Hermitian elements, and hence for all elements. The proof is complete.

If A has a unit 1, *then equality holds in* (21). Indeed, substituting $x = 1$ in (22), we have $\sigma(1) = n$, $\mathrm{Tr}(S_1^k) = \dim S^k$.

THEOREM 2.2. *Let $\{T^i\}$ be a net of n-dimensional representations in $\hat{A}$ (n finite), and let $S^1, \cdots, S^r$ be distinct elements of $\hat{A}$ such that for all subnets $\{T'^j\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have $T'^j \to S$ if and only if $S =$ some S^k. Then there exists a subnet $\{T'^j\}$ of $\{T^i\}$, and positive integers $m_1, \cdots, m_r$ such that*

$$\sum_{k=1}^r m_k \dim S^k \leq n, \tag{26}$$

and

$$\lim \operatorname{Tr}(T'^{j}_{x}) = \sum_{k=1}^{r} m_k \operatorname{Tr}(S^k_x) \tag{27}$$

for all x in A.

Proof. For fixed x in A, $|\operatorname{Tr}(T^i_x)| \leqq n\|x\|$ for all i. Hence, picking a universal subnet (see [13, p. 81]) $\{T'^{j}\}$ of $\{T^i\}$, we find that $\operatorname{Tr}(T'^{j}_x)$ converges to some limit for each x in A. Now $\{T'^{j}\}$ satisfies the hypotheses of Lemma 2.5; and the conclusion of that lemma gives the theorem.

The following example shows that in Theorem 2.2 the subnet $\{T'^{j}\}$ is unavoidable; in general, (26) and (27) cannot be satisfied with the original $\{T^i\}$.

Let A be the C^*-algebra of all sequences $x=(x^{(1)}, x^{(2)}, \cdots)$ of 2×2 complex matrices satisfying: (i) $\lim_{i\to\infty} x^{(i)}_{12} = \lim_{i\to\infty} x^{(i)}_{21} = 0$; (ii) $\lim_{i\to\infty} x^{(i)}_{11} = \lim_{i\to\infty} x^{(2i+1)}_{22}$ exists; call it $\sigma(x)$; (iii) $\lim_{i\to\infty} x^{(2i)}_{22} = 0$. Note that σ is a one-dimensional representation of A. If $T^{(n)}$ is the irreducible representation sending x into $x^{(n)}$, we have $\lim_{n\to\infty} \|T^{(n)}_x\| = |\sigma(x)|$ for $x\in A$; so that, by Theorem 2.1, the hypotheses of Theorem 2.2 are satisfied with $\{S^1, \cdots, S^r\} = \{\sigma\}$. On the other hand, $\operatorname{Tr} T^{(n)}_x \to \sigma(x)$ and $2\sigma(x)$ as $n\to\infty$ through even and odd values respectively.

Corollary 1. *Let $\{T^i\}$ be a net of elements of $\hat{A}$, all of dimension equal to or less than the integer n; and let $S^1, \cdots, S^r$ be distinct elements of $\hat{A}$ such that $T^i \to_i S^k$ for each k. Then*

$$\sum_{k=1}^{r} \dim S^k \leqq n.$$

Proof. Pick a universal subnet $\{T'^{j}\}$ of $\{T^i\}$. Then $\dim T'^{j}$ is eventually equal to some $m\leqq n$; $\operatorname{Tr}(T'^{j}_x)$ approaches a limit for each x in A; and $\{S^1, \cdots, S^r\}$ can be enlarged to a set $\{S^1, \cdots, S^t\}$ (finite by Lemma 2.4) for which (i) of Lemma 2.5 holds. Then the hypotheses of Lemma 2.5 hold for $\{T'^{j}\}$; and the conclusion of that lemma gives

$$\begin{aligned} \sum_{k=1}^{r} \dim S^k &\leqq \sum_{k=1}^{t} \dim S^k \\ &\leqq \sum_{k=1}^{t} m_k \dim S^k \leqq n. \end{aligned}$$

Corollary 2 [12, Theorem 4.2]. *If all irreducible representations of A are of the same finite dimension n, $\hat{A}$ is a Hausdorff space.*

Proof. By Corollary 1, no net of elements of $\hat{A}$ can converge to more than one limit.

Corollary 3. *If $\{T^i\}$ is a net of n-dimensional elements of $\hat{A}$ (n finite),*

and $S^1, \cdots, S^r$ *are distinct elements of* $\hat{A}$ *for which* (i) $T^i \rightarrow {}_i S^k$ for all $k=1, \cdots, r$, and (ii) $\sum_{k=1}^r \dim S^k = n$; *then, for all* x *in* A,

$$\lim_i \operatorname{Tr}(T_x^i) = \sum_{k=1}^r \operatorname{Tr}(S_x^k).$$

Proof. If the conclusion fails, there is an x in A, and a subnet $\{T'^i\}$ of $\{T^i\}$, such that $\operatorname{Tr}(T_x'^i)$ eventually lies outside some neighborhood U of $\sum_{k=1}^r \operatorname{Tr}(S_x^k)$. By (ii) and Corollary 1, no subnet of $\{T^i\}$, hence no subnet of $\{T'^i\}$, can converge to any element of $\hat{A}$ distinct from the S^k. Hence the hypotheses of Theorem 2.2 hold, and there are a subnet $\{T''^p\}$ of $\{T'^i\}$, and positive integers $m_1, \cdots, m_r$, such that

$$\sum_{k=1}^r m_k \dim S^k \leqq n, \tag{28}$$

and

$$\lim_p \operatorname{Tr}(T_x''^p) = \sum_{k=1}^r m_k \operatorname{Tr}(S_x^k). \tag{29}$$

Now (ii) and (28) give $m_k = 1$. But then (29) contradicts the definition of $\{T'^i\}$.

11. **Boundedly represented elements.** This section contains a partial converse of Theorem 2.2 (Corollary 2 of Theorem 2.3).

Let H be an arbitrary Hilbert space, with bounded linear operators A, B, C. Denote the range of A by rng A, and the dimension of rng A by dim rng A. The reader will easily verify the following lemma:

LEMMA 2.6. *Suppose* dim rng $A \leqq n$, dim rng $B \leqq m$. *Then* dim rng $A^* \leqq n$, dim rng $(A+B) \leqq n+m$, dim rng $(AC) \leqq n$, *and* dim rng $(CA) \leqq n$. *There is a projection* P *such that* dim rng $P \leqq 2n$, *and* $PAP = A$.

Now let A be a C^*-algebra, and $\mathcal{G}$ a fixed family of $*$-representations of A. An element x of A is *boundedly represented* in $\mathcal{G}$ if there is an integer n such that dim rng $T_x \leqq n$ for all T in $\mathcal{G}$.

LEMMA 2.7. *The elements of* A *which are boundedly represented in* $\mathcal{G}$ *form a self-adjoint two-sided ideal of* A (*not necessarily closed*).

Proof. By Lemma 2.6.

LEMMA 2.8. *Let* x *be a positive element of* A *which is boundedly represented in* $\mathcal{G}$. *If* $S \in \mathcal{G}$, *and* $\{T^i\}$ *is a net of representations in* $\mathcal{G}$, *such that, for all* $m = 1, 2, \cdots$, *we have*

$$\lim_i \operatorname{Tr}(T_{x^m}^i) = \operatorname{Tr}(S_{x^m}), \tag{30}$$

then

$$\lim_i \|T_x^i\| = \|S_x\|.$$

Proof. By Lemma 2.6 and bounded representedness, choose an integer n, an n-dimensional projection P^i in $H(T^i)$, and an n-dimensional projection Q in $H(S)$, so that

$$T_x^i = P^i T_x^i P^i, \qquad S_x = QS_xQ.$$

Since $x \geqq 0$, $T_x^i \geqq 0$; let $\lambda_1^i, \cdots, \lambda_n^i$ be the eigenvalues of T_x^i in rng P^i. Similarly, let $\mu_1, \cdots, \mu_n$ be the eigenvalues of S_x in rng Q. Then

$$\mathrm{Tr}(T_{x^m}^i) = \sum_{k=1}^n (\lambda_k^i)^m,$$

$$\mathrm{Tr}(S_{x^m}) = \sum_{k=1}^n \mu_k^m.$$

Taking mth roots of (30), we have for each m

$$\lim_i \left[\sum_{k=1}^n (\lambda_k^i)^m \right]^{1/m} = \left[\sum_{k=1}^n \mu_k^m \right]^{1/m} \tag{31}$$

But

$$\|T_x^i\| = \max_k \lambda_k^i \leqq \left[\sum_{k=1}^n (\lambda_k^i)^m \right]^{1/m} \leqq n^{1/m} \|T_x^i\|, \tag{32}$$

$$\|S_x\| = \max_k \mu_k \leqq \left[\sum_{k=1}^n \mu_k^m \right]^{1/m} \leqq n^{1/m} \|S_x\|. \tag{33}$$

Combining (31), (32), and (33), and letting m become arbitrarily large, we obtain the conclusion of the lemma.

THEOREM 2.3. *Let $\mathcal{G}$ be a family of *-representations of A, S an element of $\mathcal{G}$, and $\{T^i\}$ a net of elements of $\mathcal{G}$. Suppose A contains a dense self-adjoint subalgebra B such that* (i) *every x in B is boundedly represented in $\mathcal{G}$;* (ii) *for every x in B, $\lim_i \mathrm{Tr}(T_x^i) = \mathrm{Tr}(S_x)$. Then, for every x in A,*

$$\lim_i \|T_x^i\| = \|S_x\|. \tag{34}$$

Proof. For positive elements x of B, (34) follows from Lemma 2.8. This, together with $\|T_{x^*x}^i\| = \|T_x^i\|^2$ and the same for S, implies (34) for all x in B. By continuity and denseness, (34) holds for all x in A.

COROLLARY 1. *Let $\mathcal{G}$, B, $\{T^i\}$ be as in the theorem. If $\lim_i \mathrm{Tr}(T_x^i) = 0$ for all x in B, then $\lim_i \|T_x^i\| = 0$ for all x in A.*

Proof. Take S to be the zero representation in the theorem.

COROLLARY 2. *Assume that A has a dense self-adjoint subalgebra B such that every x in B is boundedly represented in $\hat{A}$. Let $\{T^i\}$ be a net of elements of $\hat{A}$, and $S^1, \cdots, S^r$ a finite sequence of elements of $\hat{A}$ (not necessarily distinct), such that for all x in B,*

$$\lim_i \operatorname{Tr}(T_x^i) = \sum_{k=1}^{r} \operatorname{Tr}(S_x^k). \tag{35}$$

Then, for all S in $\hat{A}$, $T^i \to_i S$ if and only if $S =$ some S^k.

Proof. Let $\mathcal{G}$ be the set of all direct sums of at most r elements of $\hat{A}$. Then B is boundedly represented in $\mathcal{G}$. If $S^0 = S^1 \oplus \cdots \oplus S^r$, then S^0 and T^i all belong to $\mathcal{G}$, and, by (35), we may apply the theorem to conclude, for x in A,

$$\lim_i \|T_x^i\| = \|S_x^0\| = \max_{k=1}^{r} \|S_x^k\|.$$

Now Theorem 2.1 gives the required conclusion.

CHAPTER III. THE DUALS OF THE COMPLEX UNIMODULAR GROUPS

12. **The group algebra of a semi-simple group.** Throughout this section G will be a semi-simple connected Lie group with a faithful continuous matrix representation. Let U be a maximal compact subgroup of G. Fix Haar measure dg in G and du in U (du being normalized so that U has measure 1). Then, for x in $L_1(G)$ and y in $L_1(U)$, there are natural convolutions $x * y$ and $y * x$, both lying in $L_1(G)$. This is a special case of the definition of the convolution of finite measures on G (see [5]).

Denote by E the family of all minimal central projections in $L_1(U)$ (see [14, p. 161]).

The convolution on $L_1(G) \times L_1(U)$ can be extended to a convolution on $C^*(G) \times L_1(U)$. Indeed, if $x \in L_1(G)$, $y \in L_1(U)$, and T is a unitary representation of G,

$$\|T_{y*x}\| = \|T_y T_x\| \leqq \|T_y\| \, \|T_x\| \leqq \|y\|_{L_1(U)} \|x\|_{C^*(G)}.$$

Hence $\|y * x\|_{C^*(G)} \leqq \|y\|_{L_1(U)} \|x\|_{C^*(G)}$; and similarly for $x * y$. It follows that $x * y$ can be defined on $C^*(G) \times L_1(U)$ so as to be jointly continuous in both variables; and similarly for $y * x$. The equations $T_{x*y} = T_x T_y$, $T_{y*x} = T_y T_x$ are preserved under this extension.

Since finite linear combinations of elements $e * y$ ($e \in E$, $y \in L_1(U)$) are dense in $L_1(U)$, we easily obtain:

LEMMA 3.1. *Finite linear combinations of the $e * x * f$ ($e, f \in E$, $x \in C^*(G)$) are dense in $C^*(G)$.*

Now it is proved in [7, Lemma 4, p. 505] that, for each e in E, the subalgebra $e * L_1(G) * e$ satisfies a standard polynomial identity. Since this subalgebra is dense in $e * C^*(G) * e$, we have:

LEMMA 3.2. *For each e in E, $e * C^*(G) * e$ satisfies a standard polynomial identity. Consequently there is an integer n such that every irreducible representation of $e * C^*(G) * e$ is of dimension equal to or less than n.*

Let T be an irreducible representation of G, or, equivalently, of $C^*(G)$. For e in E, T_{e*x*e} leaves the range of T_e invariant and annihilates its orthogonal subspace.

LEMMA 3.3. *T restricted to $e * C^*(G) * e$ is irreducible on the range of T_e.*

This is proved as [7, Lemma 3, p. 505].

Now let B' be the set of all finite linear combinations of elements $e * x * f$ $(e, f \in E,\ x \in C^*(G)$.

LEMMA 3.4. *B' is a dense self-adjoint subalgebra of $C^*(G)$. Every element of B' is boundedly represented in $\hat{G}$ $(=C^*(G))\hat{}$.*

Proof. By Lemmas 3.1, 3.2, and 3.3.

This lemma of course connects with Theorem 2.3 and its corollaries.

13. **The complex unimodular groups.** For the rest of this chapter we shall be considering the $n \times n$ complex unimodular group $G = SL(n)$, the group of all complex $n \times n$ matrices of determinant 1 (n being fixed). The irreducible unitary representations of G belonging to the various principal and supplementary series, as well as their characters, are described in [4]; and are proved in [17] to exhaust all of $\hat{G}$. Our goal now is to find the topology of $\hat{G}$. The principal tool for this will be Corollary 2 of Theorem 2.3, applied to the characters of the representations.

It is shown in [4] that the elements of $\hat{G}$ may be specified by parameters as follows:

By a (*proper*) *set of parameters* we shall mean a triple ν, μ, ρ, where, for some $r = 1, 2, \cdots, n$, $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_r)$, and $\rho = (\rho_1, \rho_2, \cdots, \rho_r)$ are three r-termed sequences satisfying:

(i) The ν_i are positive integers satisfying $\sum_{i=1}^{r} \nu_i = n$;

(ii) The μ_i are integers;

(iii) The ρ_i are complex;

(iv) For some permutation p of $\{1, \cdots, r\}$, and some non-negative integer τ such that $0 \leqq 2\tau \leqq r$, we have: (a) $\nu_{p(i)} = 1$ for $i = 1, \cdots, 2\tau$; (b) for $q = 1, \cdots, \tau$, $\mu_{p(2q-1)} = \mu_{p(2q)}$, $\rho_{p(2q-1)} = \bar{\rho}_{p(2q)}$, $0 < \operatorname{Im} \rho_{p(2q)} < 1$; (c) for $i = 2\tau + 1, \cdots, r$, $\rho_{p(i)}$ is real.

In case ν, μ, ρ satisfy (i), (ii), (iii) and (iv′), where (iv′) is obtained from (iv) by replacing in (b) "$0 < \operatorname{Im} \rho_{p(2q)} < 1$" by "$0 < \operatorname{Im} \rho_{p(2q)} \leqq 1$," we shall speak of ν, μ, ρ as an *extended set of parameters*.

Fix an extended set of parameters ν, μ, ρ. The sth *block of integers* (*with respect to* ν) will be the set $\{\nu_1 + \cdots + \nu_{s-1} + 1, \cdots, \nu_1 + \cdots + \nu_s\}$. If $g \in G$, let g_{st}^* be the submatrix of g consisting of the rows of the sth block and

the columns of the tth block. Define K to be the subgroup of G consisting of those k for which $k^*_{st}=0$ whenever $s>t$ (triangular block matrices). Finally, denote by X the complex homomorphism of K:

$$X(k) = \prod_{j=1}^{r} |\Lambda_j|^{\mu_j+i\rho_j}\Lambda_j^{-\mu_j}, \tag{1}$$

where $\Lambda_j=\det(k^*_{jj})$. Obviously, K and X depend on ν, μ, ρ.

If ν, μ, ρ is a *proper* set of parameters, the homomorphism X of K induces an irreducible unitary representation $T=T^{\nu,\mu,\rho}$ of G (see [4]); these exhaust all of $\hat{G}$ (see [17]).

There are equivalences among the $T^{\nu,\mu,\rho}$, as follows:

LEMMA 3.5. $T^{\nu,\mu,\rho}\cong T^{\nu',\mu',\rho'}$ *if and only if* (a) *the length* r *of the sequences* ν, μ, ρ *is equal to the length of* ν', μ', ρ', *and* (b) *there exist a permutation* p *of* $\{1, \cdots, r\}$, *an integer* m, *and a real* s, *such that, for* $i=1, \cdots, r$, *we have*

$$\nu_i' = \nu_{p(i)}, \qquad \mu_i' = \mu_{p(i)} + m, \qquad \rho_i' = \rho_{p(i)} + s.$$

The representations $T^{\nu,\mu,\rho}$ are classified into series as follows, in accordance with the values of r, $\nu_1, \cdots, \nu_r$, and τ:

In view of Lemma 3.5, we assume in this paragraph, without loss of generality, that $\nu_1\geqq\nu_2\geqq \cdots \geqq\nu_r$. If this is so, the $(\nu_1, \cdots, \nu_r; \tau)$ *series* will be the set of all $T^{\nu,\mu,\rho}$ having the given ν and the given value of τ. Each T in G lies in one and only one such series. If $\tau=0$, the series is *principal*; otherwise it is *supplementary*. If $r=n$ and all $\nu_i=1$, the series is *nondegenerate*; otherwise it is *degenerate*.

For example, if $n=2$, the elements of $\hat{G}$ are classified as follows:

(a) The principal nondegenerate series: $\tau=0$, $r=2$, $\nu_1=\nu_2=1$. By Lemma 3.5, the representations of this series are determined by $m=\mu_1-\mu_2$ and $r=\rho_1-\rho_2$. We write $T^{\nu,\mu,\rho}=T^{m,r}$ in this case where m and r run over the integers and the reals respectively. One has $T^{m,r}\cong T^{m',r'}$ if and only if either $m=m'$, $r=r'$ or $m=-m'$, $r=-r'$.

(b) The supplementary nondegenerate series: $\tau=1$, $r=2$, $\nu_1=\nu_2=1$. By Lemma 3.5, we may take $\mu_1=\mu_2=0$, $\rho_1=-is$, $\rho_2=is$, where $0<s<1$. Writing T^s for $T^{\nu,\mu,\rho}$, we have $T^s\cong T^{s'}$ if and only if $s=s'$.

(c) The principal degenerate series: $\tau=0$, $r=1$, $\nu_1=2$. This contains only the identity representation I.

14. **The parameter space.** Let Q_r be the family of all extended sets of parameters with sequences of length r. As a subset of $3r$-dimensional complex space, Q_r acquires a natural topology. The set $Q=\bigcup_{r=1}^{n} Q_r$ of all extended sets of parameters has a natural topology as the direct sum of the Q_r; with this topology we call it the *extended parameter space*. The family P of all proper sets of parameters is a dense subspace of Q, called the (*proper*) *parameter space*.

Define an equivalence relation $\sim$ in Q by requiring that $(\nu, \mu, \rho) \sim(\nu', \mu', \rho')$ if and only if conditions (a) and (b) hold in Lemma 3.5. The set $\tilde{Q}$ of equivalence classes of $\sim$ inherits a natural quotient topology from Q, which is clearly locally compact and Hausdorff. The set $\tilde{P}$ of those equivalence classes which are contained in P is a dense subset of $\tilde{Q}$. The topology of $\tilde{Q}$ relativized to $\tilde{P}$ will be called the *natural topology* of $\tilde{P}$. By Lemma 3.5 $\tilde{P}$ is in one-to-one correspondence with $\hat{G}$. The topology of $\hat{G}$ transferred to $\tilde{P}$ will be the *hull-kernel topology* of $\tilde{P}$. We may sometimes fail to distinguish corresponding elements of $\tilde{P}$ and $\hat{G}$.

15. **The characters.** It is shown in [4] that each T in $\hat{G}$ is characterized by a complex function $\gamma=\gamma^T$ on G, called the *character* of T, and given by

$$\gamma(\delta) = \frac{\sum_s' X(\delta^{(s)}) \prod_{i=1}^{r} \left| D(\delta_i^{(s)}) \right| \left| \det \delta_i^{(s)} \right|^{-\nu_i}}{\left| D(\delta) \right|}. \tag{2}$$

The notation in (2) will first be explained. Let ν, μ, ρ be a set of parameters for T, the sequences being of length r.

If δ is a diagonal $n\times n$ matrix, and s is a permutation of $\{1, \cdots, n\}$, then $\delta^{(s)}$ will be the diagonal matrix $(\delta^{(s)})_{ii}=\delta_{s(i),s(i)}$. If $i=1, \cdots, r$, then δ_i will mean δ_{ii}^* (see §13); and $\delta_i^{(s)}$ means $(\delta^{(s)})_i$. Denote by Z the group of permutations of $\{1, \cdots, n\}$ which leave setwise invariant each ith block of integers with respect to ν $(i=1, \cdots, r)$. The symbol $\sum_s'$ means summation over a set of permutations s of $\{1, \cdots, n\}$ which contains exactly one permutation from each left coset sZ. The X is the complex homomorphism defined in (1). $D(\delta)$ is the discriminant of the characteristic equation of δ, i.e., for a diagonal $n\times n$ matrix δ, $D(\delta)=\sum_{1\leq i<j\leq n} (\delta_{ii}-\delta_{jj})^2$ (if $n=1$, $D(\delta)=1$).

Clearly $\gamma(\delta^{(s)})=\gamma(\delta)$ for each permutation s of $\{1, \cdots, n\}$. If g is a matrix in G whose eigenvalues are all distinct, we may define $\gamma(g)$ without ambiguity by setting $\gamma(g)=\gamma(\delta)$, where δ is any diagonal matrix whose diagonal elements are the eigenvalues of g. The γ so defined (on almost all of G) we call γ^T or $\gamma^{\nu,\mu,\rho}$.

LEMMA 3.6. *If x is of the form $y * z$, where y, z are continuous complex functions with compact support on G, then for each proper set of parameters ν, μ, ρ,*

$$\mathrm{Tr}(T_x^{\nu,\mu,\rho}) = \int_G x(g)\gamma^{\nu,\mu,\rho}(g)dg.$$

For this lemma, see [4, Theorem 11, p. 146 and Theorem 12, p. 150].

Now (1) and (2) will be used to define $\gamma=\gamma^{\nu,\mu,\rho}$ as a function on G even when ν, μ, ρ is only an *extended* set of parameters. A routine verification gives:

LEMMA 3.7. *If (ν, μ, ρ) and (ν', μ', ρ') are in Q, and $(\nu, \mu, \rho)\sim(\nu', \mu', \rho')$, then $\gamma^{\nu,\mu,\rho}=\gamma^{\nu',\mu',\rho'}$.*

In view of this lemma, γ^q may be considered as defined for the equivalence classes q belonging to $\tilde{Q}$. Indeed, if $(\nu, \mu, \rho) \in q \in \tilde{Q}$, put $\gamma^q = \gamma^{\nu,\mu,\rho}$.

LEMMA 3.8. *If $q \in \tilde{Q}$, $\{q^i\}$ is a net of elements of $\tilde{Q}$, and $q^i \to q$ in the natural topology of $\tilde{Q}$, then, for each bounded function x on G with compact support,*

$$\int x(g)\gamma^{q^i}(g)dg \underset{i}{\to} \int x(g)\gamma^q(g)dg. \tag{3}$$

Proof. Since $\tilde{Q}$ has a countable basis of open sets, it is enough to suppose that $\{q^i\}$ is a sequence.

As g in G ranges over the compact support of x, its eigenvalues also range over a compact set. Hence $\gamma^q(g)$ and the $\gamma^{q^i}(g)$ are all majorized by a constant times $1/|D(\delta)|$; and the latter is summable over any compact subset of G. Further, it is clear that $\gamma^{q^i}(g) \to \gamma^q(g)$ for almost all g. Equation (3) now follows from the Lebesgue dominated-convergence theorem.

LEMMA 3.9. *If $\{q^i\}$ is a net of elements of $\tilde{Q}$ converging to the point at infinity of $\tilde{Q}$, then, for each bounded function x on G with compact support,*

$$\int x(g)\gamma^{q^i}(g)dg \underset{i}{\to} 0. \tag{4}$$

Proof. Δ being the diagonal subgroup of G, it is clearly possible to decompose Haar measure dg thus:

$$\int_G f(g)dg = \int_\Delta d\delta \int_G f(g^{-1}\delta g)d\mu_\delta g, \tag{5}$$

where $d\delta$ is Haar measure on Δ, and μ_δ is some measure on G depending on δ.

Let q^i contain the extended set of parameters (ν^i, μ^i, ρ^i); and let X^i be the homomorphism (1) corresponding to this set. In proving (4) it is clearly sufficient to suppose the ν^i are all the same. Substituting the definition of γ^{q^i} into (5), we find that $\int x(g)\gamma^{q^i}(g)dg$ is the sum of a finite number of terms (the number being independent of i), each of which is of the form

$$\int_\Delta X^i(\delta)L(\delta)d\delta, \tag{6}$$

L being summable on Δ and independent of i.

Now $X^i(\delta) = X_1^i(\delta)X_2^i(\delta)$, where $X_1^i(\delta) = (X^i(\delta))/(|X^i(\delta)|)$, $X_2^i(\delta) = |X^i(\delta)|$. But X_1^i is a character belonging to the dual group $\hat{\Delta}$ of the commutative group Δ. As $q^i \to \infty$ in $\tilde{Q}$, it is easy to see that $X_1^i \to \infty$ in $\hat{\Delta}$. If X_2^i were independent of i, the desired conclusion that (6) approaches 0 in i would follow from the well-known theorem that the Fourier transform of a summable function L on $\hat{\Delta}$ is 0 at ∞ (in $\hat{\Delta}$). On the other hand, though X_2^i does depend on i, we have

$|\operatorname{Im} \rho_k^i| \leqq 1$; and a uniformity argument shows that the desired conclusion follows in this case also.

16. **The sum formula.** Let q be an element of $\tilde{Q}-\tilde{P}$. Then the function γ^q is not the character of any representation. Nevertheless, we shall show in this section that γ^q is the sum of a finite number of characters of representations. It will turn out that the non-Hausdorff character of $\hat{G}$ arises from this sum formula.

Suppose that ν, μ, ρ is an extended, but not a proper, set of parameters; in fact, suppose that $\nu_1=\nu_2=1$, $\mu_1=\mu_2=\mu$, $\rho_1=\sigma-i$, $\rho_2=\sigma+i$. Z will be as in §15; and S will denote a *cross-section* of left Z cosets, i.e., a set of permutations of $\{1, \cdots, n\}$ containing exactly one element from each left coset sZ. Let δ be a diagonal matrix in G; $\delta_j^{(s)}$ means the same as in §15.

By (2), $\gamma^{\nu,\mu,\rho}=\gamma$ is given by

$$|D(\delta)|\gamma(\delta) = \sum_{s\in S} \prod_{j=1}^{r} \left\{ |\det \delta_j^{(s)}|^{\mu_j+i\rho_j-\nu_j} (\det \delta_j^{(s)})^{-\mu_j} |D(\delta_j^{(s)})| \right\}.$$

For each $u, v=1, \cdots, n$, $u\neq v$, we define $S(u, v)$ to be the set of all s in S for which $s(1)=u$, $s(2)=v$. Noting that $S=\bigcup_{u\neq v} S(u, v)$, we transform the last equation to

$$\begin{aligned} |D(\delta)|\gamma(\delta) &= \sum_{u,v=1;u\neq v}^{n} \Big[|\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu} |\delta_{vv}|^{\mu+i\sigma-2}\delta_{vv}^{-\mu} \\ &\qquad \times \sum_{s\in S(u,v)} \prod_{j=3}^{r} |\det \delta_j^{(s)}|^{\mu_j+i\rho_j-\nu_j}(\det \delta_j^{(s)})^{-\mu_j} |D(\delta_j^{(s)})| \Big] \\ &= \sum_{u<v} \Big[\Big\{ |\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu}|\delta_{vv}|^{\mu+i\sigma-2}\delta_{vv}^{-\mu} + |\delta_{vv}|^{\mu+i\sigma}\delta_{vv}^{-\mu}|\delta_{uu}|^{\mu+i\sigma-2}\delta_{uu}^{-\mu}\Big\} \\ &\qquad \times \sum_{s\in S(u,v)} \prod_{j=3}^{r} |\det \delta_j^{(s)}|^{\mu_j+i\rho_j-\nu_j}(\det \delta_j^{(s)})^{-\mu_j}|D(\delta_j^{(s)})|\Big]. \end{aligned} \tag{7}$$

Now let $\nu'=(2, \nu_3, \cdots, \nu_r)$, $\mu'=(\mu, \mu_3, \cdots, \mu_r)$, $\rho'=(\sigma, \rho_3, \cdots, \rho_r)$. If Z' is the group of permutations of $\{1, \cdots, n\}$ leaving setwise invariant the blocks with respect to ν', then $S'=\bigcup_{u,v=1;u<v}^{n} S(u, v)$ is a cross-section of left Z' cosets. Therefore, abbreviating $\gamma^{\nu',\mu',\rho'}$ to γ', we have

$$\begin{aligned} |D(\delta)|\gamma'(\delta) = \sum_{u<v} \Big\{ &|\delta_{uu}\delta_{vv}|^{\mu+i\sigma-2}(\delta_{uu}\delta_{vv})^{-\mu}|\delta_{uu}-\delta_{vv}|^2 \\ &\times \sum_{s\in S(u,v)} \prod_{j=3}^{r} |\det \delta_j^{(s)}|^{\mu_j+i\rho_j-\nu_j}(\det \delta_j^{(s)})^{-\mu_j}|D(\delta_j^{(s)})|\Big\}. \end{aligned} \tag{8}$$

Note that for any complex x, y,

$$\frac{1}{|x|^2}+\frac{1}{|y|^2}-\frac{|x-y|^2}{|xy|^2}=\frac{y}{x|y|^2}+\frac{x}{y|x|^2}.$$

Equations (7) and (8) thus give

$$\begin{aligned} |D(\delta)|(\gamma(\delta)-\gamma'(\delta)) &= \sum_{u\neq v}\Big\{|\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu-1}|\delta_{vv}|^{\mu+i\sigma-2}\delta_{vv}^{-\mu+1} \\ &\quad\times\sum_{s\in S(u,v)}\prod_{j=3}^{r}|\det\delta_j^{(s)}|^{\mu_j+i\rho_j-\nu_j}(\det\delta_j^{(s)})^{-\mu_j}|D(\delta_j^{(s)})|\Big\} \\ &= |D(\delta)|\gamma''(\delta), \end{aligned}$$

where $\gamma''=\gamma^{\nu'',\mu'',\rho''}$, and $\nu''=\nu$,

$$\mu''=(\mu+1,\mu-1,\mu_3,\cdots,\mu_r),\qquad \rho''=(\sigma,\sigma,\rho_3,\cdots,\rho_r).$$

We thus obtain

$$\gamma^{\nu,\mu,\rho}=\gamma^{\nu',\mu',\rho'}+\gamma^{\nu'',\mu'',\rho''}. \tag{9}$$

For convenience we describe an extended set of parameters ν, μ, ρ by a $3\times r$ matrix

$$A=\begin{pmatrix}\nu_1 & \nu_2\cdots\nu_r\\ \mu_1 & \mu_2\cdots\mu_r\\ \rho_1 & \rho_2\cdots\rho_r\end{pmatrix}, \tag{10}$$

and write γ^A for $\gamma^{\nu,\mu,\rho}$. Now, iterating (9), we obtain the following lemma:

LEMMA 3.10. *If*

$$A=\begin{pmatrix}1 & 1 & 1 & 1 & \cdots & 1 & 1 & \nu_{2\tau+1}\cdots\nu_r\\ \mu_1 & \mu_1 & \mu_2 & \mu_2 & \cdots & \mu_\tau & \mu_\tau & \mu_{2\tau+1}\cdots\mu_r\\ \sigma_1-i & \sigma_1+i & \sigma_2-i & \sigma_2+i & \cdots & \sigma_\tau-i & \sigma_\tau+i & \rho_{2\tau+1}\cdots\rho_r\end{pmatrix}, \tag{11}$$

we have $\gamma^A=\sum_{M_1,\cdots,M_\tau}\gamma^{(M_1\cdots M_\tau R)}$, *where*

$$R=\begin{pmatrix}\nu_{2\tau+1}\cdots\nu_r\\ \mu_{2\tau+1}\cdots\mu_r\\ \rho_{2\tau+1}\cdots\rho_r\end{pmatrix} \tag{12}$$

and each M_j $(j=1,\cdots,\tau)$ *runs over the two possibilities*

$$\begin{pmatrix}2\\ \mu_j\\ \sigma_j\end{pmatrix}\quad\textit{and}\quad\begin{pmatrix}1 & 1\\ \mu_j+1 & \mu_j-1\\ \sigma_j & \sigma_j\end{pmatrix}. \tag{13}$$

For illustration, let us apply Lemma 3.10 to the 2×2 case. Here $\tilde{Q}-\tilde{P}$ contains exactly one element q, described by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -i & i \end{pmatrix};$$

q is the limit, in the natural topology, of the representations T^s (of the supplementary series) as $s\to 1$. Since

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

corresponds to the identity representation I, and

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$$

to the representation $T^{2,0}$ of the principal series (see end of §13), Lemmas 3.8 and 3.10 give

$$\lim_{s\to 1-} \gamma^{T^s} = \gamma^q = \gamma^I + \gamma^{T^{2,0}}. \tag{14}$$

17. **The topology of $\hat{G}$.** We are now ready to describe completely the hull-kernel topology of $\tilde{P}$ $(\cong\hat{G})$.

If A is the matrix of ν, μ, ρ (see (10)), let $[A]$ be the point of $\tilde{Q}$ to which ν, μ, ρ belongs.

Theorem 3.1. *With each point q of $\tilde{Q}$, we associate one or more points of $\tilde{P}$ as follows*:

(i) *If $q\in\tilde{P}$, with q is associated just q itself*;

(ii) *If $q\in\tilde{Q}-\tilde{P}$, $q=[A]$, where A is the matrix* (11), *with* $|\operatorname{Im}\rho_j|<1$ *for $j>2\tau$, then with q are associated precisely the*

$$[M_1\cdots M_\tau R],$$

where R is as in (12), *and each M_j runs over the two alternatives* (13).

Now if $A\subset\tilde{P}$, the hull-kernel closure of A consists exactly of those p in $\tilde{P}$ which are associated with some q in the natural closure of A (*with respect to $\tilde{Q}$*).

Proof. Recalling the definition of E (§12), we define B as the set of all finite linear combinations of elements $e*x*y*f$, where $e, f\in E$, and x and y are continuous functions with compact support on G. Since the $x*y$ are dense in $C^*(G)$, B is dense in B' (see §12); hence, by Lemma 3.4, B is a dense

self-adjoint subalgebra of $C^*(G)$, all of whose elements are boundedly represented in $\hat{G}$. By Lemma 3.6,

$$\operatorname{Tr}(T_x^{\nu,\mu,\rho}) = \int_G x(g)\gamma^{\nu,\mu,\rho}(g)dg \qquad \text{for } x \in B. \tag{15}$$

Let $A \subset \tilde{P}$. If q belongs to the natural closure of A (in $\tilde{Q}$), there is a sequence $\{q^n\}$ of elements of A with $q^n \to q$. By (15) and Lemma 3.8,

$$\lim_n \operatorname{Tr}(T_x^{q^n}) = \int x(g)\gamma^q(g)dg$$

for $x \in B$. By Lemma 3.10 and the definition of associated elements, this implies that for $x \in B$,

$$\lim_n \operatorname{Tr}(T_x^{q^n}) = \operatorname{Tr}\left(\sum_{q'} \oplus T_x^{q'}\right), \tag{16}$$

where q' runs over the elements of $\tilde{P}$ which are associated with q. This combined with Corollary 2 of Theorem 2.3 shows that every q' associated with q belongs to the hull-kernel closure of A.

Conversely, let p in $\tilde{P}$ belong to the hull-kernel closure of A. Select a net $\{p^i\}$ of elements of A converging hull-kernelwise to p. Now no subnet $\{p'^i\}$ of $\{p^i\}$ converges (in the natural topology) to the point at infinity of $\tilde{Q}$. For, if it did, Lemma 3.9 and Corollary 2 of Theorem 2.3 would tell us that $\{p'^i\}$ converged hull-kernelwise to no limit at all; which is impossible.

Thus all natural cluster points of $\{p^i\}$ are in the finite part of $\tilde{Q}$. Let q be such a cluster point; and $\{p'^i\}$ a subnet of $\{p^i\}$ converging naturally to q. Again by Lemmas 3.8 and 3.10, and Corollary 2 of Theorem 2.3, $\{p'^i\}$ can converge in the hull-kernel topology to no p' except those associated with q; and the same is true of $\{p^i\}$. We have shown that every point p in the hull-kernel closure of A is associated with a point in the natural closure of A. This completes the proof.

For illustration, consider the 2×2 case. Referring to the end of §13, we see that $\tilde{Q}$ with the natural topology is homeomorphic to the subset W of the plane consisting of:

(i) all (m, r), where m is a positive integer and r is real;
(ii) all $(0, r)$ where $r \geqq 0$;
(iii) all $(-s, 0)$, where $0 < s \leqq 1$;
(iv) an isolated point, say $(-2, 0)$.

Here the (m, r) or (i) or (ii) corresponds to the representation $T^{m,r}$ of the principal series; $(-s, 0)$ corresponds to the representation T^s of the supplementary series for $0 < s < 1$; $(-2, 0)$ corresponds to the identity representation I; and $(-1, 0)$ corresponds to the one and only point of $\tilde{Q}-\tilde{P}$. By §16, $(-1, 0)$ is associated with $(-2, 0)$ and $(2, 0)$.

Corollary 1. *In the 2×2 case, transfer the hull-kernel topology of $\hat{G}$ to the subset W of the plane by means of the above correspondence. If $A\subset W-\{(-1,0)\}$ the hull-kernel closure of A is equal to*

(a) *the natural closure $\overline{A}$ of A unless $\overline{A}$ contains $(-1, 0)$;*

(b) $(\overline{A}-\{(-1, 0)\})\cup\{(-2, 0), (2, 0)\}$ *if $\overline{A}$ contains $(-1, 0)$.*

From Corollary 1 we see that *$\hat{G}$ is not Hausdorff*. In fact, if $s\to 1-$, T^s approaches both I and $T^{2,0}$. The same is true for the group G with general n. However, the deviation from the Hausdorff property is rather weak, as is shown by the next corollary:

Corollary 2. *If G is the $n\times n$ complex unimodular group, no net of elements of $\hat{G}$ converges to more than $2^{[n/2]}$ distinct limits.*

Proof. By an argument exactly similar to the second half of the proof of Theorem 3.1, for each net of elements of $\hat{G}$ there is a q in $\tilde{Q}$ such that every limit of the net is associated with q. But, by the definition of associated elements, the largest number of elements in $\tilde{P}$ associated with any one q in $\tilde{Q}$ is $2^{[n/2]}$.

Recall that each extended set of parameters ν, μ, ρ, hence each q in $\tilde{Q}$, is associated with a certain value of τ, called its *τ-value*, namely, half the number of nonreal terms in the sequence ρ. Let $\tilde{Q}_\tau$ be the set of q in Q having τ-value τ. The following fact is immediate:

Lemma 3.11. *An element q' associated with a q in $\tilde{Q}_\tau$ has τ-value $\tau'<\tau$.*

Corollary 3. *The topology of $\hat{G}$ relativized to the set A_τ of those representations having fixed τ-valued τ, is Hausdorff.*

Proof. Consider A_τ as a subset of $\tilde{P}$. If $B\subset A_\tau$, denote by $\overline{B}$ and $\overline{B}^H$ the closure of B in the natural and hull-kernel topologies respectively. It follows from Theorem 3.1 and Lemma 3.11 that

$$\overline{B}\cap A_\tau=\overline{B}^H\cap A_\tau.$$

Hence the hull-kernel and the natural topologies coincide when relativized to A_τ. Thus the former is Hausdorff on A_τ.

Corollary 4. *The topology of $\hat{G}$, relativized to the union of all the principal series, is Hausdorff.*

Proof. Put $\tau=0$ in Corollary 3.

Corollary 5. *For each fixed τ_0, the set $\mathcal{G}(\tau_0)$ of all T in $\hat{G}$ with τ-values equal to or less than τ_0 is closed in $\hat{G}$.*

Proof. By Lemma 3.11.

Note that $r=[n/2]$ is the largest permissible value of τ. Corollaries 3 and 5 show that the sequence

$$\mathcal{G}(0), \mathcal{G}(1), \cdots, \mathcal{G}(r) = \hat{G}$$

is an ascending sequence of closed subsets of $\hat{G}$ such that each $\mathcal{G}(i) - \mathcal{G}(i-1)$ is Hausdorff in the relativized topology of $\hat{G}$. An easy argument (see [12, particularly Lemma 4.1, p. 233]) now shows:

COROLLARY 6. *There is a finite increasing sequence*

$$I_0 = \{0\}, \qquad I_1, I_2, \cdots, I_r = C^*(G)$$

of closed two-sided ideals of $C^*(G)$ such that each I_i/I_{i-1} $(i=1, \cdots, r)$ has a Hausdorff structure space.

COROLLARY 7. *A subset A of $\hat{G}$ which (considered as a subset of $\tilde{Q}$) is closed in the natural topology is also closed in $\hat{G}$, and is Hausdorff as a subspace of $\hat{G}$.*

Proof. The closure of A in $\hat{G}$ follows immediately from Theorem 3.1. If $B \subset A$, the hull-kernel closure of B is equal to the natural closure. Thus the hull-kernel and natural topologies relative to A coincide, and the former is Hausdorff.

COROLLARY 8. *Each principal series is a closed subset of $\hat{G}$.*

Proof. By Corollary 7.

18. **The regular representation of** G. It is well known (see [4]) that the regular representation L of G is a direct integral of representations of the principal non-degenerate series. In fact, if $\mathcal{G}$ is the locally compact Hausdorff space consisting of the principal nondegenerate series (see Corollary 8), then

$$L = \int_{\mathcal{G}} \oplus \, (\aleph_0 \cdot T)$$

(see §6) with respect to a measure on $\mathcal{G}$ whose closed hull is $\mathcal{G}$. By Theorem 1.7, L is weakly equivalent to the set of all $\aleph_0 \cdot T$, $T \in \mathcal{G}$; hence (see remark preceding Theorem 1.6) L is weakly equivalent to $\mathcal{G}$. It follows that the spectrum of L (see Theorem 1.6) is the closure of $\mathcal{G}$ in $\hat{G}$, i.e., $\mathcal{G}$ itself.

THEOREM 3.2. *The spectrum of the regular representation of G is precisely the principal nondegenerate series.*

Thus G is an example of a locally compact group whose regular representation does not weakly contain all irreducible representations (see §7).

Theorem 3.2 also shows that the implication (i)$\rightarrow$(iii) in Theorem 1.2 fails for general Banach *-algebras. Indeed, let A be obtained by adjoining a unit element to the group algebra $L_1(G)$, G being as usual the $n \times n$ unimodular group. Let T be an element of $\hat{G}$ not belonging to the principal nondegenerate series; and consider it as acting on A. If a positive functional ϕ on A associated with T is a weak* limit of sums ψ_i of positive functionals on A associated with the regular representation L, an easy argument shows that $\psi_i \rightarrow \phi$

weakly* even after ψ_i and ϕ are extended to $C^*(G)$; and hence that the regular representation weakly contains T. But this is untrue by Theorem 3.2. Therefore condition (iii) of Theorem 1.2 fails, when A, T are as defined above, and $\mathcal{S}$ consists of L only. On the other hand, L is well known to be faithful on A; so that (i) holds. Thus the implication (i)→(iii) fails in this situation.

19. **Concluding remarks.** One naturally asks what is the relationship between the topology of $\hat{G}$ discussed in this paper and the Borel structure on $\hat{G}$ defined by Mackey in [15]. The author will deal with this question, and also with other equivalent definitions of the topology of $\hat{G}$, in a forthcoming note.

Corollaries 2 and 6 of Theorem 3.1 have shown that the departure in $\hat{G}$ from the Hausdorff property is fairly weak, when G is a complex unimodular group. Presumably the same result is true for arbitrary connected semi-simple Lie groups with faithful matrix representations; but the author has not been able to prove it.

The results of this paper form only an introduction to the detailed study of the structure of the group C^*-algebras of semi-simple groups, or, more generally, of C^*-algebras A whose irreducible representations are all completely continuous. Perhaps, as suggested by Kaplansky in [14], the cases where the structure space is Hausdorff form the appropriate building-blocks for the general case. If so, it would appear that further progress must take two directions: (a) the analysis of A, in case $\hat{A}$ is Hausdorff, in terms of fibre bundles with $\hat{A}$ as base space; (b) the extension problem—how a construct A when I and A/I are known (I being a closed two-sided ideal of A). The author has made some headway in problem (a), in the case that all irreducible representations are of the same finite dimension.

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