UNIQUE CONTINUATION FOR ELLIPTIC EQUATIONS(1)

BY

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1. **Introduction.** Let $u(x_1, x_2, \cdots, x_N)$ be a solution of an elliptic equation in a domain D. If u vanishes in an open subset of D the unique continuation principle asserts that u vanishes throughout D. A related question concerns the vanishing of a solution at a particular point of D. Can a solution u approach zero arbitrarily rapidly at a point P of D without being identically zero? Carleman [4] showed that for elliptic systems in two independent variables with coefficients that are not necessarily analytic a solution which approached zero at a point more rapidly than any power of the distance r from that point must vanish identically. Such questions for two independent variables were also investigated by Bers [2a] in conjunction with the theory of pseudo-analytic functions and by Douglis [6].

For equations in more than two variables Muller [13] showed that if $r^{-n}u \to 0$ as $r \to 0$ for every positive n and if u satisfies the equation $\Delta u = F(x_1, x_2, \cdots, x_N, u)$ then u vanishes identically in D. In this connection Heinz [9], Hartman and Wintner [8] and Lax [11] extended the result to include functions F containing first derivatives of u. The problem for general second order operators L was solved by Cordes [5] and Aronszajn [1].

The problem for higher order elliptic equations may be posed in a similar manner. Nirenberg [14] has obtained results for uniqueness of the Cauchy problem in the case of equations with constant leading coefficients. Calderón [3] obtained general results for the uniqueness of the Cauchy problem by the method of singular integrals valid quite generally for second and third order equations and for higher order equations if the number of independent variables is different from three and there are no multiple characteristics. Friedman [7] has established uniqueness for solutions of linear elliptic equations under the additional restriction that the solution be positive. Hörmander [10] has followed the method of L_2 estimates for general lower order operators in terms of higher order operators with appropriate weight functions. Mizohata [12] has obtained results for elliptic equations with constant leading terms and for a general fourth order elliptic equation.

A new and interesting approach to this problem has been given recently by Pederson [15]. In order to establish unique continuation he requires a

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stronger condition than u vanishing at a single point more rapidly than any power of r, namely that

$$e^{\alpha r^{-\beta}} u \to 0 \quad \text{as} \quad r \to 0$$

for every positive α and for some fixed sufficiently large β. In this manner he is able to establish the unique continuation property and hence uniqueness for the Cauchy problem for general second order equations. In addition, the problem for fourth order equations is solved when the principal part is the biharmonic operator.

In the present paper the starting point is the kernel employed by Pederson. We impose an even stronger restriction on u at one point, namely that

$$e^{r^{-\beta}} u \to \quad \text{as} \quad r \to 0$$

for every positive β. Of course for purposes of unique continuation and uniqueness of the Cauchy problem the speed with which a solution approaches zero at a single point is immaterial. Under such hypothesis we give in §2 a particularly simple proof of unique continuation for second order equations with Laplacian as principal part. The technique is based on that of Cordes [5] and Pederson [15]. However they both resort to spherical coordinates which in some senses make the computations more complicated. This is seen most clearly in the application to general second order equations. Here we obtain a straight-forward extension in §3 without resort to any study of local differential geometry—used in the proofs given by Aronszajn [1], Cordes [5], and Pederson [15].

In §4 a basic inequality is established which together with the particular form we have for the estimates in §§2 and 3 permit an extension to higher order equations. The method establishes unique continuation for inequalities of the form

$$\Delta^n u \leqq f(x, u, Du, \cdots, D^k u)$$

in which the right hand side may contain all derivatives of order up to $[3n/2]$. This result is obtained by certain L_2 estimates for lower order derivatives in terms of the iterated Laplacian with varying weight functions of the form $r^\gamma \exp(2r^{-\beta})$. It is interesting to compare this with the results of Hörmander [10] in which lower order operators are estimated in terms of higher order operators of a more general nature but with weight functions different from the ones considered here.

2. **Laplacian principal part.** We consider the operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$$

and a twice continuously differentiable function $u(x_1, x_2, \cdots, x_N)$ defined in a neighborhood of the origin. We employ the euclidean distance

$$r^2 = x_1^2 + x_2^2 + \cdots + x_N^2$$

and examine an expression of the form

$$r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2$$

where β is a positive integer which shall be considered a parameter. We first introduce the function $z=z(x_1, x_2, \cdots, x_N)$ by the relation

$$u = e^{-r^{-\beta}} z \tag{1}$$

and hence we have

$$r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2 = r^{\beta+2}[\Delta z + 2\nabla z\cdot \nabla(e^{-r^{-\beta}})e^{r^{-\beta}} + e^{r^{-\beta}}z\Delta(e^{-r^{-\beta}})]^2 \tag{2}$$

where the symbol ∇ represents the gradient. We have

$$2\nabla z\cdot \nabla(e^{-r^{-\beta}})e^{r^{-\beta}} = \frac{2\beta}{r^{\beta+2}}\sum_{i=1}^{N} x_i \frac{\partial z}{\partial x_i}$$

and

$$e^{r^{-\beta}}\Delta(e^{-r^{-\beta}}) = \frac{\beta}{r^{\beta+2}}\left[\frac{\beta}{r^\beta} - \beta - 2 + N\right]\cdot$$

We write the right side of (2) in the form

$$(\alpha + \delta + \gamma)^2$$

and observe that

$$(\alpha + \delta + \gamma)^2 \geqq 2\delta(\alpha + \gamma).$$

Hence we have the inequality

$$r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2 \geqq 4\beta\Delta z \sum_{i=1}^{N} x_i \frac{\partial z}{\partial x_i} + \frac{4\beta^2}{r^{\beta+2}}\left(\frac{\beta}{r^\beta} - \beta - 2 + N\right)\sum_{i=1}^{N} x_i \frac{\partial z}{\partial x_i} z. \tag{3}$$

We now make the assumption that $u(x_1, x_2, \cdots, x_N)$ vanishes outside the sphere $r \leqq r_0$ where $r_0 < 1$. Further it shall be assumed that

$$e^{r^{-\beta}}u \to 0 \quad \text{as} \quad r \to 0 \text{ for every } \beta > 0$$

and hence that the integrals considered below all exist with a finite value. From (3) we have the inequality

$$\begin{aligned}\int r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2 dx \geqq 4\beta \sum_{i=1}^{N}\int x_i \frac{\partial z}{\partial x_i}\Delta z dx \\ + 4\beta^2 \int \frac{1}{r^{\beta+2}}\left(\frac{\beta}{r^\beta} - \beta - 2 + N\right)\sum_{i=1}^{N} x_i \frac{\partial z}{\partial x_i} z dx\end{aligned} \tag{4}$$

where the integrations are over the entire x-space. An easy integration by parts shows that, since from (1) $z \equiv 0$ for $r \geqq r_0$,

$$\sum_{i=1}^{N} \int x_i \frac{\partial z}{\partial x_i} \Delta z \geqq 0$$

equality occurring if $N=2$, with the inequality otherwise. Hence (4) implies

$$\begin{aligned} \int r^{\beta+2} e^{2r^{-\beta}} (\Delta u)^2 dx \geqq 4\beta^3 \int r^{-2\beta-2} \sum_{i=1}^{N} x_i z \frac{\partial z}{\partial x_i} dx \\ - 4\beta^2(\beta + 2 - N) \int r^{-\beta-2} \sum_{i=1}^{N} x_i z \frac{\partial z}{\partial x_i} dx. \end{aligned} \tag{5}$$

We integrate the first term by parts, obtaining

$$\begin{aligned} 2\beta^3 \int r^{-2\beta-2} \sum_{i=1}^{N} x_i z \frac{\partial z}{\partial x_i} &= -\beta^3 \sum_{i=1}^{N} \int \frac{\partial}{\partial x_i}\left(\frac{x_i}{r^{2\beta+2}}\right) z^2 \\ &= 2\beta^3(\beta + 1 - N/2) \int r^{-2\beta-2} z^2. \end{aligned}$$

For $\beta+1>N/2$ the right side is positive and for $r_0<1$ and sufficiently large β the second term on the right side of (5) is negligible compared to the first term. Hence we have the inequality

$$\beta^4 \int r^{-2\beta-2} z^2 dx \leqq c_0 \int r^{\beta+2} e^{2r^{-\beta}} (\Delta u)^2 dx$$

where c_0 is a fixed constant depending on r_0 valid for all sufficiently large β. Taking (1) into account we have

$$\beta^4 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2 dx \leqq c_0 \int r^{\beta+2} e^{2r^{-\beta}} (\Delta u)^2 dx. \tag{6}$$

This is the basic inequality similar to that obtained by Pederson by a somewhat different method.

To obtain an inequality similar to (6) for the first derivatives of u we proceed in the following manner. For any function u with compact support we have the relation

$$\int au\Delta u = -\int a \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2 + \frac{1}{2} \int (\Delta a) u^2.$$

We let $a = e^{2r^{-\beta}}$. Then as before we have

$$|\Delta a| \leqq \frac{c_0 \beta^2}{r^{2\beta+2}} e^{2r^{-\beta}}$$

for $r_0<1$ and all sufficiently large β. Hence we find

$$\int e^{2r^{-\beta}} \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 \leqq \int e^{2r^{-\beta}}|u|\cdot|\Delta u| + c_0\beta^2\int r^{-2\beta-2}e^{2r^{-\beta}}u^2.$$

A rearrangement, taking (6) into account, yields

$$\int e^{2r^{-\beta}} \sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqq \int r^{-\beta-1}e^{r^{-\beta}}|u|\cdot r^{\beta+1}e^{r^{-\beta}}|\Delta u| + \frac{c_0}{\beta^2}\int r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2$$

where c_0 is used as a generic constant throughout. We now apply Cauchy's inequality and (6) in succession, getting

$$\int e^{2r^{-\beta}} \sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqq \frac{1}{2}\int r^{-2\beta-2}e^{2r^{-\beta}}u^2 + \frac{1}{2}\int r^{2\beta+2}e^{2r^{-\beta}}(\Delta u)^2 + \frac{c_0}{\beta^2}\int r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2$$

and

$$\int e^{2r^{-\beta}} \sum\left(\frac{\partial u}{\partial x_i}\right)^2 \leqq c_0\left(\frac{1}{\beta^4} + r_0^{\beta} + \frac{c_0}{\beta^2}\right)\int r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2. \tag{7}$$

We combine (6) and (7) to find

$$\int e^{2r^{-\beta}}\left[r^{-2\beta-2}u^2 + \sum\left(\frac{\partial u}{\partial x_i}\right)^2\right] \leqq c_0\left(r_0^{\beta} + \frac{1}{\beta^2}\right)\int r^{\beta+2}e^{2r^{-\beta}}(\Delta u)^2. \tag{8}$$

Inequality (8) leads directly to a unique continuation theorem for inequalities of the form

$$\Delta u \leqq f\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right)$$

where f is Lipschitzian in $u, \partial u/\partial x_1, \cdots, \partial u/\partial x_N$. The argument leading from (8) to the unique continuation theorem is the same as that given in [1; 5; 9; 13; 14] the parameter β being allowed to go to infinity. We do not repeat the details here.

3. **General second order equation.** We consider a second order operator

$$Lu = \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

with the coefficients having continuous second derivatives in a neighborhood of the origin. Without loss of generality we assume that $a_{ij}=a_{ji}$. It is supposed that there exist constants m_0 and M_0 such that

$$M_0 \sum_{i=1}^{N} \xi_i^2 \geqq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqq m_0 \sum_{i=1}^{N} \xi_i^2$$

for all ξ_i and $x_1, x_2, \cdots, x_N$ in the domain under consideration. Without loss of generality we shall suppose that the operator L coincides with the Laplacian at the origin. This may be achieved by a simple change of variables. Thus we have

$$a_{ij} = \delta_{ij} + \alpha_{ij}$$

where

$$|\alpha_{ij}| \leqq c_0 r.$$

We proceed in the same manner as in §2. We consider a function $u(x_1, x_2, \cdots, x_N)$ which vanishes for $r \geqq r_0$, where $r_0 < 1$ and set

$$u = ze^{-r^{-\beta}}$$

where β is a positive constant. We are interested in computing the expression

$$r^{\beta+2}e^{2r^{-\beta}}(Lu)^2.$$

We have

$$r^{\beta+2}e^{2r^{-\beta}}(Lu)^2 = r^{\beta+2}\left[Lz + 2e^{r^{-\beta}} \sum_{i,j=1}^{N} a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial(e^{-r^{-\beta}})}{\partial x_j} + ze^{r^{-\beta}}L(e^{-r^{-\beta}})\right]^2.$$

In exactly the same manner as previously, we have

$$r^{\beta+2}e^{2r^{-\beta}}(Lu)^2 \geqq 4Lz \sum_{i,j=1}^{N} \beta a_{ij}x_i \frac{\partial z}{\partial x_j} + 4\beta ze^{r^{-\beta}}L(e^{-r^{-\beta}}) \cdot \sum_{i,j=1}^{N} a_{ij}x_i \frac{\partial z}{\partial x_j} \cdot$$

We thus obtain the integral inequality

$$\begin{aligned} \int r^{\beta+2}e^{2r^{-\beta}}(Lu)^2 \geqq 4\beta \int Lz \sum_{i,j=1}^{N} a_{ij}x_i \frac{\partial z}{\partial x_j} \\ + 4\beta \int ze^{r^{-\beta}}L(e^{-r^{-\beta}}) \sum_{i,j=1}^{N} a_{ij}x_i \frac{\partial z}{\partial x_j} \cdot \end{aligned} \tag{9}$$

We now estimate the first integral on the right. The terms with $i=j$ give

$$\sum_{i=1}^{N} \left\{4\beta \int (1 + \alpha_{ii})x_i \frac{\partial z}{\partial x_i}\right\}\left[\Delta z + \sum \alpha_{ii} \frac{\partial^2 z}{\partial x_i^2}\right].$$

As previously we have

$$\int x_i \frac{\partial z}{\partial x_i} \Delta z \geqq 0$$

and we drop this term. For the remaining terms we write

$$4\beta \int \alpha_{ii}x_i \frac{\partial z}{\partial x_i} \frac{\partial^2 z}{\partial x_j^2} = -4\beta \int \frac{\partial}{\partial x_j}(\alpha_{ii}x_i) \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} - 2\beta \int \frac{\partial}{\partial x_i}(\alpha_{ii}x_i)\left(\frac{\partial z}{\partial x_j}\right)^2.$$

Hence

$$\sum_{i=1}^{N} 4\beta \left| \int \alpha_{ii}x_i \frac{\partial z}{\partial x_i} \frac{\partial^2 z}{\partial x_j^2} \right| \leqq 4\beta c_0 r_0 \int \sum_{i=1}^{N} \left(\frac{\partial z}{\partial x_i}\right)^2 \tag{10}$$

where now the constant c_0 depends also on the modulus of the first derivatives of the α_{ii}. A completely analogous inequality to (10) holds for the terms containing α_{ij} with $i \neq j$. A computation shows that

$$e^{r^{-\beta}} L(e^{-r^{-\beta}}) = \sum_{i,j=1}^{N} \left\{ a_{ij} \frac{\beta^2 x_i x_j}{r^{2\beta+4}} + \frac{\beta \delta_{ij}}{r^{\beta+2}} - \frac{\beta(\beta+2)x_i x_j}{r^{\beta+4}} \right\}. \tag{11}$$

We note that for $r_0 < 1$ and sufficiently large β the first term on the right dominates. Combining (9), (10), and (11) we have the inequality

$$\begin{aligned} 2\beta \int \sum_{i,j=1}^{N} a_{ij}x_i \frac{\partial z}{\partial x_j} z \sum_{k,l=1}^{N} a_{kl} \frac{\beta^2 x_k x_l}{r^{2\beta+4}} + \frac{\beta \delta_{kl}}{r^{\beta+2}} - \frac{\beta(\beta+2)x_k x_l}{r^{\beta+4}} \\ \leqq \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2 + 4\beta c_0 r_0 \int \sum_{i=1}^{N} \left(\frac{\partial z}{\partial x_i}\right)^2. \end{aligned} \tag{12}$$

We integrate by parts in the first term in the left of (12):

$$\begin{aligned} 4\beta^3 \int \sum_{i,j,k,l} a_{ij}a_{kl}x_i \frac{\partial z}{\partial x_j} z \frac{x_k x_l}{r^{2\beta+4}} \\ = -\beta^3 \sum_{i,j,k,l} \cdot \left\{ \frac{\partial}{\partial x_j} \frac{x_i x_k x_l a_{ij} a_{kl}}{r^{2\beta+4}} \right\} z^2 + \frac{\partial}{\partial x_i} \left\{ \frac{x_j x_k x_l a_{ij} a_k}{r^{2\beta+4}} \right\} z^2. \end{aligned}$$

The dominant term arises from the differentiation of $r^{-2\beta-4}$ and this yields a factor $-(2\beta+4)\cdot x_j$ in the first part with $-(2\beta+4)\cdot x_k$ in the second. Hence the ellipticity of the operator can be taken into account since

$$\sum a_{ij}x_i x_j \geqq m_0 r^2$$

and we find, for all sufficiently large β and $r_0 < 1$,

$$c_0 \beta^4 \int r^{-2\beta-2} z^2 \leqq \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2 + c_1 \beta r_0 \int \sum \left[\frac{\partial}{\partial x_i} (u e^{r^{-\beta}}) \right]^2.$$

Performing the differentiations on the right, we have

$$\beta^4 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2 \leqq c_0 \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2$$
$$+ c_1 \beta r_0 \int e^{2r^{-\beta}} \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2 + c_2 \beta^3 r_0 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2$$

and with an adjustment of the constants, for sufficiently large β,

$$\beta^4 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2 \leqq c_0 \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2 + c_1 \beta r_0 \int e^{2r^{-\beta}} \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2. \tag{13}$$

For any function $a = a(x, y)$ we have the identity

$$\int auLu = -\int a\, a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} \int (La + R) u^2$$

where as usual it is assumed that u has compact support and R contains the quantities a_{ij}, the first and second derivatives of a_{ij}, the quantity a and its first derivatives only. We select for a the value

$$a = e^{2r^{-\beta}}$$

and take into account the ellipticity of the operator L, getting

$$m_0 \int e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \leqq \int r^{-\beta-1} e^{r^{-\beta}} u r^{\beta+1} e^{r^{-\beta}} Lu + c_0 \beta^2 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2$$

or

$$\beta \int e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \leqq c_0 \beta \int r^{2\beta+2} e^{2r^{-\beta}} (Lu)^2 + c_1 \beta^3 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2. \tag{14}$$

Adding (13) and (14) we have

$$\beta \int e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 + \beta^4 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2$$
$$\leqq (c_0 + c_1 \beta r_0^{\beta}) \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2 + c_2 \beta r_0 \int e^{2r^{-\beta}} \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2$$
$$+ c_1 \beta^3 \int r^{-2\beta-2} e^{2r^{-\beta}} u^2.$$

Hence for sufficiently large β and $r_0 < 1$ we finally conclude

$$\int r^{-2\beta-2} e^{2r^{-\beta}} u^2 + \frac{c_0}{\beta^3} \int e^{2r^{-\beta}} \sum \left(\frac{\partial u}{\partial x_i}\right)^2 \leqq \frac{c_1}{\beta^4} \int r^{\beta+2} e^{2r^{-\beta}} (Lu)^2, \tag{15}$$

To obtain the uniqueness of solutions of the differential inequality

$$Lu \leqq f\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right)$$

we proceed from inequality (15) by letting $\beta \to \infty$, again as in [1; 5; 9; 14; 15].

4. **Higher order equations.** In this section we consider inequalities of the form

$$\Delta^n u \leqq f(x, u, Du, \cdots, D^k u)$$

where $D^m u$ represents an arbitrary mth derivative of u of order not exceeding $[3n/2]$ with respect to any of the variables $x_1, x_2, \cdots, x_N$. The basic identities we employ are the following:

$$\int au\Delta u = -\int a \sum \left(\frac{\partial u}{\partial x_i}\right)^2 + \frac{1}{2}\int (\Delta a)u^2 \tag{16}$$

and (using subscripts for partial derivatives now)

$$\int au_{x_1x_1}u_{x_2x_2} = \int au_{x_1x_2}^2 - \frac{1}{2}\int [a_{x_1x_1}u_{x_2}^2 - 2a_{x_1x_2}u_{x_1}u_{x_2} + a_{x_2x_2}u_{x_1}^2] \tag{17}$$

valid for any function u with compact support. In the light of (17) we consider the integral

$$\int a(\Delta u)\cdot(\Delta u) = \int \left[a \sum_{i=1}^{N} u_{x_ix_i}^2 + a \sum_{i,j=1;i\neq j}^{N} u_{x_ix_i}u_{x_jx_j}\right].$$

We apply (17) to the second term on the right above and obtain

$$\int a \sum_{i,j=1}^{N} u_{x_ix_j}^2 = \int a(\Delta u)^2 + \frac{1}{2}\int Q(a_{x_ix_j}, u_{x_i}) \tag{18}$$

in which Q is a sum of products of first derivatives of u and second derivatives of a. Relation (18) is particularly useful as it allows us to estimate second derivatives of u in terms of the Laplacian and lower order derivatives.

Formula (6) shows that the exponent of r in the kernel increases by an amount $3\beta+4$ as the function u is estimated in terms of Δu. The estimate for the first derivatives of u increases the kernel by $2\beta+2$ on the left as exhibited in inequality (8). Formulas (16) and (17) are useful for iterating the inequalities so that all derivatives of order lower than $2n$ may be estimated in terms of the nth iterate of the Laplacian. An iteration of (6) shows that

$$\beta^8 \int r^{-2\beta-2}e^{2r^{-\beta}}u^2dx \leqq c_0 \int r^{4\beta+6}e^{2r^{-\beta}}(\Delta^2 u)^2dx.$$

An estimate for the first derivatives of u in terms of $\Delta^2 u$ is obtained from (8). The second derivatives may be estimated by means of (18). The second term

in the right of (18) has r to the exponent $-2\beta-2$ if $a=e^{2r^{-\beta}}$. Hence combining this with (8) applied to Δu we get an estimate of the form

$$\beta^4 \int e^{2r^{-\beta}} \sum u_{x_i x_j}^2 dx \leqq c_0 \int r^{2\beta+2} e^{2r^{-\beta}} (\Delta^2 u)^2 dx.$$

Thus the third derivatives are estimated in terms of $\Delta^2 u$ with essentially the same powers of r occurring on both sides of the inequality. The general situation may be described by saying that to estimate each two successive derivatives the inequalities (16) and (18) require a power increase of the order 4β while inequality (6) yields an increase of the order 3β. Hence for $\Delta^3 u$ estimates through the fourth derivatives can be made, for $\Delta^4 u$ estimates through the 6th derivatives and in general for $\Delta^n u$ estimates up to the order $[3n/2]$ may be obtained. Thus we have the result: *given the differential inequality*

$$|\Delta^n u| \leqq f(x, u, Du, \cdots, D^k u) \tag{19}$$

where f is Lipschitzian, $k=[3n/2]$; if u satisfies (19) *in a neighborhood D of the origin and*

$$e^{2r^{-\beta}} u \to 0 \quad \text{as} \quad r \to 0 \tag{20}$$

for every positive β then u vanishes identically in D.

It is clear that condition (20) may be put in integrated form. Also the uniqueness of the Cauchy problem for differential inequalities (19) follows at once by standard methods.

If L is a general second order elliptic operator, we can write a more general relation than (18) by considering the expression

$$\int a(\Delta u)(Lu)$$

for functions u with compact support. If

$$Lu = \sum_1^N a_{ij} u_{x_i x_j}$$

where

$$\sum a_{ij}\xi_i\xi_j \geqq m_0 \sum \xi_i^2,$$

then we have

$$m_0 N \int a \sum u_{x_i x_j}^2 \leqq \int a \Delta u L u + \frac{1}{2} \int Q(a_{x_i x_j}, u_{x_i}), \tag{21}$$

in which Q contains first derivatives of u and second derivatives of a. With the aid of (21) it is possible to extend the unique continuation theorem to equations of the form

$$\Delta^{n-1}Lu = f$$

where f contains derivatives up to order $[3n/2]$.

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