HILBERT SPACE METHODS IN THE THEORY OF LIE ALGEBRAS[1]

BY

JOHN R. SCHUE

Introduction. In the study of the complex finite-dimensional semi-simple Lie algebras a crucial role is played by the fundamental bilinear form $\langle x, y\rangle = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$. Since the definition is meaningless when the restriction of finite-dimensionality is removed, if any of the highly desirable properties of the form are to be retained in this case they must necessarily be given *a priori.* By reconsidering the finite-dimensional situation it is possible to formulate suitable conditions in a more convenient form. To see this let L be a complex finite-dimensional semi-simple Lie algebra and let L_0 be a compact real form for L with σ as the associated involution (conjugation). If we let $x^* = -\sigma(x)$ and $(x, y) = \langle x, y^*\rangle$ then L becomes a finite-dimensional Hilbert space, the mapping x into x^* is a Hilbert space conjugation, and the connecting property $([x, y], z) = (y, [x^*, z])$ holds for all x, y, z. An L^* algebra as defined here is simply a Lie algebra whose vector space is a Hilbert space such that the connecting property above holds. This paper is a study of such algebras with emphasis, of course, on the infinite-dimensional ones. For finite dimensions nothing new is obtained and it is shown here that in this case every semi-simple L^* algebra arises essentially from a construction like that above (see the remark after 2.5).

There is an associative algebra analogue of this problem in the paper of Ambrose [1] on H^* algebras and some of his results are used here. Any H^* algebra gives rise to an L^* algebra by letting $[x, y] = xy - yx$ and the only known examples of L^* algebras are those obtained as Lie subalgebras of H^* algebras.

The main result of this paper is a classification of the (separable) simple L^* algebras which have Cartan decompositions (see §2) and it is shown that this class coincides with the simple self-adjoint Lie subalgebras of a (separable) simple H^* algebra. The results turn out to be the natural extensions of the finite-dimensional theory.

Associated with each of the Lie algebras considered here there is a gener-

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alized analytic group nucleus. For a discussion of this relationship one may refer to the paper of Birkhoff [2].

1. **Preliminaries.**

Definition. An L^* algebra is defined as a Lie algebra L over the complex field such that the vector space of L is a Hilbert space and for each $x \in L$ there is an x^* in L with $([x, y], z) = (y, [x^*, z])$ for all y, z in L.

Examples. Let A be an H^*algebra and let $[x, y] = xy - yx$. Any closed Lie subalgebra of A which is closed under the operation of taking adjoints is then an L^*algebra. Any complex finite-dimensional semi-simple Lie algebra is an L^* algebra. The Hilbert space direct sum of L^* algebras defines an L^* algebra in the obvious way.

Definitions and Remarks. L will represent an L^* algebra. For subsets M, N of L let $M^* = \{m^* : m \in M\}$, $M^\perp = \{x : (x, m) = 0$ for all m in $M\}$, $[M, N]$ = the closed subspace spanned by $\{[m, n] : m \in M, n \in N\}$. For subspaces S_1 and S_2 of L the notation $S_1 + S_2$ will be used only when $S_1 \perp S_2$.

For x in L let D_x denote the linear operator $D_x y = [x, y]$. Then D_x, D_x^* are everywhere defined (this implies both are bounded) and $D_x^* = D_{x^*}$. By using the principle of uniform boundedness it is not hard to show that the mapping x to D_x is continuous from L into the space of bounded operators on L under the uniform norm. Furthermore we may assume that $\|D_x\| \leqq \|x\|$.

L will be called semi-simple if and only if $L = [L, L]$ and this is equivalent to the mapping x to D_x being one-one. L will be called simple if and only if there are no nontrivial closed ideals. It is a simple argument to show that a closed subspace I of L is an ideal of L if and only if $I^\perp$ is an ideal. Using this one obtains the result that every L^* algebra is the direct sum of an abelian ideal (the center) and a semi-simple ideal (the derived algebra, $[L, L]$). Hence an L^* algebra is necessarily *reductive* in the sense of [3, Exposé 7].

From now on we will assume L is semi-simple. Using the fact that the adjoint representation is then faithful and the properties of adjoints for operators it follows that the mapping x to x^* is involutory, conjugate linear, and $[x, y]^* = [y^*, x^*]$. Then the connecting property implies $(x, [y, z]) = ([y, z]^*, x^*)$ for all x, y, z. By semi-simplicity, $(x, y) = (y^*, x^*)$ for all x, y so that the * mapping is a Hilbert space conjugation. L is then the complexification of the real Lie algebra formed by the skew-adjoint elements. It can be proved from all of this that every closed ideal of L is an L^* algebra.

A Cartan subalgebra of a semi-simple L is defined as a maximal self-adjoint abelian subalgebra. An application of Zorn's Lemma shows that every $x \in L$ with $[x, x^*] = 0$ is contained in a Cartan subalgebra. A Cartan subalgebra is necessarily closed.

1.1. Let H be a Cartan subalgebra of L. Then H is maximal abelian and $H^\perp = [H, L]$.

Proof. Suppose $[H, x] = 0$. Then $[H, x^*] = [H^*, x]^* = [H, x]^* = 0$. Hence $[H, x + x^*] = [H, x - x^*] = 0$. Since H is maximal self-adjoint abelian this implies $x+x^*$ and $x-x^*$ are in H, hence $x\in H$ and H is maximal abelian. If $h_1, h_2\in H$ and $x\in L$ then $(h_1, [h_2, x])=([h_2^*, h_1], x)$ implies $[h_2, x]\in H^\perp$ and $[H, L]\subset H^\perp$. If $x\in[H, L]^\perp$ then $(x, [h^*, y])=0$ for all y implies $([h, x], y)=0$ so that $[H, x]=0$ and $x\in H$.

In the event that L is finite-dimensional a Cartan subalgebra H as defined here is a Cartan subalgebra in the usual sense. For H is maximal abelian and for each $h\in H$, $[h, h^*]=0$ implies D_h is normal, hence diagonalizable. These two properties characterize the Cartan subalgebras of L (see [3, Exposé 9]). Conversely, if L is semi-simple and finite-dimensional, a Cartan subalgebra H of L in the sense of [3] is one in our sense for a suitable * mapping and inner product, for by Exposé 11 of [3] there is a compact real form L_0 of L with associated involution σ such that $\sigma(H)=H$. Applying the construction used in the introduction gives the result.

1.2. Theorem 1. *Let L be a semi-simple L^* algebra. Then there exist simple closed L^* ideals L_j, indexed by some set J, such that $L=\sum_{j\in J} L_j$, the sum being the usual Hilbert space direct sum. Every closed ideal of L is obtained by summing the L_j over some subset of J.*

Outline of Proof. Let H be a Cartan subalgebra of L and B the C^* algebra generated by $\{D_h: h\in H\}$. B is then topologically and algebraically isomorphic with the algebra of all continuous complex-valued functions vanishing at infinity on the locally compact space Δ of all homomorphisms of B onto the complex numbers. Each $\alpha\in\Delta$ defines a bounded linear functional on H and hence there is a unique $h_\alpha\neq 0$ in H such that $\alpha(D_h)=(h, h_\alpha)$ for all h. Then $\|h_\alpha\|\leqq 1$ and $\alpha(D_h^*)=[\alpha(D_h)]^-$ implies $h_\alpha^*=h_\alpha$. For $\alpha, \beta\in\Delta$ let $(\alpha, \beta)=(h_\alpha, h_\beta)$ and definite $\alpha\perp\beta$ if and only if $(\alpha, \beta)=0$. A subset M of Δ will be called indecomposable if M cannot be written as the union of nonempty orthogonal subsets. Then each $\alpha\in\Delta$ is contained in a unique maximal indecomposable subset M_α. Then either $M_\alpha=M_\beta$ or $M_\alpha\perp M_\beta$. Let $\{M_j: j\in J\}$ be the set of the distinct M_α's. For each j let H_j be the span of the h_α where α runs over M_j and let $L_j=H_j+[H_j, L]$. By a proof like that used in the finite-dimensional case each L_j is a simple closed ideal of L and $L_j\perp L_k$ for $j\neq k$. If $K=\sum L_j$ then K is a closed ideal containing H (the h_α's span H) and hence $[K^\perp, H]=0$ implies $K^\perp=0$ so that $L=\sum L_j$. The last statement is a consequence of the way the decomposition is obtained.

2. **Roots and Cartan decompositions.**

Definition. For this section L is a semi-simple L^* algebra with H as a Cartan subalgebra. For a linear mapping α of H into the complex numbers let $V_\alpha=\{v: [h, v]=\alpha(h)v$ for all $h\in H\}$. Then V_α is a closed subspace of L and α will be called a root (relative to H) if and only if $V_\alpha\neq 0$. The zero function is a root and $V_0=H$. If α is a root then necessarily it corresponds to a homomorphism of the operator algebra generated by $\{D_h: h\in H\}$. Hence α is bounded and $\alpha(h^*)=[\alpha(h)]^-$. As in the proof of Theorem 1 there is a unique

h_α in H with $\|h_\alpha\| \leqq 1$, $h_\alpha^* = h_\alpha$, and $\alpha(h) = (h, h_\alpha)$ for all h. From this it follows that if α is a root $-\alpha$ is also one and $V_\alpha^* = V_{-\alpha}$. If α, β are distinct then $V_\alpha \perp V_\beta$. By the Jacobi identity $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$.

Let $K = \sum V_\alpha$, the sum being taken over the distinct roots relative to H. Then K is a closed L^* subalgebra of L with $H \subset K \subset L$. We will say that L has a Cartan decomposition (relative to H) if and only if $K = L$, i.e. if and only if the set $\{D_h : h \in H\}$ is simultaneously diagonalizable. It is an open question as to whether or not every L has such a decomposition; however, I hope to have more complete results to be given in a later paper. Theorem 2 below settles the question if L is embedded in an H^* algebra and the later classification theory shows this is necessary as well as sufficient, at least when every simple ideal component of L is separable.

2.1. Let L be a simple L^* algebra and suppose ϕ is a continuous linear mapping of L into a Hilbert space K with $(\phi([x, y]), \phi(z)) = (\phi(y), \phi([x^*, z]))$ for all x, y, z in L. Then there is an $\epsilon \geqq 0$ such that $(\phi(x), \phi(y)) = \epsilon(x, y)$ for x and y in L.

Proof. Since ϕ is bounded there is a bounded operator B on L such that $(\phi(x), \phi(y)) = (Bx, y)$. Then $B \geqq 0$ implies B is self-adjoint. The assumption on ϕ implies B commutes with every D_x; by the spectral theorem every projection in the spectral resolution of B commutes with every D_x. The range of such a projection is then a closed ideal of L, hence is either 0 or all of L so that $B = \epsilon 1$ for some $\epsilon \geqq 0$.

2.2. THEOREM 2. *Suppose L is a semi-simple L^* subalgebra of an H^* algebra A and H is a Cartan subalgebra of L. Then L has a Cartan decomposition relative to H.*

Proof. Let $L = \sum L_j$ where each L_j is a simple closed ideal. If $H_j = H \cap L_j$ it is easily seen that H_j is a Cartan subalgebra of L_j. Hence it will be sufficient to prove the theorem when L is simple.

If I is a simple (associative) ideal of A the restriction to L of the projection P of A onto I satisfies the hypotheses of 2.1 and hence there is an $\epsilon \geqq 0$ such that $(Px, Py) = \epsilon(x, y)$ for all x, y in L. Since A is a direct sum of such simple ideals there must be some I such that the corresponding ϵ is positive. Thus L is topologically and algebraically isomorphic with a Lie subalgebra of I so that we may assume A itself is simple. Then by [1], A is the set of all Hilbert-Schmidt operators on some Hilbert space $\mathcal{H}$.

The set H is then a collection of commutative completely continuous normal operators on $\mathcal{H}$ and hence can be simultaneously diagonalized. Using a basis of $\mathcal{H}$ composed of common eigenvectors for H and regarding A as the algebra of square-convergent matrices relative to this basis, H becomes a subset of the diagonal matrices. For $h \in H$ and $y \in A$ let $T_h y = hy - yh$. Then, as in the finite-dimensional case, the operators T_h can be simultaneously diagonalized. Since L is an invariant subspace under the set of all T_h and the restriction of T_h to L is D_h then L has a Cartan decomposition relative to H.

For the remainder of this section we will assume only that L is semi-simple and H is a Cartan subalgebra.

2.3. If α is a nonzero root V_α is one-dimensional.

Proof. Choose $v_1 \in V_\alpha$ with $\|v_1\| = 1$. Let $v_2 \in V_\alpha$ with $(v_1, v_2) = 0$. It is sufficient to show that this implies $v_2 = 0$. For any $v \in V_\alpha$ we have $[v_1, v^*] \in H$. For any $h \in H$, $(h, [v_1, v^*]) = ([h, v], v_1) = (h, h_\alpha)(v, v_1)$ implies $[v_1, v^*] = (v_1, v)h_\alpha$ so that $[v_1, v_2^*] = 0$. The same argument can be used to show that $[v_2, v_2^*] = \|v_2\|^2 h_\alpha$. Then, by the Jacobi identity and the connecting property, $0 = ([v_1^*, v_2], [v_1^*, v_2]) = ([v_2, v_2^*], [v_1, v_1^*]) + ([v_1, v_2], [v_1, v_2]) = \|v_2\|^2\|h_\alpha\|^2 + \|[v_1, v_2]\|^2$ so that $\|v_2\| = 0$ and $v_2 = 0$.

Definition. Let R be the set of nonzero roots relative to H. By Zorn's lemma it is possible to decompose R as $R = R_1 \cup R_2$ where R_1, R_2 are disjoint and $\alpha \in R_1$ if and only if $-\alpha \in R_2$. For each $\alpha \in R_1$ choose $e_\alpha \in V_\alpha$ such that $\|e_\alpha\| = 1$. Then $e_\alpha^* \in V_{-\alpha}$ and $\|e_\alpha^*\| = 1$. For $\alpha \in R_2$ let $e_\alpha = e_{-\alpha}^*$. Thus $e_\alpha^* = e_{-\alpha}$ for all α in R and the set $\{e_\alpha\}$ is an orthonormal set. By the proof of 2.2, $[e_\alpha, e_\alpha^*] = h_\alpha$.

Suppose $\alpha, \beta \in R$ and $\beta \neq -\alpha$. If $\alpha + \beta$ is a root let $c_{\alpha,\beta}$ be defined by the equation $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$, otherwise let $c_{\alpha,\beta} = 0$ and $e_{\alpha+\beta} = 0$.

If β is any root and α a nonzero root the sequence $\{\beta - k\alpha : k = 0, \pm 1, \cdots\}$ contains only finitely many roots for if $\beta - k\alpha$ is a root then $1 \geqq \|h_{\beta - k\alpha}\| = \|h_\beta - kh_\alpha\| \geqq |k| \|h_\alpha\| - \|h_\beta\|$. Thus it is possible to define the integers $k_1(\alpha, \beta)$ and $k_2(\alpha, \beta)$ by the conditions $\beta + k\alpha$ is a root for $-k_1 \leqq k \leqq k_2$ while $\beta - (k_1 + 1)\alpha$ and $\beta + (k_2 + 1)$ are not roots. Then, by the same proof as used in [3], $(h_\alpha, h_\beta) = (1/2)[k_1(\alpha, \beta) - k_2(\alpha, \beta)]\|h_\alpha\|^2$ for any roots α, β with $\alpha \neq 0$.

2.4. Suppose $\alpha_1, \cdots, \alpha_k \in R$. Let M be the set of all roots which are linear combinations with integral coefficients of $\alpha_1, \cdots, \alpha_k$. Let V be the span of the e_α's where $\alpha \in M$ and let H_1 be the span of $h_{\alpha_1}, \cdots, h_{\alpha_k}$. Then $L_1 = H_1 + V$ is a finite-dimensional semi-simple L^* algebra with H_1 as a Cartan subalgebra and M is the complete set of roots relative to H_1.

Proof. The proof is straightforward except, perhaps, for the statement that the dimension of L_1 is finite. Since $\dim H_1 \leqq k < \infty$, $\dim L_1$ is infinite if and only if $\{e_\alpha : \alpha \in M\}$ is infinite and this can occur only if $\{h_\alpha : \alpha \in M\}$ is infinite. In this event the latter set is an infinite bounded set in the unitary space H_1 and must then contain an infinite convergent sequence h_{β_n}. Letting $h_i = 2\|h_{\alpha_i}\|^{-2} h_{\alpha_i}$ for $i = 1, \cdots, k$, (h_{β_n}, h_i) is an integer for all n and i and H_1 is spanned by $h_1, \cdots, h_k$. From this it is clear that no such sequence exists and consequently L_1 is finite-dimensional.

2.5. Suppose L is finite-dimensional and simple. Let $\langle x, y\rangle = \mathrm{Tr}(D_x D_y)$ for all x, y. Then there is an $\epsilon > 0$ such that $\langle x, y\rangle = (x, y^*)$.

Proof. Define the operator B on L by the equation $(Bx, y) = \langle x, y^*\rangle$ for all

x, y. Then $(Bx, x) = \mathrm{Tr}(D_x D_x^*)$ implies B is positive definite. The condition $\langle [x, y], z\rangle = \langle x, [y, z]\rangle$ implies B commutes with every D_x and, by the argument used in 2.1, B must be a positive multiple of the identity.

REMARK. The result of 2.5 justifies the remarks of the introduction for finite-dimensional L^* algebras. If L is simple and ϵ is as in 2.5 let L_0 be the set of skew-adjoint elements of L and $\sigma(x) = -x^*$ for all x. Then σ is an involution and 2.5 shows that L_0 is a compact real form for L. The extension to semi-simple algebras is immediate. An immediate consequence of this relationship is the result 2.6 below which will be needed in the later classification theory.

2.6. Let L be finite-dimensional and simple with H as a Cartan subalgebra.

(i) If $\alpha, \beta, \gamma \in R$ and $\alpha+\beta+\gamma=0$ then $c_{\alpha,\beta} = c_{\beta,\gamma} = c_{\gamma,\alpha}$.

(ii) If $\alpha, \beta, \gamma, \delta \in R$, $\alpha+\beta+\gamma+\delta=0$, and the sum of no pair is zero, then $c_{\alpha,\beta}c_{\gamma,\delta} + c_{\beta,\gamma}c_{\alpha,\delta} + c_{\gamma,\alpha}c_{\beta,\delta} = 0$.

(iii) If $\alpha, \beta \in R$ and $\beta \neq -\alpha$ then

$$c_{\alpha,\beta}c_{-\alpha,-\beta} = -(1/2)k_2(\alpha, \beta)(1 + k_1(\alpha, \beta))\|h_\alpha\|^2.$$

Proof. See [3, Exposé 11, Lemmas 1, 2, 3].

3. **The classification theory.** For this section L will be a simple infinite-dimensional L^* algebra and H a Cartan subalgebra such that L has a Cartan decomposition relative to H. We further require that the space of L be separable.

DEFINITION. For a finite subset $F = \{\alpha_1, \cdots, \alpha_k\}$ of R let $L(F)$ denote the finite-dimensional semi-simple algebra defined in 2.4. Then $F_1 \subset F_2$ implies $L(F_1) \subset L(F_2)$. A subset G of R will be called a root system if and only if $\alpha \in G$ implies $-\alpha \in G$ and $\alpha, \beta \in G$, $\alpha+\beta \in R$ implies $\alpha+\beta \in G$. Then $L(F)$ is the subalgebra generated by the e_α where α ranges over the root system generated by F. Using the notion of indecomposability as in the proof of Theorem 1, if F is an indecomposable finite subset of R then the root system generated by F is indecomposable and $L(F)$ is simple. Furthermore it is clear that R is indecomposable since L is simple. A subset $\alpha_0, \cdots, \alpha_n$ of R will be called a chain from α_0 to α_α if $(h_{\alpha_{i-1}}, h_{\alpha_i}) \neq 0$ for $i = 1, \cdots, n$. Since R is indecomposable any $\alpha, \beta \in R$ must be connected by a finite chain. Any chain is indecomposable.

3.1. For any finite subset F of R there exists a finite indecomposable root system containing F.

Proof. Let $F = \{\alpha_1, \cdots, \alpha_n\}$. For each i, $1 \leqq i \leqq n-1$, let F_i be a chain from α_i to α_{i+1}. Let $F_1 = \bigcup F_i$. Then F_1 is indecomposable and finite. If F_2 is the root system generated by F_1, F_2 is indecomposable and 2.4 implies F_2 is finite.

DEFINITION. Since L is separable the orthonormal set $\{e_\alpha : \alpha \in R\}$ is countable and hence R is countably infinite. Let $R=\{\alpha_1, \alpha_2, \cdots\}$ and let $F_n=\{\alpha_1, \cdots, \alpha_n\}$.

3.2. There is a sequence G_n of finite subsets of R such that the following are true:

(i) $F_n \subset G_n \subset G_{n+1}$.

(ii) G_n is an indecomposable root system.

(iii) $R=\bigcup G_n$.

(iv) The simple subalgebras $L(G_n)$ form a strictly increasing sequence with $L=$closure of $\bigcup L(G_n)$. All of the $L(G_n)$ are of the same Cartan type A, B, C, or D.

Proof. The sequence $\{G_n\}$ can be defined inductively. Let G_1 be a finite indecomposable root system containing F_1. Having chosen $G_1, \cdots, G_{n-1}$ satisfying (i) and (ii) let $F=G_{n-1}\cup F_n$ and choose G_n to be a finite indecomposable root system containing F. The G_n obtained in this way will then satisfy (i) and (ii). Since $R=\bigcup F_n$, (iii) will hold and $G_n \subset G_{n+1}$ implies $L(G_n) \subset L(G_{n+1})$. An $h \in H$ such that $(h, h_\alpha)=0$ for all α in R would then have $D_h=0$, hence $h=0$ and H is spanned by the set of h_α. Since the set of e_α spans $H^\perp$ then $L=$closure of $\bigcup L(G_n)$. Now dim L is infinite and each $L(G_n)$ is finite-dimensional so that there are infinitely many distinct $L(G_n)$. Then any infinite subsequence of the G_n will also satisfy (i)–(iii) and the first part of (iv). By passing to subsequences if necessary it is possible to eliminate any duplications and furthermore obtain a sequence whose elements are all of the same type. Since their dimensions are unbounded there can be no exceptional algebras.

DEFINITION. Let K_n be the real linear subspace of the conjugate space of H spanned by $\{\alpha: \alpha \in G_n\}$. Let $p_1=\dim K_1$ and $p_n=\dim(K_n/K_{n-1})$ for $n=2, 3, \cdots$. Then each p_i is a positive integer and the rank of the simple algebra $L(G_n)$ is $p_1+\cdots+p_n$.

Since G_1 is a root system for $L(G_1)$ there exist $\alpha_{1,1}, \cdots, \alpha_{1,p_1}$ in G_1 which form a linear basis of K_1. Since G_2 is a root system for $L(G_2)$ there exist $\alpha_{2,1}, \cdots, \alpha_{2,p_2}$ in G_2 such that $\alpha_{1,1}, \cdots, \alpha_{1,p_1}, \alpha_{2,1}, \cdots, \alpha_{2,p_2}$ form a linear basis for K_2. Necessarily $\alpha_{2,i} \notin G_1$. Continuing this process we can find, for each $n \geqq 2$, $\alpha_{n,1}, \cdots, \alpha_{n,p_n} \in G_n - G_{n-1}$ such that the set

$$\{\alpha_{i,j}: i = 1, \cdots, n; j = 1, \cdots, p_i\}$$

is a linear basis for K_n. Order this basis as follows:

$$\alpha_{n,p_n}, \cdots, \alpha_{n,1}, \alpha_{n-1,p_{n-1}}, \cdots, \alpha_{n-1,1}, \cdots, \alpha_{2,1}, \alpha_{1,p_1}, \cdots, \alpha_{1,1}.$$

Suppose $\tau \in K_n$, $\tau \neq 0$. Then τ is a linear combination with real coefficients of the elements of this ordered basis. Define $\tau > 0$ or $\tau < 0$ according as the first nonzero coefficient is positive or negative. For $\tau_1 \neq \tau_2$ let $\tau_1 > \tau_2$ if and only if $\tau_1 - \tau_2 > 0$. This then gives a total ordering of K_n and induces an ordering of G_n. By the choice of basis for each K_n, for integers n, m and $\alpha, \beta \in G_n \cap G_m$, $\alpha > \beta$ in the ordering of G_n if and only if $\alpha > \beta$ in the ordering of G_m.

Now suppose α, β are any roots. Choose n such that $\alpha, \beta \in G_n$ and define $\alpha > \beta$ if and only if they are so related in the ordering of G_n. This gives a well-defined total ordering on the set of all roots and has the following properties:

(i) $\alpha > 0$ implies $-\alpha < 0$.

(ii) $\alpha > 0$, $\beta > 0$ implies $\alpha + \beta > 0$.

(iii) If $\alpha > 0$ and $\alpha \notin G_n$ then $\alpha > \beta$ for every $\beta \in G_n$.

(iv) The ordering induced on G_n is a lexicographical ordering with respect to a basis of roots.

Let R^+ be the set of positive roots. Then, since G_n is finite, property (iii) implies that R^+ is well-ordered. An $\alpha \in R^+$ will be called simple if α cannot be written as the sum of two positive roots. Let S denote the set of all simple roots.

3.3. (1) $S \cap G_n$ is a complete set of simple roots (in the sense of [3]) for $L(G_n)$.

(2) For α, β in S, $\alpha - \beta$ is a root only if $\alpha = \beta$. Thus $k_1(\alpha, \beta) = k_1(\beta, \alpha) = 0$.

(3) S is linearly independent over the reals and every α in R^+ is a linear combination of elements of S with non-negative integral coefficients which are almost all zero.

(4) If $\tau = \sum n_i \alpha_i$ where $\alpha_i \in S$ and almost all n_i are zero there is an algorithm to determine whether or not τ is a root. To apply the algorithm it is sufficient to know (h_α, h_β) for all $\alpha, \beta \in S$.

Proof. (1) If $\alpha, \beta, \gamma \in R^+$ and $\alpha = \beta + \gamma$ then $\alpha > \beta > 0$ and $\alpha > \gamma > 0$. If $\alpha \in G_n$ then (iii) of the definition above implies $\beta, \gamma \in G_n$. Hence an $\alpha \in G_n$ is simple in G_n if and only if α is simple in R.

(2) and (3) can be deduced from the corresponding properties for the finite-dimensional case proved in [3, Exposé 10].

(4) Choose n such that $\tau \in K_n$. The statement then follows from the result proved in [3, Exposé 16], applied to the algebra $L(G_n)$, using the fact that the fundamental bilinear form is determined up to a constant multiple.

Definition. Define the graph of S to be the set G of all (h_α, h_β) where α, β vary over S. Then knowing the graph is equivalent to determining $\|h_\alpha\|$ and $k_2(\alpha, \beta)$ for α, β in S. If L, L' are two algebras of the type considered in this section with H and H' as Cartan subalgebras and G, G' as the corresponding graphs we will say that G is isomorphic to G' if and only if there is a mapping α to α' of S onto S' with $(h_\alpha, h_\beta) = (h'_{\alpha'}, h'_{\beta'})$ for all α, β in S.

3.4. Let L, L' be as above and suppose G is isomorphic to G'. Then there is an algebraic isomorphism ϕ of L onto L' such that:

(1) $\phi(h_\alpha) = h'_{\alpha'}$ for all $\alpha \in R$.

(2) $\phi(x)^* = \phi(x^*)$ for all $x \in L$.

(3) $(\phi(x), \phi(y)) = (x, y)$ for all x, y in L.

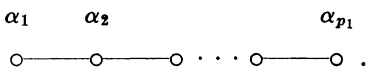
Proof. By using the algorithm of 3.3, (4) it is possible to extend the map of S onto S' to a mapping α to α' of R onto R' which preserves inner products for the h_α. This mapping then necessarily preserves all of the algebraic structure of R. For a complex linear combination $h=\sum c_i h_{\alpha_i}$, where $\alpha_i \in R$, let $\phi(h)=\sum c_i h'_{\alpha_i'}$. Then ϕ is well-defined and preserves inner products so that it extends uniquely to an isometry of H onto H' and satisfies (1). Since $\phi(h_\alpha^*) =\phi(h_\alpha)^*$ for all h_α, ϕ will satisfy (2) for any $x\in H$.

Let $\{f_{\alpha'}: \alpha' \in R'\}$ be a fixed set of elements in L' with $f_{\alpha'} \in V_{\alpha'}$, $\|f_{\alpha'}\|=1$, and $f_{\alpha'}^* = f_{-\alpha'}$. Let $c_{\alpha',\beta'}$ be the structure constants for L' defined by the set of $f_{\alpha'}$. To extend ϕ to all of L with the required properties it is then sufficient to find a set $\{e_\alpha : \alpha \in R\}$ in L with $e_\alpha \in V_\alpha$, $\|e_\alpha\|=1$, $e_\alpha^* = e_{-\alpha}$, and such that the structure constants $c_{\alpha,\beta}$ for L defined by this set satisfy $c_{\alpha,\beta}=c_{\alpha',\beta'}$.

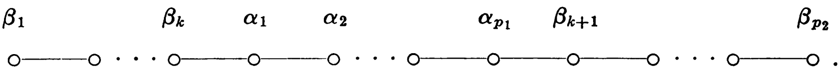
Thus the problem is reduced to finding the set of e_α. A corresponding result appears in [3, Exposé 11, Théorème 1]. An examination of the proof there shows the essential features are a well-ordering of R^+ compatible with the algebraic structure and the relations on the structure constants which were proved here in 2.6. (These hold for L since there is always an n such that α, β, γ, δ all lie in G_n.) Using these, the proof in [3] can be repeated here word for word.

3.5. Because of 3.4 it only remains to determine the possible graphs for L and give examples of each type in order to complete the classification.

First, suppose all of the $L(G_n)$ in 3.2 are of type A. Then the root diagram for the simple system $S\cap G_1$ has the form:

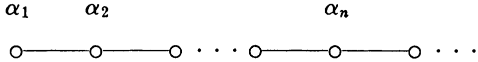


This means, of course, that $k_2(\alpha_i, \alpha_{i-1})=k_2(\alpha_i, \alpha_{i+1})=1$ and otherwise $k_2(\alpha_i, \alpha_j)=0$. Furthermore, by the remark after 2.5, $\|h_{\alpha_i}\|=\|h_{\alpha_j}\|$ for $1\leqq i, j\leqq p_1$. Now let $S\cap G_2$ be written as $\alpha_1, \cdots, \alpha_{p_1}, \beta_1, \cdots, \beta_{p_2}$. After any necessary reordering the diagram for $S\cap G_2$ will have the form:



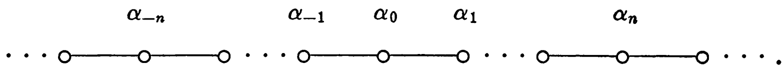
It is possible that all the β_j may be at one end of the chain. Again it follows that $\|h_{\alpha_i}\|=\|h_{\beta_j}\|$ for α_i, β_j. Continuing this process and introducing the necessary new notation for the α's in S we will obtain one of the following two possibilities:

Type A.



or

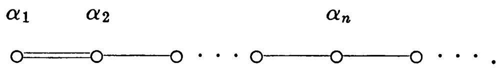
Type A′.



In either case $\|h_{\alpha_i}\| = \|h_{\alpha_j}\|$ for all i, j, $k_2(\alpha_i, \alpha_j) = 0$ for $j \neq i-1$, $i+1$ while $k_2(\alpha_i, \alpha_{i-1}) = k_2(\alpha_i, \alpha_{i+1}) = 1$. Thus the graph is completely determined up to a constant multiple.

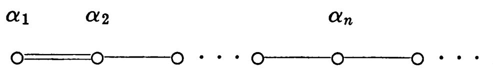
Entirely similar arguments for the other possibilities give the following types:

Type B.



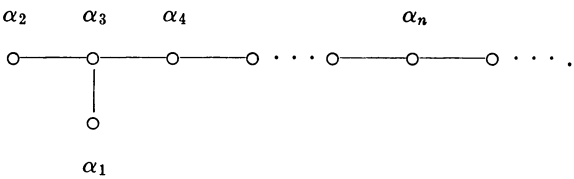
Here $2^{1/2}\|h_{\alpha_1}\| = \|h_{\alpha_i}\|$ for $i = 2, 3, \cdots$ and $k_2(\alpha_1, \alpha_2) = 1$, $k_2(\alpha_2, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

Type C.



Here $\|h_{\alpha_1}\| = 2^{1/2}\|h_{\alpha_i}\|$ for $i = 2, 3, \cdots$ and $k_2(\alpha_1, \alpha_2) = 2$, $k_2(\alpha_2, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

Type D.



Here $\|h_{\alpha_i}\| = \|h_{\alpha_j}\|$ for all i and j and $k_2(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1) = 0$, $k_2(\alpha_3, \alpha_1) = 1$ while otherwise $k_2(\alpha_i, \alpha_j)$ is as above.

3.6. In this paragraph it will be shown that each of the five types A, A′, B, C, D occurs as the graph of an L^* algebra. However, these algebras are not all distinct and give rise to only three nonisomorphic types. More explicitly, A and A′ are isomorphic and so are B and D.

All of these examples are Lie subalgebras of the associative H^* algebra K of Hilbert-Schmidt operators on a separable Hilbert space $\mathcal{H}$. For descriptive purposes it is convenient to choose an orthonormal basis of $\mathcal{H}$ and regard K as a matrix algebra relative to this basis. In each case a Cartan subalgebra H is obtained by taking the intersection of the algebra in question with the set of diagonal matrices. Having done this we will let λ_i denote the linear functional on H which assigns the ith diagonal entry to every element of H. Determination of a set of simple roots and the associated graph is analogous to the finite-dimensional case and the computations will be omitted here. After choosing the proper norm on L an application of 3.4 and 3.5 will show that L is isomorphic in all respects to one of the algebras described here.

In the following discussion a conjugate linear transformation J of $\mathcal{H}$ onto

$\mathcal{H}$ such that $(Jx, Jy)=(y, x)$ will be called a conjugation if $J^2=1$ and an anti-conjugation if $J^2=-1$.

TYPE A. Let $\{\phi_n: n=1, 2, \cdots\}$ be a basis of $\mathcal{H}$ and let A be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. A is simple since the center of K is trivial. A simple system of roots is given by

$$\{\lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

TYPE A′. Let $\{\phi_n: n=0, \pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let A′ be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. A simple system of roots is given by $\{\lambda_i-\lambda_{i+1}: i=0, \pm 1, \pm 2, \cdots\}$.

The algebras A and A' are isomorphic since there is a unitary operator U on $\mathcal{H}$ such that $X\in A$ if and only if UXU^{-1} is in A'.

TYPE B. Let $\{\phi_n: n=0, \pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let J_1 be the conjugation of $\mathcal{H}$ such that $J_1\phi_n=\phi_{-n}$. Let B be the set of Hilbert-Schmidt operators T such that $T^*J_1=-J_1T$. If $\langle x, y\rangle$ is defined by $\langle x, y\rangle=(x, J_1y)$ for $x, y\in\mathcal{H}$ then $\langle\ ,\ \rangle$ is a symmetric bilinear form and B is the set of T in K which are skew-adjoint with respect to this form. A simple system of roots is given by $\{\lambda_1, \lambda_i-\lambda_{i+1}: i=1, 2, \cdots\}$.

TYPE D. Let $\{\phi_n: n=\pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$ and let J_2 be the conjugation on $\mathcal{H}$ such that $J_2\phi_n=\phi_{-n}$. Let D be the set of T in K such that $T^*J_2=-J_2T$. A simple system of roots is given by

$$\{\lambda_1 + \lambda_2, \lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

Since J_1 and J_2 are two conjugations of $\mathcal{H}$ there is a unitary U on $\mathcal{H}$ such that $UJ_1=J_2U$. Then for any $T\in K$, $T\in B$ if and only if UTU^{-1} is in D. Hence B is isomorphic to D.

TYPE C. Let $\{\phi_n: n=\pm 1, \pm 2, \cdots\}$ be a basis of $\mathcal{H}$. Let J be the anti-conjugation on $\mathcal{H}$ such that $J\phi_n=-\phi_{-n}$ for all positive n. Let C be the set of all Hilbert-Schmidt operators on $\mathcal{H}$ such that $T^*J=-JT$. Then C is the set of all $T\in K$ which are skew-symmetric with respect to the skew-symmetric form $\langle x, y\rangle=(x, Jy)$. A simple system of roots is given by

$$\{2\lambda_1, \lambda_i - \lambda_{i+1}: i = 1, 2, \cdots\}.$$

3.7. THEOREM 3. *Let L be a separable simple L^* algebra which has a Cartan decomposition relative to some Cartan subalgebra. Then (up to a multiple of the inner product on L) L is isomorphic to one of the following algebras:*

(1) *A, the algebra of all Hilbert-Schmidt operators on a separable Hilbert space $\mathcal{H}$.*

(2) *B, the algebra of all Hilbert-Schmidt operators T on $\mathcal{H}$ such that $T^*J=-JT$ for some fixed conjugation J of $\mathcal{H}$.*

(3) *C, the algebra of all Hilbert-Schmidt operators T on $\mathcal{H}$ such that $T^*J=-JT$ for some fixed anti-conjugation J of $\mathcal{H}$.*

REMARK. It still should be shown that the remaining three algebras A,

B, C are nonisomorphic. For two algebras L, L' of the type described in 3.7 and acting on the same space $\mathfrak{H}$ let L be equivalent to L' if and only if there is a unitary U on $\mathfrak{H}$ with $ULU^{-1}=L'$. We will show that L and L' are isomorphic only if they are equivalent. Since A, B, and C are clearly not equivalent this will be sufficient.

An $x\in L$ will be called primitive if (i) $x=x^*\neq 0$, (ii) $D_x^3=D_x$, and (iii) x cannot be written $x=y+z$ where y and z satisfy (i) and (ii). By using the fact (see the proof of Theorem 2) that every Cartan subalgebra of L is a set of diagonal matrices relative to some basis of $\mathfrak{H}$ it follows that each such subalgebra has a basis of primitive elements and the vectors h_α are obtained from these by linear operations in a unique way according to the type of the associated graph. Since any isomorphism of L will preserve primitive elements the set $\{h_\alpha : \alpha\in R\}$, and hence the graph of L, is determined up to equivalence and the same will then hold for L.

4. Some remarks on derivations.

DEFINITION. Let L be a semi-simple L^* algebra. A bounded operator D on L will be called a derivation of L if and only if $D[x, y]=[Dx, y]+[x, Dy]$ for all x, y in L.

If dim L is finite it is known that every derivation of L is inner, i.e. equal to D_x for some $x\in L$ [3, Exposé 7]. However, this is not true in general. To see this let A be the L^* algebra of all Hilbert-Schmidt operators on a separable infinite-dimensional Hilbert space. Then A is an associative ideal in the algebra of all bounded operators (see [4, pp. 73–75]). For a bounded operator B let T_B be the operator on A defined by $T_BX=BX-XB$. Then T_B is a bounded derivation of A and $T_B=0$ if and only if B is a scalar multiple of the identity. Hence T_B is inner only if it differs from a Hilbert-Schmidt operator by a multiple of the identity and this implies A has outer derivations. Similar arguments can be used for B and C.

The same example can be used to show that the image of L under the adjoint representation need not be closed. By the closed graph theorem this is equivalent to proving that the norms on L and its image are not equivalent. Letting $L=A$ as above and regarding A as a matrix algebra with the usual unit matrices as a basis let $X_k=k^{-1/2}\sum_1^k E_{ii}$. Then $\|X_k\|=1$ while $\|D_{X_k}\|=k^{-1/2}$. Thus $\|D_{X_k}\|$ tends to zero as k becomes large.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS