CONVERGENCE OF INTERPOLATORY POLYNOMIALS, (0, 1, 2, 4) INTERPOLATION

BY

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1. **Introduction.** In most of the problems of interpolation we prescribe the values of a function at some points x_i, and of some of its consecutive derivatives there. Thus in the Hermite interpolation formula, the value of the function and its first derivative are prescribed at some points in a given interval. A general problem of interpolation was treated by Birkhoff in 1906, who considered a system of pairs of numbers (k_i, x_i) $(i=1, 2, \cdots, n)$ where k_i are integers $\geqq 0$ and x_i are any points in a given interval. Birkhoff's main theorem concerns the number of changes of sign of the kernel in the integral representation of the remainder in a general interpolatory formula when the system is a "normal" point system. For this purpose he has to classify the system of points according as they are "conservative" or "nonconservative." For greater details see [7]. A more particular case $n=2$ was treated directly by Polya when the system is not "normal" in the sense of Birkhoff. A similar case in the complex plane has been treated by Cinquini [8]. But this general point of view does not bring out the character of the interpolatory polynomials.

Recently in a series of papers [1; 2; 3], Turán has treated Lacunary interpolation from a different point of view by considering what he calls (0, 2)-interpolation, where the value of the function and its second derivative are given at some points. He considers the problem of their existence, uniqueness, and explicit representation and the problem of convergence.

In the papers [4; 5] we have been dealing with so called "Lacunary interpolation." The terms (0, 1, 3) and (0, 1, 2, 4) interpolations have been defined therein. By (0, 1, 2, 4) interpolation we mean the interpolation which concerns n given points in $[-1, 1]$,

$$1 \geqq t_1 > t_2 > t_3 > \cdots > t_{n-1} > t_n \geqq -1 \tag{1.1}$$

when the values of the function, its first, second and fourth derivatives are prescribed at these n points. In other words

$$f(t_i) = a_i, \qquad f'(t_i) = b_i, \qquad f''(t_i) = c_i, \qquad f^{(4)}(t_i) = d_i \tag{1.2}$$

for $i=1, 2, \cdots, n$; we want to determine the explicit forms of the polynomials of degree $\leqq 4n-1$ which take the values a_i at t_i, whose first and second derivatives at t_i are respectively equal to b_i and c_i and whose fourth derivative at t_i is equal to d_i. It has been shown that when we choose the n points to be the real zeros of

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$$\pi_n(x) = n(n-1)\int_{-1}^{x} P_{n-1}(t)dt = (1-x^2)P'_{n-1}(x) \tag{1.3}$$

where $P_k(x)$ is the kth Legendre polynomial such that $P_k(1)=1$; these polynomials exist if and only if n is even. So we take $n=2k\geqq 4$.

The object of this paper is to study the convergence of the polynomials $R_n(x)$ of interpolation for which we showed in [5] that

$$R_n(x) = \sum_{\nu=1}^{n} a_\nu A_\nu(x) + \sum_{\nu=1}^{n} b_\nu B_\nu(x) + \sum_{\nu=1}^{n} c_\nu C_\nu(x) + \sum_{\nu=1}^{n} d_\nu D_\nu(x)$$

where $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$, $D_\nu(x)$ are polynomials each of degree $\leqq 4n-1$. These polynomials are uniquely determined by the conditions:

$$A_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & j = \nu \end{cases}, \quad A'_\nu(x_j) = 0, \quad A''_\nu(x_j) = 0, \quad A^{(4)}_\nu(x_j) = 0, \tag{1.4}$$

$$B_\nu(x_j) = 0, \quad B'_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & j = \nu \end{cases}, \quad B''_\nu(x_j) = 0, \quad B^{(4)}_\nu(x_j) = 0, \tag{1.5}$$

$$C_\nu(x_j) = 0, \quad C'_\nu(x_j) = 0, \quad C''_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & j = \nu \end{cases}, \quad C^{(4)}_\nu(x_j) = 0. \tag{1.6}$$

$$D_\nu(x_j) = 0, \quad D'_\nu(x_j) = 0, \quad D''_\nu(x_j) = 0, \quad D^{(4)}_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & j = \nu \end{cases} \tag{1.7}$$

where $j = 1, 2, \cdots, n$.

2. We now consider the sequence of points

$$1 = x_{1n} > x_{2n} > \cdots > x_{n-1,n} > x_{nn} = -1, \quad (n = 4, 6, \cdots, 2k, \cdots) \tag{2.1}$$

where $x_{\nu n}$'s stand for the zeros of $\pi_n(x)$. Then forming the interpolatory polynomials for each $n=2k$, we shall write the fundamental polynomials (1.4), (1.5), (1.6) and (1.7) as $A_{\nu n}(x)$, $B_{\nu n}(x)$, $C_{\nu n}(x)$, $D_{\nu n}(x)$ respectively. Let $f(x)$ be defined for $[-1, 1]$: we consider the sequence of polynomials

$$\begin{aligned} R_n(x,f) = &\sum_{\nu=1}^{n} f(x_{\nu n})A_{\nu n}(x) + \sum_{\nu=1}^{n} f'(x_{\nu n})B_{\nu n}(x) \\ &+ \sum_{\nu=1}^{n} c_{\nu n}C_{\nu n}(x) + \sum_{\nu=1}^{n} d_{\nu n}D_{\nu n}(x) \end{aligned} \tag{2.2}$$

with arbitrary numbers $c_{\nu n}$ and $d_{\nu n}$. We shall prove the following

THEOREM. *Let $f(x)$ have the continuous derivative of order 2 in $[-1, 1]$ with continuity modulus $\omega(\delta)$ of $f''(x)$ such that*

$$\int_0 \frac{\omega(t)}{t^{3/2}}dt \tag{2.3}$$

exists. Supposing that for arbitrary small $\epsilon>0$ *we have for* $n>n_0(\epsilon)$ *and* $\nu=1, 2, \cdots, n$;

$$|c_{\nu n}| \leqq \epsilon n, \qquad |d_{\nu n}| \leqq \epsilon n^3, \tag{2.4}$$

the sequence $R_n(x, f)$ *converges to* $f(x)$ *uniformly in* $[=1, 1]$.

3. **Preliminaries.** The explicit forms of the fundamental functions $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ and $D_\nu(x)$ which we have found in [5] are the following:

(a)

$$D_1(x) = -\frac{\overset{3}{\pi}_n(x)}{60n^4(n-1)^4}\left\{P_{n-1}(x) + \frac{5}{3}\right\}, \tag{3.1}$$

$$D_n(x) = \frac{\overset{3}{\pi}_n(x)}{60n^4(n-1)^4}\left\{P_{n-1}(x) - \frac{5}{3}\right\}, \tag{3.2}$$

and for $2 \leqq \nu \leqq n-1$,

$$\begin{aligned} D_\nu(x) = \frac{\overset{3}{\pi}_n(x)}{24P''_{n-1}(x_\nu)\pi'^3_n(x_\nu)}\Bigg[\int_{-1}^{x} \frac{P'_{n-1}(t)}{t-x_\nu}\,dt - \Bigg\{ & \frac{x_\nu}{1-x_\nu^2} \\ - \frac{3}{5(1-x_\nu^2)P_{n-1}(x_\nu)}\Bigg\} P_{n-1}(x) - \frac{2+3x_\nu}{3(1-x_\nu^2)} + & \frac{1}{(1-x_\nu^2)P_{n-1}(x_\nu)}\Bigg]. \end{aligned} \tag{3.3}$$

(b) Denoting $\pi_n^2(x)$ by $Q_{2n}(x)$,

$$C_1(x) = \frac{Q_{2n}(x)}{Q''_{2n}(1)}\left\{r_1(x) + \left(\frac{5}{72} - \frac{1}{18n(n-1)}\right)\pi_n(x) - \frac{Q'_{2n}(x)}{16n(n-1)}\right\}, \tag{3.4}$$

where

$$\begin{aligned} r_1(x) = \frac{3+x}{4}\,l_1^2(x) - \frac{1-x^2}{4}\,l_1(x)l_1'(x) \\ + \left\{\frac{5}{16} + \frac{1}{8n(n-1)}\right\}\pi_n(x)\left(1 + \frac{1}{3}P_{n-1}(x)\right). \end{aligned} \tag{3.5}$$

$$C_n(x) = \frac{Q_{2n}(x)}{Q''_{2n}(-1)}\left\{r_n(x) - \left(\frac{5}{72} - \frac{1}{18n(n-1)}\right)\pi_n(x) + \frac{Q'_{2n}(x)}{16n(n-1)}\right\}, \tag{3.6}$$

where

$$\begin{aligned} r_n(x) = \frac{3-x}{4}\,l_n^2(x) + \frac{1-x^2}{4}\,l_n(x)l_n'(x) \\ + \left\{\frac{5}{16} + \frac{1}{8n(n-1)}\right\}\pi_n(x)\left(1 - \frac{1}{3}P_{n-1}(x)\right), \end{aligned} \tag{3.7}$$

and for $2 \leqq \nu \leqq n-1$,

$$C_\nu(x) = \frac{Q_{2n}(x)}{Q''_{2n}(x_\nu)}\left[r_\nu(x) + \frac{2n(n-1)}{3(1-x_\nu^2)}\rho_\nu(x) + \frac{4\pi_n(x)}{9n(n-1)(1-x_\nu^2)P_{n-1}^2(x_\nu)} - \frac{Q'_{2n}(x)}{9n^2(n-1)^2(1-x_\nu^2)P_{n-1}^3(x_\nu)}\right], \tag{3.8}$$

where

$$r_\nu(x) = l_\nu^2(x) + \frac{\pi_n(x)}{2n(n-1)P_{n-1}(x_\nu)}\left\{\int_1^x \frac{l'_\nu(t)}{t-x_\nu}dt + \int_{-1}^x \frac{l'_\nu(t)}{t-x_\nu}dt + \frac{P_{n-1}(x)}{3(1-x_\nu^2)P_{n-1}^2(x_\nu)}\right\}, \tag{3.9}$$

$$l_\nu(x) = \frac{\pi_n(x)}{(x-x_\nu)\pi_n'(x_\nu)}, \tag{3.9a}$$

and

$$\rho_\nu(x) = \frac{\pi_n(x)}{4n^2(n-1)^2P_{n-1}^2(x_\nu)}\left\{(1-x_\nu^2)\int_1^x \frac{P'_{n-1}(t)}{t-x_\nu}dt + (1-x_\nu^2)\int_{-1}^x \frac{P'_{n-1}(t)}{t-x_\nu}dt + 2P_{n-1}(x)\left(-x_\nu + \frac{1}{3P_{n-1}(x_\nu)}\right) - 4\right\}; \tag{3.10}$$

(c)

$$B_1(x) = \frac{\pi_n(x)}{\pi_n'(1)}\left\{u_1(x) + \frac{n(n-1)}{576}(71n^4 - 142n^3 + 151n^2 - 80n - 12)\omega_1(x) + \frac{n(n-1)}{576}(73n^4 - 146n^3 + 137n^2 - 64n + 12)\omega_n(x)\right\} - \frac{\pi_n''(1)}{\pi_n'(1)}C_1(x), \tag{3.11}$$

where

$$\omega_1(x) = \frac{Q_{2n}(x)}{6n^3(n-1)^3}\left\{1 + \frac{1}{2}P_{n-1}(x)\right\}, \tag{3.12}$$

$$\omega_n(x) = -\frac{Q_{2n}(x)}{6n^3(n-1)^3}\left\{1 - \frac{1}{2}P_{n-1}(x)\right\}, \tag{3.13}$$

$$
\begin{aligned}
u_1(x) = {} & l_1^3(x) + \frac{\pi_n(x)}{3\pi_n'(1)}\left[l_1(x)l_1'(x) - 10l_1'(1)r_1(x)\right] \\
& - \frac{n(n-1)}{32}(n^4 - 2n^3 - 17n^2 + 18n + 8)\omega_1(x) \\
& + \frac{n(n-1)}{96}(25n^4 - 50n^3 + 35n^2 - 10n - 48)\omega_n(x).
\end{aligned} \tag{3.14}
$$

and $C_1(x)$ is the polynomial of degree $\leqq 4n-1$ given by (3.4).

$$
\begin{aligned}
B_n(x) = {} & \frac{\pi_n(x)}{\pi_n'(-1)} \\
& \cdot \left\{u_n(x) - \frac{n(n-1)}{576}(73n^4 - 146n^3 + 137n^2 - 64n + 12)\omega_1(x)\right. \\
& \left. - \frac{n(n-1)}{576}(71n^4 - 142n^3 + 151n^2 - 80n - 12)\omega_n(x)\right\} \\
& - \frac{\pi_n''(-1)}{\pi_n'(-1)} C_n(x),
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
u_n(x) = {} & l_n^3(x) + \frac{\pi_n(x)}{3\pi_n'(-1)}\left[l_n(x)l_n'(x) - 10l_n'(-1)r_n(x)\right] \\
& - \frac{n(n-1)}{96}(25n^4 - 50n^3 + 35n^2 - 10n - 48)\omega_1(x) \\
& + \frac{n(n-1)}{32}(n^4 - 2n^3 - 17n^2 + 18n + 8)\omega_n(x).
\end{aligned} \tag{3.16}
$$

and $C_n(x)$ is the polynomial of degree $\leqq 4n-1$ given by (3.6). For $2 \leqq \nu \leqq n-1$,

$$
\begin{aligned}
B_\nu(x) = {} & \frac{\pi_n(x)}{\pi_n'(x_\nu)}\left[u_\nu(x) + \frac{n(n-1)x_\nu}{(1-x_\nu^2)^2}\omega_\nu(x)\right. \\
& + \frac{n^4(n-1)^4 x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left\{\frac{3}{(1-x_\nu^2)P_{n-1}(x_\nu)} + \frac{12}{P_{n-1}(x_\nu)} + \frac{5}{1-x_\nu^2}\right\}\omega_1(x) \\
& \left. + \frac{n^4(n-1)^4 x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left\{\frac{3}{(1-x_\nu^2)P_{n-1}(x_\nu)} + \frac{12}{P_{n-1}(x_\nu)} - \frac{5}{1-x_\nu^2}\right\}\omega_n(x)\right],
\end{aligned} \tag{3.17}
$$

where

$$
\begin{aligned}
w_\nu(x) = {} & \frac{Q_{2n}(x)}{6\pi_n'^{\,2}(x_\nu)P''_{n-1}(x_\nu)}\Bigg[\int_{-1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt - P_{n-1}(x)\Bigg\{\frac{x_\nu}{2(1-x_\nu^2)} \\
& + \frac{1}{4}\int_{-1}^{1}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt\Bigg\} - \Bigg\{\frac{1}{1-x_\nu^2} + \frac{1}{2}\int_{-1}^{1}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt\Bigg\}\Bigg].
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
u_\nu(x) = {} & l_\nu^3(x) + \frac{4n(n-1)x_\nu}{(1-x_\nu^2)^2}\,\omega_\nu(x) + \frac{2n^4(n-1)^4}{\pi_n'^{\,3}(x_\nu)(1-x_\nu)^2}\,\omega_1(x) \\
& + \frac{2n^4(n-1)^4\omega_n(x)}{\pi_n'^{\,3}(x_\nu)(1+x_\nu)^2} + \frac{1}{3\pi_n'(x_\nu)}\cdot\pi_n(x)l_\nu(x)l_\nu'(x).
\end{aligned}
\tag{3.19}
$$

(d) Lastly, for $2 \leqq \nu \leqq n-1$ we have

$$
\begin{aligned}
A_\nu(x) = {} & l_\nu^4(x) + \frac{\pi_n(x)}{4\pi_n'(x_\nu)}\left[l_\nu^2(x)l_\nu'(x) - 9l_\nu''(x_\nu)v_\nu(x)\right] \\
& - \frac{3n(n-1)}{(1-x_\nu^2)^2}\left\{(n-2)(n+1) - \frac{8x_\nu^2}{1-x_\nu^2}\right\}D_\nu(x) \\
& + \frac{3n^2(n-1)^2}{4(1-x_\nu)^2P_{n-1}^3(x_\nu)\pi_n'(x_\nu)}\left\{\frac{10}{1-x_\nu} + \frac{n(n-1)}{(1+x_\nu)^2}\left(1+\frac{2}{P_{n-1}(x_\nu)}\right)\right\}D_1(x) \\
& + \frac{3n^2(n-1)^2}{4(1+x_\nu)^2P_{n-1}^3(x_\nu)\pi_n'(x_\nu)}\left\{\frac{10}{1+x_\nu} + \frac{n(n-1)}{(1-x_\nu)^2}\left(1-\frac{2}{P_{n-1}(x_\nu)}\right)\right\}D_n(x),
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
v_\nu(x) = {} & \frac{\pi_n(x)}{\pi_n'(x_\nu)}\Bigg[r_\nu(x) + \frac{n(n-1)}{3(1-x_\nu^2)}\,\rho_\nu(x) + \frac{\pi_n(x)}{6n(n-1)(1-x_\nu^2)P_{n-1}^2(x_\nu)} \\
& - \frac{Q'_{2n}(x)}{18n^2(n-1)^2(1-x_\nu^2)P_{n-1}^3(x_\nu)}\Bigg]
\end{aligned}
\tag{3.21}
$$

and $D_1(x)$, $D_n(x)$ and $D_\nu(x)$ are the polynomials each of degree $\leqq 4n-1$ given by (3.1), (3.2) and (3.3).

$$
\begin{aligned}
A_1(x) = {} & l_1^4(x) + \frac{\pi_n(x)}{4\pi_n'(1)}\left[l_1^2(x)l_1'(x) - (9l_1''(1) + 10l_1'^{\,2}(1))v_1(x)\right. \\
& \left. - 17l_1'(1)u_1(x)\right] - \frac{n(n-1)}{1536}(804n^6 - 1913n^5 - 1049n^4 + 5353n^3 \\
& - 1215n^2 + 744n - 864)D_1(x) + \frac{n(n-1)}{1536}(653n^6 - 1959n^5 \\
& + 3035n^4 - 2805n^3 + 1352n^2 - 276n - 1440)D_n(x),
\end{aligned}
\tag{3.22}
$$

where

$$(3.23)\quad v_1(x) = \frac{\pi_n(x)}{\pi_n'(x_1)}\left\{r_1(x) + \pi_n(x)\cdot\frac{(n-2)(n+1)}{48n(n-1)} - \frac{Q_{2n}'(x)}{32n(n-1)}\right\},$$

$$(3.24)\quad \begin{aligned} A_n(x) = {} & l_n^4(x) + \frac{\pi_n(x)}{4\pi_n'(-1)}\left[l_n^2(x)l_n'(x) - (9l_n''(-1) + 10l_n'^2(-1))v_n(x)\right. \\ & \left. - 17l_n'(-1)u_n(x)\right] \\ & + \frac{n(n-1)}{1536}(653n^6 - 1959n^5 + 3035n^4 \\ & - 2805n^3 + 1352n^2 - 276n - 1440)D_1(x) \\ & - \frac{n(n-1)}{1536}(804n^6 - 1913n^5 - 1049n^4 + 5353n^3 - 1215n^2 \\ & + 744n - 864)D_n(x), \end{aligned}$$

where

$$(3.25)\quad v_n(x) = \frac{\pi_n(x)}{\pi_n'(-1)}\left\{r_n(x) + \frac{(n-2)(n+1)}{48n(n-1)}\pi_n(x) + \frac{Q_{2n}'(x)}{32n(n-1)}\right\}.$$

4. For our purposes we shall need some other representations of $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ and $D_\nu(x)$.

Using the integral

$$(4.1)\quad \int_{-1}^{1}\frac{P_{n-1}'(t)}{t-x_\nu}dt = \frac{2x_\nu}{1-x_\nu^2} - \frac{2}{1-x_\nu^2}\cdot\frac{1}{P_{n-1}(x_\nu)}$$

we have for $\nu = 2, 3, \cdots, n-1$,

(a)

$$(4.2)\quad \begin{aligned} D_\nu(x) = {} & \frac{\pi_n^3(x)}{24P_{n-1}''(x_\nu)\pi_n'^3(x_\nu)}\left[\int_{-1}^{x}\frac{P_{n-1}'(t)}{t-x_\nu}dt - \left\{\frac{x_\nu}{1-x_\nu^2}\right.\right. \\ & \left.\left. - \frac{3}{5(1-x_\nu^2)P_{n-1}(x_\nu)}\right\}P_{n-1}(x) - \frac{2+3x_\nu}{3(1-x_\nu^2)} + \frac{1}{(1-x_\nu^2)P_{n-1}(x_\nu)}\right], \end{aligned}$$

and

$$(4.3)\quad \begin{aligned} D_\nu(x) = {} & \frac{\pi_n^3(x)}{24P_{n-1}''(x_\nu)\pi_n'^3(x_\nu)}\left[\int_{1}^{x}\frac{P_{n-1}'(t)}{t-x_\nu}dt - \left\{\frac{x_\nu}{1-x_\nu^2}\right.\right. \\ & \left.\left. - \frac{3}{5(1-x_\nu^2)P_{n-1}(x_\nu)}\right\}P_{n-1}(x) - \frac{2-3x_\nu}{3(1-x_\nu^2)} - \frac{1}{(1-x_\nu^2)P_{n-1}(x_\nu)}\right]. \end{aligned}$$

(b) Since

$$Q_{2n}''(1) = 2n^2(n-1)^2 = Q_{2n}''(-1),$$

and $Q_{2n}''(x_\nu) = 2\pi_n'^2(x_\nu)$ we have from (3.4), (3.8), (3.6), (1.3) and (5.3)

$$C_1(x) = \frac{\pi_n^2(x)}{2n^2(n-1)^2}\left[r_1(x) + \left\{\frac{5}{72} - \frac{1}{18n(n-1)}\right\}\pi_n(x) + \frac{1}{8}\,P_{n-1}(x)\pi_n(x)\right], \tag{4.4}$$

$$C_n(x) = \frac{\pi_n^2(x)}{2n^2(n-1)^2}\left[r_n(x) - \left\{\frac{5}{72} - \frac{1}{18n(n-1)}\right\}\pi_n(x) - \frac{1}{8}\,P_{n-1}(x)\pi_n(x)\right], \tag{4.5}$$

and for $2 \leqq \nu \leqq n-1$,

$$C_\nu(x) = \frac{\pi_n^2(x)}{2n^2(n-1)^2P_{n-1}^2(x_\nu)}\left[r_\nu(x) + \frac{2n(n-1)}{3(1-x_\nu^2)}\,\rho_\nu(x) + \frac{2\pi_n(x)}{9n(n-1)(1-x_\nu^2)P_{n-1}^2(x_\nu)}\left\{2 - \frac{P_{n-1}(x)}{P_{n-1}(x_\nu)}\right\}\right]. \tag{4.6}$$

(c) To simplify $B_1(x)$ and $B_n(x)$ we use the following results [1]:

$$\pi_n'(1) = -n(n-1) = (-1)^{n-1}\pi_n'(-1),$$

$$\pi_n''(1) = -\frac{n^2(n-1)^2}{2} = (-1)^n\pi_n''(-1),$$

and then we have

$$B_1(x) = -\frac{\pi_n(x)}{n(n-1)}\left[u_1(x) + \frac{n(n-1)}{576}(73n^4 - 146n^3 + 137n^2 - 64n + 12)\omega_n(x) + \frac{n(n-1)}{576}(71n^4 - 142n^3 + 151n^2 - 80n - 12)\omega_1(x)\right] + \frac{n(n-1)}{2}\,C_1(x). \tag{4.7}$$

$$B_n(x) = \frac{\pi_n(x)}{n(n-1)}\left[u_n(x) - \frac{n(n-1)}{576}(73n^4 - 146n^3 + 137n^2 - 64n + 12)\omega_1(x) - \frac{n(n-1)}{576}(71n^4 - 142n^3 + 151n^2 - 80n - 12)\omega_n(x)\right] - \frac{n(n-1)}{2}\,C_n(x). \tag{4.8}$$

From (3.17), (3.12), (3.13) etc. we have for $2 \leqq \nu \leqq n-1$,

$$
\begin{aligned}
B_\nu(x) = &- \frac{\pi_n(x)}{n(n-1)P_{n-1}(x_\nu)} \\
&\cdot \left[u_\nu(x) + \frac{n(n-1)x_\nu}{(1-x_\nu^2)^2}\,\omega_\nu(x) - \frac{x_\nu \pi_n^2(x)}{180(1-x_\nu^2)n^2(n-1)^2 P_{n-1}^3(x_\nu)} \right. \\
&\left. \cdot \left\{ \frac{10}{1-x_\nu^2} + \frac{P_{n-1}(x)}{P_{n-1}(x_\nu)} \left(\frac{3}{1-x_\nu^2} + 12 \right) \right\} \right].
\end{aligned} \tag{4.9}
$$

(d) To simplify $A_1(x)$ and $A_n(x)$ we need the following results which are easy to verify from (3.9a) (see [4]).

$$
l_1'(1) = \frac{n(n-1)}{4}, \qquad l_n'(-1) = -\frac{n(n-1)}{4},
$$

$$
l_1''(1) = \frac{(n-2)(n+1)}{6}\, l_1'(1), \qquad l_n''(-1) = -\frac{(n-2)(n+1)}{6}\, l_n'(-1).
$$

Thus we have

$$
\begin{aligned}
A_1(x) = &\frac{3}{4}\, l_1^4(x) + \frac{1}{4}\, P_{n-1}(x) l_1^3(x) \\
&+ \frac{\pi_n(x)}{16} \left\{ (4n^2 - 4n - 3) v_1(x) - 17 u_1(x) \right\} \\
&- \frac{n(n-1)}{1536} (804n^6 - 1913n^5 - 1049n^4 + 5353n^3 - 1215n^2 \\
&\qquad + 744n - 864) D_1(x) \\
&+ \frac{n(n-1)}{1536} (653n^6 - 1559n^5 + 3035n^4 - 2805n^3 + 1352n^2 \\
&\qquad - 276n - 1440) D_n(x).
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
A_n(x) = &\frac{3}{4}\, l_n^4(x) - \frac{1}{4}\, P_{n-1}(x) l_n^3(x) \\
&- \frac{\pi_n(x)}{16} \left\{ (4n^2 - 4n - 3) v_n(x) - 17 u_n(x) \right\} \\
&+ \frac{n(n-1)}{1536} (653n^6 - 1559n^5 + 3035n^4 - 2805n^3 + 1352n^2 \\
&\qquad - 276n - 1440) D_1(x) \\
&- \frac{n(n-1)}{1536} (804n^6 - 1913n^5 - 1049n^4 + 5353n^3 - 1215n^2 \\
&\qquad + 744n - 864) D_n(x).
\end{aligned} \tag{4.11}
$$

Since $l_\nu''(x_\nu) = -n(n-1)/3(1-x_\nu^2)$ (see [5]), a little calculation shows that

$$A_\nu(x) = \frac{3}{4} l_\nu^4(x) + \frac{1}{4} \frac{P_{n-1}(x)}{P_{n-1}(x_\nu)} l_\nu^3(x) - \frac{3}{4} \frac{\pi_n(x) v_\nu(x)}{(1-x_\nu^2) P_{n-1}(x_\nu)}$$

$$(4.12) \quad + \frac{\pi_n^3(x)}{8n^3(n-1)^3(1-x_\nu^2)^2 P_{n-1}^4(x_\nu)} \left[\frac{2}{5} \left\{ \frac{5x_\nu(3+x_\nu^2)}{1-x_\nu^2} + \frac{n(n-1)}{P_{n-1}(x_\nu)} \right\} P_{n-1}(x) \right.$$

$$\left. + \frac{1}{3} \left\{ \frac{10(1+3x_\nu^2)}{1-x_\nu^2} + n(n-1) \right\} \right]$$

$$+ \frac{3n(n-1)}{(1-x_\nu^2)^2} \left\{ -(n-2)(n+1) + \frac{8x_\nu^2}{1-x_\nu^2} \right\} D_\nu(x).$$

5. We shall use some well-known facts about Legendre polynomials. For $-1 \leqq x \leqq 1$ we have [3]

$$(5.1) \qquad |\pi_n(x)| \leqq (2n/\pi)^{1/2},$$

$$(5.2) \qquad |P_m(x)| \leqq 1,$$

the equation

$$(5.3) \qquad (1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$$

and for the polynomials $l_j(x)$

$$(5.4) \qquad l_j^2(x) \leqq \sum_{\nu=1}^{n} |l_\nu^2(x)| \leqq 1, \qquad (-1 \leqq x \leqq 1, j = 1, 2, \cdots, n).$$

We shall also use Markov's and S. Bernstein's inequalities according to which, for a polynomial $g(x)$ of degree $\leqq m$ and real coefficients, we have for $-1 \leqq x \leqq 1$

$$(5.5) \qquad |g'(x)| \leqq m^2 \max_{-1 \leqq x \leqq 1} |g(x)|,$$

and

$$(5.6) \qquad |g'(x)| \leqq m/(1-x^2)^{1/2} \max_{-1 \leqq x \leqq 1} |g(x)|$$

respectively.

The following results have been proved by Turán and Balázs [3].

For $n = 2, 4, \cdots, \nu = 2, 3, 4, \cdots, n/2$.

$$(5.7) \qquad |P_{n-1}(x_\nu)| \geqq 1/(8\nu\pi)^{1/2},$$

$$(5.8) \qquad (1-x_\nu^2) > \nu^2/4(n-1)^2.$$

We shall further make use of the estimates of $\rho_\nu(x)$ and $r_\nu(x)$ as obtained by Turán and Balázs [3].

For $n=4, 6, \cdots$, and $-1 \leqq x \leqq 1$ we have

$$(5.9)\quad |r_1(x)| < \frac{n}{4} + 2n^{1/2}, \qquad |r_n(x)| < \frac{n}{4} + 2n^{1/2},$$

$$(5.10)\quad |r_\nu(x)| \leqq 87\pi\left(\frac{n}{\nu}\right)^{1/2}, \qquad \text{for } 2 \leqq \nu \leqq \frac{n}{2},$$

$$(5.11)\quad |r_\nu(x)| \leqq 87\pi\left(\frac{n}{n-\nu}\right)^{1/2}, \qquad \text{for } \frac{n}{2} + 1 \leqq \nu \leqq n-1,$$

$$(5.12)\quad |\rho_\nu(x)| \leqq (8\pi)^{1/2}\frac{|l_\nu(x)|(1-x_\nu^2)}{n(n-1)}\nu^{1/2} + 128\pi\left(\frac{\nu}{n}\right)^{3/2}\frac{1}{(n-1)^2}, \qquad \text{for } 2 \leqq \nu \leqq \frac{n}{2},$$

$$(5.13)\quad |\rho_\nu(x)| \leqq (8\pi)^{1/2}\frac{|l_\nu(x)|(1-x_\nu^2)(n-\nu)^{1/2}}{n(n-1)} + 128\pi\left(\frac{n-\nu}{n}\right)^{3/2}\frac{1}{(n-1)^2}, \qquad \text{for } n/2 + 1 \leqq \nu \leqq n-1.$$

Lastly we shall require the following results which have been obtained by Dr. A. Sharma and myself [6]:

$$(5.14)\quad |w_1(x)| \leqq \frac{1}{n^2(n-1)^3}, \qquad |w_n(x)| \leqq \frac{1}{n^2(n-1)^3},$$

$$(5.15)\quad |w_\nu(x)| \leqq \frac{8(2\pi)^{1/2}}{3}\cdot\frac{(1-x_\nu^2)|l_\nu(x)|}{n^{3/2}(n-1)^2}\cdot\nu + 256\pi\left(\frac{\nu}{n}\right)^2\frac{1}{(n-1)^3}, \qquad \text{for } 2 \leqq \nu \leqq n/2,$$

and

$$(5.16)\quad |\omega_\nu(x)| \leqq \frac{8(2\pi)^{1/2}}{3}\cdot\frac{(1-x_\nu^2)|l_\nu(x)|}{n^{3/2}(n-1)^2}(n-\nu) + 256\pi\left(\frac{n-\nu}{n}\right)^2\frac{1}{(n-1)^3}, \qquad \text{for } n/2 + 1 \leqq \nu \leqq n-1.$$

$$(5.17)\quad |u_1(x)| < 3n^{1/2} + 2n, \qquad |u_n(x)| < 3n^{1/2} + 2n,$$

$$(5.18)\quad |u_\nu(x)| \leqq \frac{128(2\pi)^{1/2}}{3}n^{1/2}\cdot\frac{|l_\nu(x)|}{\nu} + 2^{13}\pi\frac{n}{\nu^2} + \frac{2(2\pi\nu)^{1/2}}{3}l_\nu^2(x) + \frac{2}{3}|l_\nu^3(x)|, \qquad \text{for } 2 \leqq \nu \leqq n/2,$$

and

$$(5.19)\quad \begin{aligned} |u_\nu(x)| \leqq \frac{128(2\pi)^{1/2}}{3} n^{1/2} \frac{|l_\nu(x)|}{n-\nu} + 2^{13} \frac{\pi n}{(n-\nu)^2} + \frac{2(2\pi)^{1/2}}{3} (n-\nu)^{1/2} l_\nu^2(x) \\ + \frac{2}{3} |l_\nu^3(x)|, \qquad \text{for } n/2+1 \leqq \nu \leqq n-1, \end{aligned}$$

$$(5.20)\quad |v_1(x)| < 4/n; \qquad |v_n(x)| < 4/n,$$

$$(5.21)\quad \begin{aligned} |v_\nu(x)| \leqq \frac{981\pi}{n} + \frac{8(2\pi)^{1/2}}{3n^{1/2}(n-1)} \cdot \nu |l_\nu(x)| + \frac{64(2\pi)^{1/2}}{3n\nu^{1/2}}, \\ \text{for } 2 \leqq \nu \leqq n/2, \end{aligned}$$

$$(5.22)\quad \begin{aligned} |v_\nu(x)| \leqq \frac{981\pi}{n} + \frac{8(2\pi)^{1/2}}{3n^{1/2}(n-1)} (n-\nu) |l_\nu(x)| + \frac{64(2\pi)^{1/2}}{3n(n-\nu)^{1/2}}, \\ \text{for } n/2+1 \leqq \nu \leqq n-1. \end{aligned}$$

6. **Investigation of the fundamental polynomials of the fourth kind.** In this section we shall estimate the quantity

$$\sum_{\nu=1}^{n} |D_\nu(x)|.$$

For $n=4, 6, 8, \cdots$ and $-1 \leqq x \leqq 1$,

$$(6.1)\quad |D_1(x)| \leqq \frac{1}{n^{5/2}(n-1)^4}, \qquad |D_n(x)| \leqq \frac{1}{n^{5/2}(n-1)^4}.$$

For the further fundamental polynomials we have

LEMMA 6.1. *For $n=4, 6, 8, \cdots$ and $-1 \leqq x \leqq 1$ we have*

$$(6.2)\quad \begin{aligned} |D_\nu(x)| \leqq \frac{4(8\pi)^{1/2}}{3} \cdot \frac{|l_\nu(x)| (1-x_\nu^2)\nu^{3/2}}{n^2(n-1)^3} + 128\pi \left(\frac{\nu}{n}\right)^{5/2} \frac{1}{(n-1)^4}, \\ \text{for } 2 \leqq \nu \leqq n/2, \end{aligned}$$

$$(6.3)\quad \begin{aligned} |D_\nu(x)| \leqq \frac{4(8\pi)^{1/2}}{3} \cdot \frac{|l_\nu(x)| (1-x_\nu^2)(n-\nu)^{3/2}}{n^2(n-1)^3} \\ + 128\pi \left(\frac{n-\nu}{n}\right)^{5/2} \frac{1}{(n-1)^4}, \qquad \text{for } n/2+1 \leqq \nu \leqq n-1. \end{aligned}$$

Proof. Obviously it suffices to prove the first assertion. First let $x < x_\nu < 1$. Since

$$\left| \int_{-1}^{x} \frac{P'_{n-1}(t)}{t-x_\nu} dt \right| = \frac{P_{n-1}(x)}{x-x_\nu} - \frac{1}{1+x_\nu} + \int_{-1}^{x} \frac{P_{n-1}(t)}{(t-x_\nu)^2} dt,$$

we have owing to (5.2)

$$\left|\int_{-1}^{x} \frac{P'_{n-1}(t)}{t-x_\nu}\,dt\right| \leqq \frac{2}{x_\nu - x} + \frac{1}{1+x_\nu} < 2\left(\frac{1}{x_\nu - x} + \frac{1}{1-x_\nu^2}\right).$$

Then (4.2) gives, owing to (3.9a) and (5.2),

$$\begin{aligned} |D_\nu(x)| \leqq{}& \frac{1}{12}\cdot\frac{\pi_n^2(x)}{\pi_n'^2(x_\nu)}\cdot\frac{|l_\nu(x)|}{|P''_{n-1}(x_\nu)|} + \frac{1}{4}\cdot\frac{|\pi_n^3(x)|}{|\pi_n'^3(x_\nu)|\,|P''_{n-1}(x_\nu)|} \\ &\cdot\frac{1}{(1-x_\nu^2)\,|P_{n-1}(x_\nu)|}. \end{aligned} \tag{6.4}$$

Since from (5.3) $(1-x_\nu^2)P''_{n-1}(x_\nu) = -n(n-1)P_{n-1}(x_\nu)$ and from (1.3)

$$\pi_n'(x_\nu) = -n(n-1)P_{n-1}(x_\nu),$$

we have from (6.4), owing to (5.1) and (5.7)

$$|D_\nu(x)| \leqq \frac{4(8\pi)^{1/2}}{3}\cdot\frac{|l_\nu(x)|\,(1-x_\nu^2)}{n^2(n-1)^3}\nu^{3/2} + 128\pi\left(\frac{\nu}{n}\right)^{5/2}\cdot\frac{1}{(n-1)^4}.$$

For $-1 < x_\nu < x$ the estimate runs similarly, since we have again

$$\left|\int_{x}^{1} \frac{P'_{n-1}(t)}{t-x_\nu}\,dt\right| < 2\left\{\frac{1}{x-x_\nu} + \frac{1}{1-x_\nu^2}\right\}.$$

For $x = x_\nu$ the lemma obviously holds. From Lemma 6.1, we can deduce

LEMMA 6.2. *We have for* $n = 4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$ *the estimation*

$$\sum_{\nu=1}^{n} |D_\nu(x)| \leqq \frac{97\pi}{n^3}.$$

Proof. From (6.1) and Lemma 6.1 we obtain

$$\begin{aligned} \sum_{\nu=1}^{n} |D_\nu(x)| \leqq{}& \frac{2}{n^{5/2}(n-1)^4} + \frac{4(8\pi)^{1/2}}{3}\cdot\frac{1}{n^2(n-1)^3} \\ &\cdot\left[\sum_{\nu=1}^{n/2}\nu^{3/2}|l_\nu(x)| + \sum_{\nu=n/2+1}^{n-1}(n-\nu)^{3/2}|l_\nu(x)|\right] + \frac{80\pi}{n^3} \\ <{}& \frac{81\pi}{n^3} + \frac{8(8\pi)^{1/2}}{3}\cdot\frac{1}{n^2(n-1)^3}\left(\sum_{\nu=1}^{n/2}\nu^3\right)^{1/2}\left(\sum_{\nu=1}^{n} l_\nu^2(x)\right)^{1/2} \\ <{}& \frac{81\pi}{n^3} + \frac{16\pi}{n^3} = \frac{97\pi}{n^3}. \end{aligned}$$

7. Investigation of the fundamental polynomials of the third kind. We have the following

LEMMA 7.1. *For* $n=4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$ *the estimation*

$$|C_1(x)| < 1/(n-1)^2, \qquad |C_n(x)| < 1/(n-1)^2 \tag{7.1}$$

hold.

Proof. It is enough to consider $C_1(x)$ only. Knowing the estimates of $r_1(x)$ (5.9), we have from (4.4), using (5.1) and (5.2)

$$|C_1(x)| \leqq \frac{1}{\pi n(n-1)^2}\left(\frac{n}{4} + 2n^{1/2} + \left(\frac{2n}{\pi}\right)^{1/2}\right) < \frac{1}{(n-1)^2}\cdot$$

Further we have

LEMMA 7.2. *For* $n=4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$ *we have*

$$|C_\nu(x)| \leqq \frac{8}{n(n-1)^2}\left\{230\pi\cdot(n\nu)^{1/2} + \frac{2(8\pi)^{1/2}}{3}\cdot|l_\nu(x)|\,\nu^{3/2}\right\} \text{ for } 2 \leqq \nu \leqq n/2$$

and

$$|C_\nu(x)| \leqq \frac{8}{n(n-1)^2}\left\{230\pi(n(n-\nu))^{1/2} + \frac{2(8\pi)^{1/2}}{3}\,|l_\nu(x)|\,(n-\nu)^{3/2}\right\}$$

$$\text{for } n/2+1 \leqq \nu \leqq n-1.$$

Proof. We confine ourselves to the case $2 \leqq \nu \leqq n/2$. From (4.6), using the estimates of $r_\nu(x)$ and $\rho_\nu(x)$ in (5.10) and (5.12), and also using (5.1), (5.2), (5.7) and (5.8) we have

$$\begin{aligned}
|C_\nu(x)| &\leqq \frac{\pi_n^2(x)}{2n^2(n-1)^2P_{n-1}^2(x_\nu)}\left[|r_\nu(x)| + |\rho_\nu(x)|\,\frac{2n(n-1)}{3(1-x_\nu^2)}\right. \\
&\qquad\qquad \left. + \frac{2}{3}\,\frac{\pi_n(x)}{n(n-1)(1-x_\nu^2)\,|P_{n-1}^3(x_\nu)|}\right] \\
&\leqq \frac{8\nu}{n(n-1)^2}\left\{87\pi\left(\frac{n}{\nu}\right)^{1/2} + \frac{2(8\pi)^{1/2}}{3}\,|l_\nu(x)|\,\nu^{1/2}\right. \\
&\qquad\qquad \left. + 107\,\frac{\nu^{3/2}}{n^{1/2}(n-1)(1-x_\nu^2)}\right\} \\
&< \frac{8}{n(n-1)^2}\left\{230\pi(n\nu)^{1/2} + \frac{2(8\pi)^{1/2}}{3}\,|l_\nu(x)|\,\nu^{3/2}\right\}.
\end{aligned}$$

From Lemma 7.1 and Lemma 7.2 it follows

LEMMA 7.3. *For* $n=4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$ *we have the estimation*

$$\sum_{\nu=1}^{n} | C_\nu(x) | \leqq \frac{9368}{n}.$$

Proof.

$$\sum_{\nu=1}^{n} | C_\nu(x) | \leqq \frac{2}{(n-1)^2} + \frac{1840\pi}{n^{1/2}(n-1)^2}\left(\sum_{\nu=2}^{n/2} \nu^{1/2} + \sum_{\nu=n/2+1}^{n-1} (n-\nu)^{1/2}\right)$$

$$+ \frac{16(8\pi)^{1/2}}{3n(n-1)^2}\left(\sum_{\nu=2}^{n/2} | l_\nu(x) | \nu^{3/2} + \sum_{\nu=n/2+1}^{n-1} |l_\nu(x)|(n-\nu)^{3/2}\right)$$

$$< \frac{2}{(n-1)^2} + \frac{3680\pi}{n^{1/2}(n-1)^2}\sum_{\nu=1}^{n/2} \nu^{1/2} + \frac{32(8\pi)^{1/2}}{3n(n-1)^2}\left(\sum_{\nu=1}^{n/2} \nu^3\right)^{1/2}\left(\sum_{\nu=1}^{n} l_\nu^2(x)\right)^{1/2}$$

$$< \frac{2}{(n-1)^2} + \frac{1840\pi}{2^{1/2}n(n-1)^2}\cdot\frac{n}{(n-1)^2} + \frac{4(8\pi)^{1/2}}{3}\cdot\frac{n+2}{(n-1)^2} < \frac{9368}{n}.$$

8. **Investigation of the fundamental polynomials of the second kind.** In this section we shall estimate the quantity

$$\sum_{\nu=1}^{n} | B_\nu(x) | .$$

We need

LEMMA 8.1. *For* $n=4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$

$$| B_1(x) | \leqq 4; \qquad | B_n(x) | \leqq 4.$$

Proof. From (4.7), (5.1), (5.14), (5.17), and (7.1)

$$|B_1(x)| \leqq \left(\frac{2n}{\pi}\right)^{1/2}\cdot\frac{1}{n(n-1)}\left\{3n^{1/2} + 2n + \frac{5}{12}\cdot\frac{n^6}{n^2(n-1)^3}\right\} + \frac{n}{2(n-1)}$$

$$< \frac{3}{n^{1/2}(n-1)}(n^{1/2}+n) + \frac{n}{2(n-1)} < 4.$$

Similarly $|B_n(x)| \leqq 4$. For the further fundamental polynomials we have

LEMMA 8.2. *For* $n=4, 6, 8, \cdots, -1 \leqq x \leqq 1$ *we have*

$$| B_\nu(x) | \leqq \frac{4}{n^{1/2}(n-1)}\left[160 | l_\nu(x) | \left(\frac{n}{\nu}\right)^{1/2} + k_{19}\frac{n}{\nu^{3/2}} + 2\nu\cdot l_\nu^2(x) + \nu^{1/2}\right],$$

$$\text{for } 2 \leqq \nu \leqq n/2,$$

and

$$|B_\nu(x)| \leqq \frac{4}{n^{1/2}(n-1)}\left[160\,|l_\nu(x)|\left(\frac{n}{n-\nu}\right)^{1/2} + k_{19}\frac{n}{(n-\nu)^{3/2}}\right.$$
$$\left. + 2(n-\nu)l_\nu^2(x) + (n-\nu)^{1/2}\right], \qquad \text{for } n/2+1 \leqq \nu \leqq n-1.$$

Proof. Again we confine ourselves to prove the first assertion. From (4.9) using (5.15), (5.18) we have for $2 \leqq \nu \leqq n/2$

$$|B_\nu(x)| \leqq \frac{|\pi_n(x)|}{n(n-1)\,|P_{n-1}(x_\nu)|}\left[\frac{128(2\pi)^{1/2}}{3}n^{1/2}\frac{|l_\nu(x)|}{\nu} + \frac{2^{13}\pi n}{\nu^2}\right.$$
$$+ \frac{2(2\pi)^{1/2}}{3}\nu^{1/2}\cdot l_\nu^2(x) + \frac{2}{3}\,|l_\nu^3(x)| + \frac{8(2\pi)^{1/2}}{3}\cdot\frac{|l_\nu(x)|\,\nu}{n^{1/2}(n-1)(1-x_\nu^2)}$$
$$+ \frac{256\pi\nu^2}{n(n-1)^2(1-x_\nu^2)^2} + \frac{\pi_n^2(x)}{180(1-x_\nu^2)n^2(n-1)^2\,|P_{n-1}^3(x_\nu)|}$$
$$\left.\cdot\left\{\frac{10}{1-x_\nu^2} + \frac{|P_{n-1}(x)|}{|P_{n-1}(x_\nu)|}\left(12 + \frac{3}{1-x_\nu^2}\right)\right\}\right].$$

Then using (5.1), (5.2), (5.4), (5.7) and (5.8) we have

$$|B_\nu(x)| \leqq \frac{4\nu^{1/2}}{n^{1/2}(n-1)}\left[\frac{128(2\pi)^{1/2}}{3}\cdot\frac{|l_\nu(x)|}{\nu}\cdot n^{1/2} + \frac{2^{13}\pi n}{\nu^2} + \frac{2(2\pi)^{1/2}}{3}\nu^{1/2}l_\nu^2(x)\right.$$
$$+ \frac{2}{3}\,|l_\nu^3(x)| + \frac{32(2\pi)^{1/2}}{3}\cdot\frac{|l_\nu(x)|\,(n-1)}{\nu n^{1/2}}$$
$$\left. + 2^{12}\cdot\frac{\pi(n-1)^2}{n\nu^2} + \frac{10.2^8}{9}\pi\frac{(n-1)^2}{\nu^2}\right]$$
$$\leqq \frac{4\nu^{1/2}}{n^{1/2}(n-1)}\left[1 + 160n^{1/2}\cdot\frac{|l_\nu(x)|}{\nu} + k_{19}\cdot\frac{n}{\nu^2} + 2\nu^{1/2}l_\nu^2(x)\right],$$

since

$$\frac{10}{1-x_\nu^2} + \frac{|P_{n-1}(x)|}{|P_{n-1}(x_\nu)|}\left(12 + \frac{3}{1-x_\nu^2}\right) \leqq \frac{25}{(1-x_\nu^2)\,|P_{n-1}(x_\nu)|}\,.$$

From Lemma 8.1 and Lemma 8.2 it follows

LEMMA 8.3. *For $n=4, 6, 8, \cdots$ and $-1 \leqq x \leqq 1$ we have the estimation*

$$\sum_{\nu=1}^{n} |B_\nu(x)| \leqq k_{23}.$$

Proof. From Lemma 8.1 and Lemma 8.2.

$$\sum_{\nu=1}^{n} |B_\nu(x)| \leqq 8 + \sum_{\nu=2}^{n-1} |B_\nu(x)| \leqq 8 + \frac{640}{n-1}\left[\sum_{\nu=2}^{n/2} \frac{|l_\nu(x)|}{\nu^{1/2}} + \sum_{\nu=n/2+1}^{n-1} \frac{|l_\nu(x)|}{(n-\nu)^{1/2}}\right]$$

$$+ \frac{4n^{1/2}}{n-1} k_{19}\left[\sum_{\nu=2}^{n/2} \nu^{-3/2} + \sum_{\nu=n/2+1}^{n-1} (n-\nu)^{-3/2}\right]$$

$$+ \frac{8}{n^{1/2}(n-1)}\left[\sum_{\nu=2}^{n/2} \nu l_\nu^2(x) + \sum_{\nu=n/2+1}^{n-1} (n-\nu) l_\nu^2(x)\right]$$

$$+ \frac{4}{n^{1/2}(n-1)}\left[\sum_{\nu=2}^{n/2} \nu^{1/2} + \sum_{\nu=n/2+1}^{n-1} (n-\nu)^{1/2}\right]$$

$$< 8 + \frac{1280}{n-1}\left(\sum_{\nu=1}^{n/2} \frac{1}{\nu}\right)^{1/2}\left(\sum_{\nu=1}^{n} l_\nu^2(x)\right)^{1/2}$$

$$+ \frac{k_{20}}{n^{1/2}} + \frac{16}{n^{1/2}(n-1)}\left(\sum_{\nu=1}^{n/2} \nu^2\right)^{1/2}\left(\sum_{\nu=1}^{n} l_\nu^4(x)\right)^{1/2} + \frac{8}{n-1}\left(\sum_{\nu=1}^{n/2} \nu\right)^{1/2}$$

$$< k_{21}\frac{\log n}{n} + \frac{k_{20}}{n^{1/2}} + k_{22} < k_{23}.$$

9. **Investigation of the fundamental polynomials of the first kind.** Here we shall estimate the quantity

$$\sum_{\nu=1}^{n} |A_\nu(x)|.$$

We prove

LEMMA 9.1. *For $n=4, 6, 8, \cdots$ and $-1 \leqq x \leqq 1$ we have*

$$|A_1(x)| \leqq 16n^{3/2}; \qquad |A_n(x)| \leqq 16n^{3/2}.$$

Proof. From (4.10), (5.20), (5.17), (6.1), (5.1), (5.2) and (5.4) we at once get

$$|A_1(x)| \leqq 1 + \frac{1}{16}\left(\frac{2n}{\pi}\right)^{1/2}\left(6n^2\cdot\frac{4}{n} + 17(3n^{1/2} + 2n)\right) + \frac{5n^8}{2n^{5/2}(n-1)^4}$$

$$< 1 + \frac{84n^{3/2}}{16} + 10n^{3/2} < 16n^{3/2}.$$

LEMMA 9.2. *For $n=4, 6, 8, \cdots$ and $-1 \leqq x \leqq 1$ we have*

$$(9.1) \qquad |A_\nu(x)| \leqq k_1 + k_2\nu^{1/2} + k_3\left(\frac{n}{\nu}\right)^{3/2} + k_4 n\frac{|l_\nu(x)|}{\nu^{1/2}}, \quad \text{for } 2 \leqq \nu \leqq \frac{n}{2},$$

and

$$(9.2)\qquad |A_\nu(x)| \leqq k_1 + k_2(n-\nu)^{1/2} + k_3\left(\frac{n}{n-\nu}\right)^{3/2} + k_4 n \frac{|l_\nu(x)|}{(n-\nu)^{1/2}}$$

$$\text{for } n/2+1 \leqq \nu \leqq n-1.$$

Proof. As before we shall prove the first assertion. From (4.12), (6.1), (6.2) etc. we have

$$|A_\nu(x)| \leqq \frac{3}{4} + \frac{1}{4}(8\nu\pi)^{1/2} + \frac{12n^{5/2}}{\nu^{3/2}}|v_\nu(x)|$$

$$+ 32\pi\left(2+\frac{32}{\nu^2}\right)\left[n\cdot\frac{|l_\nu(x)|}{\nu^{1/2}} + 192\left(\frac{n}{\nu}\right)^{3/2}\right]$$

$$+ 256(2\pi)^{1/2}\left[\frac{86}{\nu^4} + \frac{1}{\nu^2} + \frac{2(8\pi)^{1/2}}{5\nu^{3/2}}\right]n^{3/2}.$$

Using (5.21), we get

$$|A_\nu(x)| \leqq k_1 + k_2\nu^{1/2} + k_3\left(\frac{n}{\nu}\right)^{3/2} + k_4 n\cdot\frac{|l_\nu(x)|}{\nu^{1/2}}, \qquad \text{for } 2 \leqq \nu \leqq n/2,$$

with numerical k_1, k_2, k_3 and k_4.

From Lemma 9.1 and Lemma 9.2 it follows

LEMMA 9.3. *For* $n=4, 6, 8, \cdots$ *and* $-1 \leqq x \leqq 1$ *we have*

$$\sum_{\nu=1}^{n} |A_\nu(x)| \leqq k_6 n^{3/2}.$$

Proof.

$$\sum_{\nu=1}^{n} |A_\nu(x)| \leqq 32n^{3/2} + (n-3)k_1 + k_2\left\{\sum_{\nu=2}^{n/2}\nu^{1/2} + \sum_{\nu=n/2+1}^{n-1}(n-\nu)^{1/2}\right\}$$

$$+ k_3\left[\sum_{\nu=2}^{n/2}\left(\frac{n}{\nu}\right)^{3/2} + \sum_{\nu=n/2+1}^{n-1}\left(\frac{n}{n-\nu}\right)^{3/2}\right]$$

$$+ k_4\cdot n\left[\sum_{\nu=2}^{n/2}\frac{|l_\nu(x)|}{\nu^{1/2}} + \sum_{\nu=n/2+1}^{n-1}\frac{|l_\nu(x)|}{(n-\nu)^{1/2}}\right]$$

$$\leqq k_5\cdot n^{3/2} + k_4\cdot 2n\left(\sum_{\nu=1}^{n/2}\frac{1}{\nu}\right)^{1/2}\left(\sum_{\nu=1}^{n} l_\nu^2(x)\right)^{1/2} < k_6 n^{3/2}.$$

10. **Lemmas on Jackson means.** Let $\phi(\theta)$ have the period 2π, then the Jackson means of $\phi(\theta)$ is

$$(10.1)\qquad J_n(\theta, \phi) = \frac{3}{2\pi n(2n^2+1)} \int_{-\pi}^{\pi} \phi(t) \left[\frac{\sin n \dfrac{t-\theta}{2}}{\sin \dfrac{t-\theta}{2}} \right]^4 dt.$$

An alternative form of $J_n(\theta, \phi)$ is well-known:

$$(10.2)\qquad J_n(\theta, \phi) = \frac{3}{\pi n(2n^2+1)} \int_0^{\pi/2} \{\phi(\theta+2t) + \phi(\theta-2t)\} \left(\frac{\sin nt}{\sin t}\right)^4 dt.$$

From (10.2) it follows

$$(10.3)\qquad 1 = \frac{6}{\pi n(2n^2+1)} \int_0^{\pi/2} \left(\frac{\sin nt}{\sin t}\right)^4 dt.$$

Under the hypothesis in our theorem, if $\phi(\theta)$ is continuously derivable in $[-1, 1]$, and $\omega_1(\delta)$, the modulus of continuity of $\phi'(\theta)$ satisfying the condition that

$$(10.4)\qquad \int_0 \frac{\omega_1(t)}{t^{3/2}}\, dt$$

exists, then we can prove the following modification of Balázs-Turán's Lemma (5.1) in [3].

LEMMA 10.1 *If $\phi(\theta)$ is everywhere continuously derivable and satisfies the condition* (10.4) *then for an arbitrary $\epsilon > 0$ we have for $n > n_0(\epsilon)$*

$$|\phi(\theta) - J_n(\theta, \phi)| \leqq \epsilon/n^{3/2}.$$

From this we at once have

LEMMA 10.2. *Let*

$$\Phi(\theta) = \int_0^{\theta} \phi(\theta) \sin \theta d\theta,$$

then

$$\left|\Phi(\theta) - \int_0^{\theta} J_n(\theta, \phi) \sin \theta d\theta\right| < (\epsilon/n^{3/2})(1 - \cos\theta).$$

Now let $f(x)$ be continuously derivable of order 2 in $[-1, 1]$ and

$$(10.4)\qquad \phi(\theta) = f'(\cos\theta).$$

Let $\omega(\delta)$ be the continuity module of $f''(x)$ for $-1 \leqq x \leqq 1$ with existing

$$\int_0 \frac{\omega(\vartheta)}{v^{3/2}}\, d\vartheta.$$

If $\max_{-1\leq x\leq 1}|f''(x)|=M$ then denoting the continuity module of $\phi'(\theta)$ by $\omega_1(\delta)$ we have

$$\begin{aligned}\omega_1(\delta) &= \max_{|\theta''-\theta'|\leq\delta}\left|\left[\frac{d}{d\theta}\phi(\theta)\right]_{\theta=\theta''}-\left[\frac{d}{d\theta}\phi(\theta)\right]_{\theta=\theta'}\right| \\ &= \max_{|\theta''-\theta'|\leq\delta}\left|\left[\frac{d}{d\theta}f'(\cos\theta)\right]_{\theta=\theta''}-\left[\frac{d}{d\theta}f'(\cos\theta)\right]_{\theta=\theta'}\right| \\ &= \max_{|\theta''-\theta'|\leq\delta}\left|\left[\frac{d^2f(x)}{dx^2}(1-x^2)^{1/2}\right]_{x=\cos\theta'}-\left[\frac{d^2f(x)}{dx^2}(1-x^2)^{1/2}\right]_{x=\cos\theta''}\right| \\ &\leq \omega(\delta)+M\delta,\end{aligned}$$

i.e., the integral

$$\int_0\frac{\omega(\vartheta)}{\vartheta^{3/2}}d\vartheta \tag{10.5}$$

exists.

We consider the polynomials $J_n(\theta,\phi)$ belonging to $\phi(\theta)=f'(\cos\theta)$. Since $\phi(\theta)$ is now even, $J_n(\theta,\phi)$ is a pure cosine polynomial of order $2n-2$ i.e., $J_n(\text{arc}\cos x,\phi)=\pi_{2n-2}(x)$ is a rational polynomial of degree $(2n-2)$ and

$$\begin{aligned}\pi_{2n-2}(x) = \frac{3}{\pi n(2n^2+1)}\int_0^{\pi/2}\{&\phi(\text{arc}\cos x+2t) \\ &+\phi(\text{arc}\cos x-2t)\}\left(\frac{\sin nt}{\sin t}\right)^4dt.\end{aligned} \tag{10.6}$$

Lemma 10.1 gives for $-1\leq x\leq 1$, arbitrary small $\epsilon>0$ and $n>n_1(\epsilon)$

$$|\pi_{2n-2}(x)-f'(x)| = \epsilon/n^{3/2}. \tag{10.7}$$

Putting

$$P_{2n-1}(x) = \int_1^x\pi_{2n-2}(x)dx. \tag{10.8}$$

Lemma 10.2 at once gives for $-1\leq x\leq 1$, arbitrary small $\epsilon>0$ and $n>n_2(\epsilon)$

$$|P_{2n-1}(x)-f(x)+f(1)| < 2\epsilon/n^{3/2}. \tag{10.9}$$

Differentiating (10.6) one times according to x we get

$$\begin{aligned}\pi'_{2n-2}(x) = -\frac{3}{\pi n(2n^2+1)(1-x^2)^{1/2}}\int_0^{\pi/2}\Bigg\{&\left(\frac{d\phi(u)}{du}\right)_{u=\text{arc}\cos x+2t} \\ &+\left(\frac{d\phi(u)}{du}\right)_{u=\text{arc}\cos x-2t}\Bigg\}\left(\frac{\sin nt}{\sin t}\right)^4dt.\end{aligned} \tag{10.10}$$

From (10.10) we have the

LEMMA 10.3. *For* $-1<x<1$, *we have for* $n>n_3(\epsilon)$ *for the* $\pi_{2n-2}(x)$ *defined in* (10.6) *the estimation*

$$|\pi'_{2n-2}(x)| \leqq \frac{1}{(1-x^2)^{1/2}} \cdot \max_\theta \left| \frac{df'(\cos\theta)}{d\theta} \right| .$$

Proof. Denoting

$$\max_\theta \left| \frac{df'(\cos\theta)}{d\theta} \right| = M_1,$$

we have from (10.10) and (10.3)

$$(10.11) \qquad |\pi'_{2n-2}(x)| \leqq \frac{M_1}{(1-x^2)^{1/2}} \cdot \frac{6}{\pi n(2n^2+1)} \int_0^{\pi/2} \left(\frac{\sin nt}{\sin t}\right)^4 dt = \frac{M_1}{(1-x^2)^{1/2}}.$$

Lemma 10.3 does not give any information for $x=\pm 1$. This is given by

LEMMA 10.4. *We have for the* $\pi_{2n-2}(x)$ *defined in* (10.6) *the estimation*

$$|\pi'_{2n-2}(\pm 1)| \leqq k_{19} n^{1/2}.$$

The proof of this Lemma is obvious from Lemma 5.3 in [3].

The following Lemmas have been proved exactly as by Turán and Balázs [3].

LEMMA 10.5. *For* $-1<x<1$ *we have for* $n>n_4(\epsilon)$ *for* $\pi_{2n-2}(x)$ *defined in* (10.6) *the estimation*

$$|\pi''_{2n-2}(x)| \leqq \frac{1}{(1-x^2)^{3/2}} \cdot \max_\theta \left| \frac{df'(\cos\theta)}{d\theta} \right| + \frac{\epsilon n^{1/2}}{(1-x^2)^{1/2}} .$$

LEMMA 10.6. *We have for the* $\pi_{2n-2}(x)$ *defined in* (10.6) *the estimation*

$$|\pi''_{2n-2}(\pm 1)| \leqq k_7 n^{5/2}.$$

From Lemma 10.5 and using the S. Bernstein's inequality (5.6) we have for $-1<x<1$ for $\pi_{2n-2}(x)$ defined in (10.6) the estimation

$$(10.12) \qquad |\pi'''_{2n-2}(x)| \leqq \frac{2n}{(1-x^2)^2} \cdot \max_\theta \left| \frac{df'(\cos\theta)}{d\theta} \right| + \frac{\epsilon n^{3/2}}{(1-x^2)^{3/2}} .$$

From Lemma 10.6 and using Markov's inequality (5.5) we have

$$(10.13) \qquad |\pi'''_{2n-2}(\pm 1)| \leqq k_8 n^{9/2}.$$

11. **The proof of the theorem.** The lemmas given before very easily prove

our theorem. Owing to the uniqueness theorem in [5] we have for the polynomials $P_{2n-1}(x)$ in (10.8)

$$
\begin{aligned}
P_{2n-1}(x) \equiv & \sum_{\nu=1}^{n} P_{2n-1}(x_{\nu n}) A_{\nu n}(x) + \sum_{\nu=1}^{n} P'_{2n-1}(x_{\nu n}) B_{\nu n}(x) \\
& + \sum_{\nu=1}^{n} P''_{2n-1}(x_{\nu n}) C_{\nu n}(x) + \sum_{\nu=1}^{n} P^{(4)}_{2n-1}(x_{\nu n}) D_{\nu n}(x),
\end{aligned} \tag{11.1}
$$

$$
\begin{aligned}
P_{2n-1}(x) \equiv & \sum_{\nu=1}^{n} P_{2n-1}(x_{\nu n}) A_{\nu n}(x) + \sum_{\nu=1}^{n} \pi_{2n-2}(x_{\nu n}) B_{\nu n}(x) \\
& + \sum_{\nu=1}^{n} \pi'_{2n-2}(x_{\nu n}) C_{\nu n}(x) + \sum_{\nu=1}^{n} \pi'''_{2n-2}(x_{\nu n}) D_{\nu n}(x).
\end{aligned} \tag{11.2}
$$

Hence owing to (11.2), (2.2), (2.4) and the remark that in (2.2)

$$
\sum_{\nu=1}^{n} A_{\nu}(x) = 1 \tag{11.3}
$$

we have for $n > n_6(\epsilon)$

$$
\begin{aligned}
|f(x) - R_n(x,f)| \leqq & \; |f(x) - f(1) - P_{2n-1}(x)| \\
& + \Big| \sum_{\nu=1}^{n} [P_{2n-1}(x_{\nu n}) - f(x_{\nu n}) + f(1)] A_{\nu n}(x) \\
& + \sum_{\nu=1}^{n} [\pi_{2n-2}(x_{\nu n}) - f'(x_{\nu n})] B_{\nu n}(x) \\
& + \sum_{\nu=1}^{n} [\pi'_{2n-2}(x_{\nu n}) - c_{\nu n}] C_{\nu n}(x) \\
& + \sum_{\nu=1}^{n} [\pi'''_{2n-2}(x_{\nu n}) - d_{\nu n}] D_{\nu n}(x) \Big| \\
\leqq & \; |f(x) - f(1) - P_{2n-1}(x)| \\
& + \sum_{\nu=1}^{n} |A_{\nu n}(x)| \max_{-1 \leqq u \leqq 1} |f(u) - P_{2n-1}(u) + f(1)| \\
& + \sum_{\nu=1}^{n} |B^{(x)}_{\nu n}| \max_{-1 \leqq v \leqq 1} |f'(v) - \pi_{2n-2}(v)| + \epsilon n \sum_{\nu=1}^{n} |C_{\nu n}(x)| \\
& + \sum_{\nu=1}^{n} |\pi'_{2n-2}(x_{\nu n})| \, |C_{\nu n}(x)| + \epsilon n^3 \sum_{\nu=1}^{n} |D_{\nu n}(x)| \\
& + \sum_{\nu=1}^{n} |\pi'''_{2n-2}(x_{\nu n})| \, |D_{\nu n}(x)| .
\end{aligned} \tag{11.4}
$$

Using (10.9) and Lemma 9.3 the first two terms tend to 0 uniformly for

$-1 \leqq x \leqq 1$; so does the third term owing to (10.7) and Lemma 8.3. Similarly fourth and sixth terms tend to 0 uniformly in $[-1, 1]$ owing to Lemma 7.3 and Lemma 6.2 respectively. To estimate the fifth and last terms in (11.4)

$$\sum_{\nu=1}^{n} \left| \pi'_{2n-2}(x_{\nu n}) \right| \left| C_{\nu n}(x) \right| \equiv S,$$

$$\sum_{\nu=1}^{n} \left| \pi'''_{2n-2}(x_{\nu n}) \right| \left| D_{\nu n}(x) \right| \equiv T,$$

we write owing to (2.1)

$$\begin{aligned} S = {} & \left| \pi'_{2n-2}(1) \right| \left| C_{1n}(x) \right| + \left| \pi'_{2n-2}(-1) \right| \left| C_{nn}(x) \right| \\ & + \sum_{\nu=2}^{n-1} \left| \pi'_{2n-2}(x_{\nu n}) \right| \left| C_{\nu n}(x) \right| . \end{aligned} \tag{11.5}$$

Lemma 10.4 and (7.1) obviously give for the first two terms in (11.5) the upper bound $k_9 n^{-3/2}$. In order to estimate

$$S_1 = \sum_{\nu=2}^{n-1} \left| \pi'_{2n-2}(x_{\nu n}) \right| \left| C_{\nu n}(x) \right|$$

we use Lemma 10.3 and Lemma 7.2. These give

$$\begin{aligned} \left| S_1 \right| \leqq k_{10} \sum_{\nu=2}^{n-1} \frac{M_1}{n(n-1)^2(1-x_{\nu n}^2)^{1/2}} \left\{ 230\pi(n\nu)^{1/2} + \frac{2(8\pi)^{1/2}}{3} \left| l_\nu(x) \right| \nu^{3/2} \right\} \\ \equiv S_1' + S_1''. \end{aligned}$$

Using (5.8) we have

$$\begin{aligned} S_1' &= k_{10} \cdot \frac{230\pi M_1}{n^{1/2}(n-1)^2} \sum_{\nu=2}^{n-1} \frac{\nu^{1/2}}{(1-x_{\nu n}^2)^{1/2}} \\ &< \frac{k_{11}}{n^{1/2}(n-1)^{1/2}} \sum_{\nu=2}^{n-1} \frac{1}{\nu^{1/2}} < \frac{k_{12}}{n^{1/2}}, \\ S_1'' &= k_{10} \cdot \frac{2(8\pi)^{1/2}}{3} \cdot M_1 \cdot \frac{1}{n(n-1)^2} \sum_{\nu=2}^{n-1} \frac{\left| l_\nu(x) \right| \nu^{3/2}}{(1-x_{\nu n}^2)^{1/2}} \\ &< k_{13} \cdot \frac{1}{n(n-1)} \sum_{\nu=2}^{n-1} \left| l_\nu(x) \right| \nu^{1/2}. \end{aligned} \tag{11.6}$$

The Schwarz's inequality gives

$$S_1'' < k_{13} \cdot \frac{1}{n(n-1)} \left(\sum_{\nu=1}^{n} \nu \right)^{1/2} \left(\sum_{\nu=1}^{n} l_\nu^2(x) \right)^{1/2} < \frac{k_{14}}{n} . \tag{11.7}$$

Similarly

$$T = |\pi'''_{2n-2}(1)|\,|D_{1n}(x)| + |\pi'''_{2n-2}(-1)|\,|D_{nn}(x)| + \sum_{\nu=2}^{n-1} |\pi'''_{2n-2}(x_{\nu n})|\,|D_{\nu n}(x)|. \tag{11.8}$$

(10.13) and (6.1) give for the first two terms in (11.8) the upper bound $k_{15}n^{-2}$. In order to estimate $T_1 = \sum_{\nu=2}^{n-1} |\pi'''_{2n-2}(x_{\nu n})|\,|D_{\nu n}(x)|$ we use (10.12) and Lemma 6.1. These give

$$\begin{aligned} T_1 &\leqq k_{16} \sum_{\nu=2}^{n-1} \left\{ M_1 \cdot \frac{2n}{(1-x_{\nu n}^2)^2} + \frac{\epsilon n^{1/2}}{(1-x_{\nu n}^2)^{3/2}} \right\} \\ &\qquad \cdot \left\{ \frac{|l_{\nu n}(x)|\,(1-x_{\nu n}^2)\nu^{3/2}}{n^2(n-1)^3} + \frac{\nu^{5/2}}{n^{5/2}(n-1)^4} \right\} \\ &= k_{16} \sum_{\nu=2}^{n-1} \left\{ \frac{2M_1}{1-x_{\nu n}^2} + \frac{\epsilon n^{1/2}}{(1-x_{\nu n}^2)^{1/2}} \right\} \frac{|l_{\nu n}(x)|\,\nu^{3/2}}{n(n-1)^3} \\ &\quad + k_{16} \sum_{\nu=2}^{n-1} \left\{ \frac{2M_1}{(1-x_{n\nu}^2)^2} + \frac{\epsilon n^{1/2}}{(1-x_{\nu n}^2)^{3/2}} \right\} \frac{\nu^{5/2}}{n^{3/2}(n-1)^4} \\ &\equiv T_1' + T_1'' + T_1''' + T_1'''', \end{aligned} \tag{11.9}$$

$$\begin{aligned} T_1' &= k_{16} \frac{2M_1}{n(n-1)^3} \sum_{\nu=2}^{n-1} \frac{|l_{\nu n}(x)|\,\nu^{3/2}}{(1-x_{\nu n}^2)} < 8k_{16} \frac{M_1}{n(n-1)} \sum_{\nu=1}^{n} \frac{|l_{\nu n}(x)|}{\nu^{1/2}} \\ &< k_{17} \frac{M_1 \log n}{n^2}. \end{aligned} \tag{11.10}$$

$$\begin{aligned} T_1'' &= k_{16} \frac{\epsilon}{n^{1/2}(n-1)^3} \sum_{\nu=2}^{n-1} \frac{|l_{\nu n}(x)|\,\nu^{3/2}}{(1-x_{\nu n}^2)^{1/2}} \\ &< 4k_{16} \frac{\epsilon}{n^{1/2}(n-1)^2} \sum_{\nu=1}^{n} \nu^{1/2} |l_{\nu n}(x)|, \end{aligned} \tag{11.11}$$

$$T_1''' = k_{16} \frac{2M_1}{n^{3/2}(n-1)^4} \sum_{\nu=2}^{n-1} \frac{\nu^{5/2}}{(1-x_{\nu n}^2)^2} < 16k_{16} \frac{2M_1}{n^{3/2}} \sum_{\nu=2}^{n-1} \frac{1}{\nu^{3/2}} < k_{18} \frac{M_1}{n^{3/2}}, \tag{11.12}$$

$$\begin{aligned} T_1'''' &= k_{16} \frac{\epsilon}{n(n-1)^4} \sum_{\nu=2}^{n-1} \frac{\nu^{5/2}}{(1-x_{\nu n}^2)^{3/2}} \\ &< 8k_{16} \frac{\epsilon}{n(n-1)} \sum_{\nu=2}^{n-1} \frac{1}{\nu^{1/2}} < k_{19} \cdot \frac{\epsilon}{n}. \end{aligned} \tag{11.13}$$

(11.4) to (11.13) complete the proof of the theorem.

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