ORTHONORMAL SETS WITH NON-NEGATIVE DIRICHLET KERNELS

BY

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1. **Introduction.** With any orthonormal system $\{f_n(s)\}_{n=0}^{\infty}$ of real-valued functions on a measure space S are associated the Dirichlet kernels

$$D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t), \qquad n \geqq 1.$$

If these kernels happen to be non-negative, the Lebesgue functions of the system, i.e. the functions

$$L_n(s) = \int_S | D_n(s, t) | dt, \qquad n \geqq 1,$$

are much more accessible than otherwise. In particular, if $S=[0, 1]$ and $f_0(s)\equiv 1$, it is immediate that $L_n(s)\equiv 1$ for all n. This is exactly the situation one finds in the case of the classical Haar functions (see [1] or [3]). The uniform boundedness of the functions $L_n(s)$ leads to a proof that the expansion of a continuous function $g(x)$ on $[0, 1]$ in terms of Haar functions converges uniformly to $g(x)$.

It seems natural to ask whether there are other orthonormal sets with the property that $D_n(s, t)\geqq 0$ for all n. We shall prove that the answer to this question is essentially negative. It will be shown that only the Haar functions and certain minor modifications of them possess non-negative Dirichlet kernels.

2. **Definitions.** It will be assumed throughout, except in §5, that μ is a totally finite measure(1) on a space S normalized so that $\mu(S)=1$.

Using the terminology of [2], a *partition* of S is a finite set P of disjoint subsets of S whose union is S. Given two partitions P_1 and P_2, $P_1\geqq P_2$ if each element of P_2 is contained in an element of P_1.

Consider a sequence of partitions of S having the following two properties.

(i) $P_0\geqq P_1\geqq P_2\geqq \cdots$.

(ii) For each $n\geqq 0$, $P_n=\{S_{n,1}, S_{n,2}, \cdots, S_{n,n+1}\}$ where all sets have positive μ-measure and $S_{0,1}=S$.

These conditions amount to assuming that $P_0=\{S\}$ and P_{n+1} is obtained from P_n by splitting one of the sets $S_{n,j}$, say for $j=k(n)$, into two subsets of

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(1) See [2] for measure-theoretic definitions.

positive measure. The enumeration can always be arranged so that $S_{n,k(n)}$ splits to form $S_{n+1,1}$ and $S_{n+1,2}$. The sets $S_{n,j}$ ($1 \leqq j \leqq n+1$, $j \neq k(n)$) are relabeled and become the sets $S_{n+1,i}$ ($3 \leqq i \leqq n+2$). To avoid cumbersome subscripts, we shall use k instead of $k(n)$.

With such a sequence of partitions, we associate an orthonormal set $\{f_n(s)\}_{n=0}^{\infty}$ in $L^2(S, \mu)$ as follows. Put $f_0(s) \equiv 1$. Let $\mu(S_{n,j}) = \mu_{n,j}$ and define for $n \geqq 1$

$$
f_n(s) = \begin{cases} \left(\dfrac{\mu_{n,2}}{\mu_{n,1}\mu_{n-1,k}}\right)^{1/2}, & s \in S_{n,1}, \\ -\left(\dfrac{\mu_{n,1}}{\mu_{n,2}\mu_{n-1,k}}\right)^{1/2}, & s \in S_{n,2}, \\ 0, & \text{otherwise}. \end{cases} \tag{1}
$$

A set of functions defined in this way will be called a *Haar system on* (S, μ). It is easy to verify that a Haar system is an orthonormal set in $L^2(S, \mu)$.

These are obvious generalizations of the orthonormal sets introduced by Haar in [1]. There, S is the unit interval, μ is Lebesgue measure, and the $S_{n,j}$ are sub-intervals. That set in which $\mu_{n,1} = \mu_{n,2}$ ($n \geqq 1$) is the one usually called "the Haar functions." (Strictly speaking, the above definition may yield only a subset of the Haar functions. In order to insure completeness, some additional assumption is needed. For instance if μ is nonatomic, one might require that $\lim_{n\to\infty} \max_j \mu_{n,j} = 0$.)

3. **Main theorem.** For the sake of simplicity, we are going to avoid "almost everywhere" statements. We shall tacitly identify two functions which differ on a set of measure zero. In this way, for example, an essentially bounded function will be considered bounded.

THEOREM 1. *Let $H = \{f_n(s)\}_{n=0}^{\infty}$ be a real orthonormal set in $L^2(S, \mu)$ with $f_0(s) \equiv 1$. The Dirichlet kernels associated with H are non-negative if and only if H is a Haar system on (S, μ).*

The following lemma is needed for the proof of this theorem.

LEMMA. *Let T be a subset of S having measure μ_T, and let c be a positive constant. Define $\mathfrak{F}$ to be the class of functions $f \in L^2(T, \mu)$ satisfying*

$$\int_T f\,d\mu = 0 \tag{A}$$

and

$$f(s)f(t) \geqq -c, \qquad (s, t) \in T \times T. \tag{B}$$

Then

$$\int_T f^2\,d\mu \leqq c\mu_T, \qquad f \in \mathfrak{F}. \tag{C}$$

The equality in (C) *holds for* $f^* \in \mathfrak{F}$ *if and only if there are two complementary subsets* T_1 *and* T_2 *of* T *with measures* μ_1 *and* μ_2 *such that*

$$f^*(s) = \begin{cases} \left(\dfrac{c\mu_2}{\mu_1}\right)^{1/2}, & s \in T_1, \\ -\left(\dfrac{c\mu_1}{\mu_2}\right)^{1/2}, & s \in T_2. \end{cases} \tag{D}$$

Proof. Given $f \in \mathfrak{F}$, $f = f_1 - f_2$ where $f_1(s) = \max \{f(s), 0\}$ and $f_2(s) = \max \{-f(s), 0\}$. For $i=1, 2$, let $T_i = \{s \mid f_i(s) > 0\}$ and let $T_3 = \{s \mid f(s) = 0\}$. Then the sets T_i are disjoint ($i=1, 2, 3$) and $\mu_1+\mu_2+\mu_3=\mu_T$ where $\mu_i=\mu(T_i)$.

Because of assumption (A),

$$\int_{T_1} f_1 d\mu = \int_{T_2} f_2 d\mu = I(f).$$

Because of (A) and (B), f is bounded. In fact, if $M_i = \sup_{s\in T} f_i(s)$, ($i=1, 2$), then $M_1M_2 \leqq c$.

We first maximize $I(f)$. Since $I(f) \leqq M_i\mu_i$, ($i=1, 2$),

$$I(f) \leqq \min \{M_1\mu_1, M_2\mu_2\} = \min \{M_1\mu_1, M_2(\mu_T - \mu_1 - \mu_3)\}.$$

The function of μ_1 on the right above has a unique maximum which occurs when $M_1\mu_1 = M_2\mu_2$. One easily obtains

$$I(f) \leqq \max_{\mu_1} \min \{M_1\mu_1, M_2\mu_2\} = \frac{M_1M_2}{M_1+M_2}(\mu_T - \mu_3). \tag{2}$$

Now

$$\int_T f^2 d\mu = \int_{T_1} f_1^2 d\mu + \int_{T_2} f_2^2 d\mu \leqq M_1 \int_{T_1} f_1 d\mu + M_2 \int_{T_2} f_2 d\mu = (M_1 + M_2)I(f),$$

and it follows from (2) that

$$\int_T f^2 d\mu \leqq M_1M_2(\mu_T - \mu_3) \leqq c\mu_T. \tag{3}$$

This establishes assertion (C). Furthermore, both equalities hold in (3) if and only if the conditions

$$f(s) \equiv \begin{cases} M_1, & s \in T_1, \\ -M_2, & s \in T_2, \end{cases} \tag{4}$$

$$M_1\mu_1 = M_2\mu_2, \qquad M_1M_2 = c, \tag{5}$$

$$\mu_3 = 0, \tag{6}$$

are satisfied simultaneously. (6) shows that T_1 and T_2 must be complementary subsets of T. From (5),

$$M_1 = (M_1M_2)^{1/2}\left(\frac{M_1}{M_2}\right)^{1/2} = \left(\frac{c\mu_2}{\mu_1}\right)^{1/2};$$

$$M_2 = \left(\frac{c\mu_1}{\mu_2}\right)^{1/2}.$$

Assertion (D) now follows from (4).

4. **Proof of Theorem** 1. Let $\{f_n(s)\}_{n=0}^{\infty}$ be a Haar system on (S, μ). We shall prove by induction that for $n \geqq 0$,

$$D_{n+1}(s, t) = \begin{cases} \dfrac{1}{\mu_{n,j}}, & (s, t) \in S_{n,j} \times S_{n,j}, \ (1 \leqq j \leqq n+1), \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

This is true for $n=0$ since $f_0(s)\equiv 1$, $D_1(s, t)\equiv 1$, and $\mu_{0,1}=\mu(S)=1$. In general

$$D_{n+2}(s, t) = D_{n+1}(s, t) + f_{n+1}(s)f_{n+1}(t). \tag{8}$$

By definition $f_{n+1}(s)\equiv 0$ outside $S_{n,k}$. Hence $D_{n+2}(s, t)=D_{n+1}(s, t)$ outside $S_{n,k}^2=S_{n,k}\times S_{n,k}$. From the induction hypothesis (7) this means

$$\text{if } (s, t) \notin S_{n,k}^2, \ D_{n+2}(s, t) = \begin{cases} \dfrac{1}{\mu_{n+1,j}}, & (s, t) \in S_{n+1,j}^2, \ (3 \leqq j \leqq n+2), \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Now consider $(s, t)\in S_{n,k}^2=(S_{n+1,1}\cup S_{n+1,2})^2$. From the definition (1),

$$f_{n+1}(s)f_{n+1}(t) = \begin{cases} \dfrac{\mu_{n+1,2}}{\mu_{n+1,1}\mu_{n,k}}, & (s, t) \in S_{n+1,1}^2, \\ \dfrac{\mu_{n+1,1}}{\mu_{n+1,2}\mu_{n,k}}, & (s, t) \in S_{n+1,2}^2, \\ -\dfrac{1}{\mu_{n,k}}, & \text{otherwise}. \end{cases} \tag{10}$$

But by the induction hypothesis and (8),

$$D_{n+2}(s, t) = f_{n+1}(s)f_{n+1}(t) + \frac{1}{\mu_{n,k}} \tag{11}$$

when $(s, t)\in S_{n,k}^2$. From (10) and (11), we easily find that

$$\text{(12)} \qquad \text{if } (s,t) \in S^2_{n,k},\ D_{n+2}(s,t) = \begin{cases} \dfrac{1}{\mu_{n+1,1}}, & (s,t) \in S^2_{n+1,1}, \\ \dfrac{1}{\mu_{n+1,2}}, & (s,t) \in S^2_{n+1,2}, \\ 0, & \text{otherwise.} \end{cases}$$

Statements (9) and (12) together are equivalent to equation (7) with $n+1$ replaced by $n+2$, completing the induction.

Suppose now that $H=\{f_n(s)\}_{n=0}^{\infty}$ is an orthonormal set in $L^2(S, \mu)$, $f_0(s)\equiv 1$, and the associated Dirichlet kernels are non-negative. We shall prove by induction that for each $n\geqq 0$ there is a sequence of partitions $P_0\geqq P_1\geqq P_2\geqq \cdots \geqq P_n$ satisfying (i) and (ii), that $f_0(s), f_1(s), \cdots, f_n(s)$ are the associated Haar functions, and that (7) holds.

For $n=0$, these assertions are obvious. Assume they are true for n. $D_{n+2}(s, t)\geqq 0$. Therefore, from (8) and the induction hypothesis

$$\text{(13)} \qquad f_{n+1}(s)f_{n+1}(t) \geqq -D_{n+1}(s,t) = \begin{cases} -\dfrac{1}{\mu_{n,j}}, & (s,t) \in S^2_{n,j},\ (1 \leqq j \leqq n+1), \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $f_{n+1}(s)\equiv 0$ on all but one of the sets $S_{n,j}$ $(1\leqq j\leqq n+1)$. Since $f_{n+1}(s)\not\equiv 0$ on S, we can find a set $S_{n,k}$ on which $f_{n+1}(s)\not\equiv 0$. Suppose $f_{n+1}(s)\not\equiv 0$ outside $S_{n,k}$. Then there are points $s\in S_{n,k}$ and points $t\notin S_{n,k}$ where the function is nonzero. For any pair (s, t) of such points, we have from (13) that $f_{n+1}(s)f_{n+1}(t)>0$. It follows that either $f_{n+1}(s)\geqq 0$ or $f_{n+1}(s)\leqq 0$ for all $s\in S$. But this is impossible since

$$\int_S f_{n+1}d\mu = \int_S f_0 f_{n+1} d\mu = 0.$$

Therefore, $f_{n+1}(s)\equiv 0$ outside $S_{n,k}$.

Consider now the restriction of f_{n+1} to the subset $S_{n,k}$. The following three statements hold.

$$\text{(A)} \qquad \int_{S_{n,k}} f_{n+1} d\mu = 0.$$

$$\text{(B)} \qquad f_{n+1}(s)f_{n+1}(t) \geqq -\frac{1}{\mu_{n,k}}\cdot$$

$$\text{(C)} \qquad \int_{S_{n,k}} f^2_{n+1} d\mu = 1.$$

(A) and (C) express orthonormality; (B) is a restatement of part of (13). Now (A) and (B) are precisely the conditions of the lemma with $\mu_T=\mu_{n,k}$ and $c=1/\mu_{n,k}$. We conclude

$$\int_{S_{n,k}} f_{n+1}^2 d\mu \leqq c\mu_T = 1.$$

But now (C) shows that f_{n+1} is maximal in the sense of the lemma. Therefore, there exist two complementary subsets $S_{n+1,1}$ and $S_{n+1,2}$ of $S_{n,k}$ having measures $\mu_{n+1,1}$ and $\mu_{n+1,2}$ such that

$$f_{n+1}(s) = \begin{cases} \left(\dfrac{\mu_{n+1,2}}{\mu_{n+1,1}\mu_{n,k}}\right)^{1/2}, & s \in S_{n+1,1}, \\ -\left(\dfrac{\mu_{n+1,1}}{\mu_{n+1,2}\mu_{n,k}}\right)^{1/2}, & s \in S_{n+1,2}, \\ 0, & \text{otherwise.} \end{cases}$$

For $3 \leqq j \leqq n+2$, define sets $S_{n+1,j}$ to be the sets $S_{n,i}$ $(1 \leqq i \leqq n+1,\ i \neq k)$ taken in any order. Clearly $P_{n+1} = \{S_{n+1,j}\}_{j=1}^{n+2}$ is a partition of (S, μ), $P_0 \geqq P_1 \geqq \cdots \geqq P_n \geqq P_{n+1}$ is an extension of the sequence of partitions already defined, $f_{n+1}(s)$ is the Haar function associated with P_{n+1}, and (7) follows for $n+2$ exactly as in the first part of the proof. This completes the proof by induction that H is a Haar system on (S, μ).

5. **An extension of Theorem** 1. We now drop the assumption that (S, μ) is totally finite and weaken the restriction that $f_0(s) \equiv 1$.

THEOREM 2. *Let $\{f_n(s)\}_{n=0}^{\infty}$ be a real orthonormal set in $L^2(S, \mu)$. The associated Dirichlet kernels are non-negative if and only if*

(A) $f_0(s)$ *is either non-negative or nonpositive*

and

(B) *for each $n \geqq 0$, $f_n(s) = f_0(s)\phi_n(s)$ where $\{\phi_n(s)\}_{n=0}^{\infty}$ is a Haar system on (S, ν) and $d\nu = f_0^2(s)d\mu$.*

Proof. To prove sufficiency of these conditions, note that (S, ν) is totally finite and $\nu(S) = 1$. Therefore, by Theorem 1,

$$\sum_{j=0}^{n-1} \phi_j(s)\phi_j(t) \geqq 0.$$

But

$$D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t) = f_0(s)f_0(t) \sum_{j=0}^{n-1} \phi_j(s)\phi_j(t).$$

Since $f_0(s)f_0(t) \geqq 0$ from condition (A), $D_n(s, t) \geqq 0$.

Next, we show necessity. Since $f_0(s)f_0(t) = D_1(s, t) \geqq 0$ condition (A) holds.

Let $Z = \{s \mid f_0(s) = 0\}$. We shall prove by induction that $f_n(s) \equiv 0$ on Z for $n \geqq 0$. Suppose this is true for $0, 1, 2, \cdots, n$.

$$0 \leqq D_{n+2}(s, t) = f_{n+1}(s)f_{n+1}(t) + \sum_{j=0}^{n} f_j(s)f_j(t).$$

If $s \in Z$, the second term on the right vanishes because of the induction hypothesis, so that $f_{n+1}(s)f_{n+1}(t) \geqq 0$. Suppose $f_{n+1}(s) \not\equiv 0$ on Z. Let $s \in Z$ and $t \in S$ be points where the function is nonzero. Then $f_{n+1}(s)f_{n+1}(t) > 0$. It follows that either $f_{n+1}(s) \geqq 0$ or $f_{n+1}(s) \leqq 0$ for all $s \in S$. But this is impossible because then either $f_0(s)f_{n+1}(s) \geqq 0$ or $f_0(s)f_{n+1}(s) \leqq 0$ for all s contradicting the fact that

$$\int_S f_0 f_{n+1} d\mu = 0.$$

Therefore $f_{n+1}(s) \equiv 0$ on Z.

We may now define

$$\phi_n(s) = \begin{cases} \dfrac{f_n(s)}{f_0(s)}, & s \notin Z, \\ 0, & s \in Z. \end{cases}$$

Let $d\nu = f_0^2(s)d\mu$. Then the set $\{\phi_n(s)\}_{n=0}^{\infty}$ is an orthonormal set in $L^2(S, \nu)$ for

$$\int_S \phi_j \phi_k d\nu = \int_{S-Z} \frac{f_j f_k}{f_0^2} f_0^2 d\mu = \int_S f_j f_k d\mu = \delta_{jk}$$

and

$$\int_S \phi_j^2 d\nu = \int_{S-Z} \left(\frac{f_j}{f_0}\right)^2 f_0^2 d\mu = \int_S f_j^2 d\mu = 1.$$

Furthermore, the Dirichlet kernels associated with this set are non-negative since

$$\sum_{j=0}^{n-1} \phi_j(s)\phi_j(t) = \begin{cases} 0, & s \in Z \text{ or } t \in Z, \\ \dfrac{1}{f_0(s)f_0(t)} \displaystyle\sum_{j=1}^{n-1} f_j(s)f_j(t) = \dfrac{D_n(s, t)}{D_1(s, t)} \geqq 0, & \text{otherwise.} \end{cases}$$

By Theorem 1, $\{\phi_n(s)\}_{n=0}^{\infty}$ is a Haar system on (S, ν). Since $f_n(s) = f_0(s)\phi_n(s)$ for $n \geqq 0$, the theorem is proved.

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