PROJECTION CONSTANTS[1]

BY

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1. There are two main problems connected with the notion of projections in Banach spaces. The first, which grew out of attempts to generalize results known to hold in Hilbert space, is the following: For a subspace X of a given Banach space Y, does there exist a (continuous) projection of Y onto X? Moreover, under what conditions is it possible to choose for each $X \subset Y$ a projection $P_X: Y \rightarrow X$ such that the norms $\|P_X\|$ are uniformly bounded for all $X \subset Y$? For results in this direction see, in particular, Kakutani [6], Murray [8], and Sobczyk [11].

The second problem is the following: Given a Banach space X, under what conditions does there exist a projection onto X of any space $Y \supset X$? Moreover, what may be said on the norms of such projections? An important concept in this connection is that of the class $\mathfrak{P}_\lambda$, for a fixed $\lambda \geqq 1$: A Banach space X belongs to $\mathfrak{P}_\lambda$ if, whenever X is imbedded in a space Y, there exists a projection of Y onto X of norm less than or equal to λ. Of particular interest is class $\mathfrak{P}_1$ which has been completely characterized (see Day [4] for a summary of known results on the classes $\mathfrak{P}_\lambda$ and for references to original papers), but little of a positive nature is known about $\mathfrak{P}_\lambda$ for $\lambda > 1$.

In the hope that more precise knowledge of properties of Minkowski spaces (i.e. Banach spaces of finite dimensions) will be helpful in solving problems pertaining to projections in infinite dimensional Banach spaces, we report in the present paper some results on projections onto Minkowski spaces. We find it convenient to define, for any Banach space X, the *projection constant* $\mathcal{P}(X)$ as the greatest lower bound of the numbers λ such that $X \in \mathfrak{P}_\lambda$. If, moreover $X \in \mathfrak{P}_{\mathcal{P}(X)}$, we say that $\mathcal{P}(X)$ is exact.

It is well known [4] that any Banach space may be imbedded in a member of $\mathfrak{P}_1$, and that each $X \in \mathfrak{P}_1$ has the following extension property: Given any Banach spaces Y and $Z \supset Y$, and any linear transformation f from Y to X, there exists a linear transformation F from Z to X coinciding on Y with f such that $\|F\| = \|f\|$.

These facts imply immediately that *we can find the projection constant* $\mathcal{P}(X)$ *of a Banach space* X *by taking a member* Y *of* $\mathfrak{P}_1$ *which contains* X *and taking the greatest lower bound* (*or minimum, if exact*) *of the norms of all projec-*

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tions from Y *onto* X. In some cases (see, e.g. Theorems 1 and 3) this characterization allows the determination of $\mathcal{P}(X)$ in a direct way.

In the present paper we shall determine the projection constants for some particularly simple Minkowski spaces and obtain bounds for other Minkowski spaces. Our results are in a certain sense complementary to those of Murray [8] and Sobczyk [11].

2. **Notation and statement of results.** Throughout the paper the following notation will be used:

n: a (fixed) natural number, $N=2^{n-1}$;

E^n: n-dimensional Euclidean space;

m^n, l^n: n-dimensional Minkowski space with points $x=(x_1, \cdots, x_n)$ and norm $\|x\|=\max_{1\leq i\leq n}|x_i|$ resp. $\|x\|=\sum_{i=1}^n|x_i|$.

R_n: Minkowski plane whose unit cell is an affine-regular 2^n-gon (i.e. the transform of a regular 2^n-gon under a nonsingular affine map).

M^n: any n-dimensional Minkowski space.

The main results of the present paper are:

THEOREM 1. $\mathcal{P}(R_n)=2^{2-n}\operatorname{ctg} 2^{-n}\pi$ *and is exact.*

THEOREM 2. $\mathcal{P}(E^2)=4/\pi$.

THEOREM 3.

$$\mathcal{P}(l^n) = 2^{1-n}n\, C_{n-1,\,(n-1)/2}$$

and is exact.

THEOREM 4.

$$\mathcal{P}(E^n) \leq \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}.$$

THEOREM 5.

$$\mathcal{P}(M^n) \leq (2/\pi)^{1/2}n + O(n^{-1});$$
$$\mathcal{P}(M^2) < 3/2.$$

The following sections contain the proofs of the above results and some additional remarks.

3. **Proof of Theorems** 1 **and** 2. The proof of Theorem 1 will be divided into three parts:

(i) Applying an idea of Naumann [9] we shall represent R_n as a (2-dimensional) subspace of the $N=2^{n-1}$ dimensional Minkowski space m^N.

(ii) Next we shall exhibit a projection $P: m^N \to R_n$ such that

$$\|P\| = \frac{2}{N} \operatorname{ctg} \frac{\pi}{2N} \cdot$$

(iii) We shall show that any projection $P^*: m^N \to R_n$ satisfies $\|P^*\| \geqq \|P\|$. Since $m^N \in \mathfrak{P}_1$, this will (according to the alternate definition of $\mathcal{P}(X)$) prove Theorem 1.

In order to establish (i), let the points of R_n be given by (x_0, x_1), where the system of coordinates is chosen in such a fashion that G_n, the unit cell of R_n, has the representation

$$G_n = \left\{ (x_0, x_1);\ \max_{0 \leqq i \leqq N-1} \left| x_0 \cos 2i\phi + x_1 \sin 2i\phi \right| \leqq 1 \right\}, \tag{3.1}$$

with $\phi = \pi/2N$. The vertices A_j, $0 \leqq j \leqq N-1$, of G_n are then given by

$$A_j = \left(\frac{\cos (2j+1)\phi}{\cos \phi};\ \frac{\sin (2j+1)\phi}{\cos \phi} \right). \tag{3.2}$$

Let $D = (d_{ij})_{i,j=0}^{N-1}$ be an N by N matrix with elements

$$d_{ij} = \begin{cases} \cos 2i(j+1)\phi, & \text{for } 0 \leqq i \leqq N-1 \text{ and } j = 2k,\ 0 \leqq k \leqq N/2 - 1, \\ \sin 2ij\phi, & \text{for } 0 \leqq i \leqq N-1 \text{ and } j = 2k+1,\ 0 \leqq k \leqq N/2 - 1. \end{cases}$$

It is easily verified that $DD' = (1/2)NI_N$, where D' denotes the transpose of D and I_N is the N by N unit matrix. Therefore, in the N-dimensional vector space with points $(x_0, \cdots, x_{N-1})$, a hypercube H^N is given by

$$H^N = \left\{ (x_0, \cdots, x_{N-1});\ \max_{0 \leqq i \leqq N-1} \left| \sum_{j=0}^{N-1} d_{ij} x_j \right| \leqq 1 \right\}. \tag{3.3}$$

Taking H^N as the unit cell, we obtain m^N. It is obvious that R_n is the subspace of m^N consisting of points $(x_0, \cdots, x_{N-1})$ for which

$$x_i = 0, \qquad 2 \leqq i \leqq N-1.$$

To establish (ii), let $P: m^N \to R_n$ be the projection defined by

$$P(x_0, \cdots, x_{N-1}) = (x_0, x_1, 0, 0, \cdots, 0) = (x_0, x_1).$$

In other words, denoting by e_i the unit vector in the direction of the positive x_i axis, $e_i = (\delta_i^0, \cdots, \delta_i^{N-1})$, we have

$$P(e_i) = \begin{cases} e_i, & \text{for } i = 0, 1, \\ 0, & \text{for } 2 \leqq i \leqq N-1. \end{cases}$$

In order to compute the norm $\|P\|$ of P it is obviously sufficient to find

$$\max\{\|P(x)\|;\ x \in m^N,\ \|x\| \leqq 1\},$$

which, in turn, equals

$$\max\{\|P(V)\|;\ V \text{ a vertex of } H^N\}.$$

Now, it follows from (3.3) that any vertex V of H^N is representable in the form

$$V = (v_0, v_1, \cdots, v_{N-1}) = \frac{2}{N}(\epsilon_0, \epsilon_1, \cdots, \epsilon_{N-1})\cdot D,$$

where each ϵ_i is either $+1$ or -1. In particular

$$(3.4)\qquad \begin{aligned} v_0 &= \frac{2}{N}\sum_{i=0}^{N-1} \epsilon_i \cos 2i\phi, \\ v_1 &= \frac{2}{N}\sum_{i=0}^{N-1} \epsilon_i \sin 2i\phi. \end{aligned}$$

We are especially interested in those vertices $V^{(k)}$, $0 \leqq k \leqq 2N-1$, for which the corresponding $\epsilon_i^{(k)}$ satisfy

$$\epsilon_i^{(k)} = \begin{cases} +1 & \text{for } i \geqq k \\ -1 & \text{for } i < k \end{cases} \qquad \text{if } 0 \leqq k \leqq N-1,$$

$$\epsilon_i^{(k)} = \begin{cases} +1 & \text{for } i < k-N \\ -1 & \text{for } i \geqq k-N \end{cases} \qquad \text{if } N \leqq k \leqq 2N-1.$$

It is obvious from (3.4) that $\|P(V)\|$ is maximal for these and only these vertices.

Now, as is easily computed,

$$(3.5)\qquad \begin{aligned} v_{2i}^{(k)} &= \frac{2}{N}\cdot\frac{\sin(1-2k)(2i+1)\phi}{\sin(2i+1)\phi}, \\ v_{2i+1}^{(k)} &= \frac{2}{N}\cdot\frac{\cos(1-2k)(2i+1)\phi}{\sin(2i+1)\phi}, \end{aligned}$$

for $0 \leqq i \leqq N/2-1$, $0 \leqq k \leqq 2N-1$.

Therefore, assuming the convention $A_{k+2N} = A_k$ for the vertices of G_n, we see, on comparing (3.5) and (3.2), that

$$(3.6)\qquad P(V^{(k)}) = \frac{2}{N}\operatorname{ctg}\phi\cdot A_{k+N/2-1}, \quad \text{for } 0 \leqq k \leqq 2N-1,$$

and thus $\max\|P(V)\| = \|P(V^{(k)})\| = \|P\| = (2/N)\operatorname{ctg}\phi = (2/N)\operatorname{ctg}(\pi/2N)$ as claimed.

(iii) Let us assume that there exists a projection $P^*: m^N \to R_n$ such that

$$\|P^*\| < \|P\|. \tag{3.7}$$

We shall reach a contradiction in the following way:

Let P^* be represented by

$$\begin{aligned} P^*(e_{2j}) &= (a_{2j} \sin (2j+1)\phi;\, b_{2j} \sin (2j+1)\phi) \in R_n, \\ P^*(e_{2j+1}) &= (a_{2j+1} \sin (2j+1)\phi;\, b_{2j+1} \sin (2j+1)\phi) \in R_n \end{aligned} \tag{3.8}$$

for $0 \leqq j \leqq N/2-1$, where obviously (since P^* is a projection onto R_n)

$$a_0 = b_1 = 1/\sin\phi; \qquad a_1 = b_0 = 0. \tag{3.9}$$

By (3.7) we have

$$\|P^*(V^{(k)})\| < \frac{2}{N} \operatorname{ctg}\phi, \qquad \text{for } 0 \leqq k \leqq 2N-1,$$

and therefore, using (3.8) and (3.1), it follows that

$$\begin{aligned} &\left(\sum_{i=0}^{N/2-1} [\overset{(k)}{v_{2i}} a_{2i} + \overset{(k)}{v_{2i+1}} a_{2i+1}] \sin (2i+1)\phi\right) \cos (N+2k)\phi \\ &\qquad + \left(\sum_{i=0}^{N/2-1} [\overset{(k)}{v_{2i}} b_{2i} + \overset{(k)}{v_{2i+1}} b_{2i+1}] \sin (2i+1)\phi\right) \cos (N+2k)\phi \\ &\qquad < \frac{2}{N} \operatorname{ctg}\phi \qquad\qquad \text{for each } k,\ 0 \leqq k \leqq 2N-1. \end{aligned}$$

Taking into account (3.5) and (3.9) we obtain (after elementary simplifications),

$$\begin{aligned} &-\left(\sum_{i=1}^{N/2-1} [a_{2i} \sin (1-2k)(2i+1)\phi + a_{2i+1} \cos (1-2k)(2i+1)\phi]\right) \sin 2k\phi \\ &+\left(\sum_{i=1}^{N/2-1} [b_{2i} \sin (1-2k)(2i+1)\phi + b_{2i+1} \cos (1-2k)(2i+1)\phi]\right) \cos 2k\phi < 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leqq k \leqq 2N-1. \end{aligned}$$

Adding these inequalities for all values of k we obtain

$$\sum_{j=2}^{N-1} (g_j a_j + h_j b_j) < 0 \tag{3.10}$$

where

$$\begin{aligned} g_{2j} &= \sum_{i=0}^{2N-1} \sin 2i\phi \cdot \sin (2j+1)(2i-1)\phi, \\ g_{2j+1} &= \sum_{i=0}^{2N-1} \sin 2i\phi \cdot \cos (2j+1)(2i-1)\phi, \end{aligned}$$

$$h_{2j} = \sum_{i=0}^{2N-1} \cos 2i\phi \cdot \sin (2j+1)(2i-1)\phi,$$

$$h_{2j+1} = \sum_{i=0}^{2N-1} \cos 2i\phi \cdot \cos (2j+1)(2i-1)\phi,$$

$$\text{for } 1 \leqq j \leqq N/2 - 1.$$

But, on evaluating the above sums, we find that $g_j = h_j = 0$ for all j; therefore (3.10) reduces to $0 < 0$. This contradiction proves assertion (iii) and thus also Theorem 1.

REMARKS. (i) It is easily seen that for $n > 2$ there exist projections $P^*: m^N \to R_n$ different from P but satisfying $\|P^*\| = \|P\|$.

(ii) Applying arguments of the same nature as those used in the above proof the projection constant of any Minkowski plane whose unit cell is a polygon may be determined. But even in the case of a regular k-gon, if $k \neq 2^n$, the relation (3.6) fails and with it the estimate of the projection constant given in Theorem 1. Thus, for $k = 6$ the projection constant equals $4/3$.

Theorem 2 follows easily from Theorem 1. In order to establish it we need the following lemma, whose obvious proof we omit:

LEMMA. *Let X be a linear space in which two norms $\| \ \|_1$ and $\| \ \|_2$ are defined; let X_1 and X_2 denote the corresponding normed spaces. Then*

$$\|x\|_1 \leqq \|x\|_2 \leqq \mu \cdot \|x\|_1, \qquad \textit{for all } x \in X,$$

implies

$$\frac{1}{\mu} \mathcal{P}(X_1) \leqq \mathcal{P}(X_2) \leqq \mu \mathcal{P}(X_1).$$

Now, let E^2 be the Euclidean plane with norm $\| \ \|$ and unit cell S, and let $\| \ \|_n$ denote another norm in the plane, according to which the unit cell is a regular 2^n-gon circumscribed to S. It is evident that

$$\|x\|_n \leqq \|x\| \leqq \frac{1}{\cos (\pi/2^n)} \|x\|_n, \qquad \text{for all } x \in E^2.$$

Therefore, by the lemma and Theorem 1, we have

$$2^{2-n} \cos \frac{\pi}{2^n} \operatorname{ctg} \frac{\pi}{2^n} \leqq \mathcal{P}(E^2) \leqq 2^{2-n}/\sin \frac{\pi}{2^n}$$

for any $n \geqq 2$. Since both estimates of $\mathcal{P}(E^2)$ tend, as $n \to \infty$, to $4/\pi$, this proves Theorem 2.

4. **Proof of Theorem 3.** The proof of Theorem 3 parallels closely that of Theorem 1, although in the present case the technical details are more com-

plicated. The proof is again divided into the following three parts, which are sufficient in view of the alternate definition of $\mathcal{P}(X)$.

(i) We represent l^n as a subspace of m^N;

(ii) A projection $P: m^N \to l^n$ is defined such that

$$\|P\| = \frac{n}{N} C_{n-1,\,(n-1)/2};$$

(iii) We show that any projection $P^*: m^N \to l^n$ satisfies $\|P^*\| \geqq \|P\|$.

In order to find the imbedding of l^n in m^N we need, let us assume that m^N is given in the usual representation, according to which the unit cell H^N is given by

$$H^N = \left\{(x_0, \cdots, x_{N-1});\ \max_{0 \leqq i \leqq N-1} |x_i| \leqq 1\right\}.$$

Let $\{e_i\}_{i=0}^{N-1}$ denote the usual basis of m^N, given by $e_i = (\delta_i^0, \cdots, \delta_i^{N-1})$. The vertices V of H^N are obviously of the form $V = (\epsilon_0, \cdots, \epsilon_{N-1}) = \sum_{i=0}^{N-1} \epsilon_i e_i$, where each ϵ_i is either $+1$ or -1. We shall need another basis for m^N, which we proceed to define.

Let A_2 denote the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let A_k, $k>2$, be the matrix obtained from A_{k-1} by substituting A_2 for $+1$ and $-A_2$ for -1. Thus, e.g.,

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Obviously A_{n+2} is an N by N matrix with elements ± 1. It is well known (see, e.g., Sobczyk [11] and the references given there) and easily proved that $A_n = A_n'$ and

$$A_n A_n' = N \cdot I_N. \tag{4.1}$$

Let the elements of A_n, $n>0$, be denoted by a_{ij}, $0 \leqq i, j \leqq N-1$. Since A_n is nonsingular, the vectors $\{b_i\}$ (which correspond to some of the vertices of H^N) given by

$$b_i = \sum_{j=0}^{N-1} a_{ij} e_j, \qquad 0 \leqq i \leqq N-1$$

form a basis for m^N. (In the sequel we shall use the notation $\sum_{j<N}$ instead of $\sum_{j=0}^{N-1}$.)

Let I denote the set of n integers $\{0, 1, 2, 4, 8, \cdots, 2^{n-2}\}$. From the definition of the matrices A_n it is immediate that the components of b_0 according to the base $\{e_j\}$ are all $+1$, while for $i \in I$, $i \neq 0$, the components of b_i are i $+1$'s alternating with the same number of -1's. Therefore (see the more detailed discussion in the third part of the proof; see also Sobczyk [10, p. 940], we have

$$\left\| \sum_{i\in I} x_i b_i \right\| = \sum_{i \in I} | x_i | . \tag{4.2}$$

This relation implies that the n-dimensional subspace of m^N spanned by the vectors $\{b_i, i \in I\}$ is the space l^n; in the sequel we shall use this (and only this) imbedding of l^n in m^N.

We define now the projection $P: m^N \rightarrow l^n$ by

$$P(b_i) = \begin{cases} b_i & i \in I, \\ 0 & i \notin I. \end{cases}$$

As in §3, $\|P\| = \max\{\|P(V)\|;\ V \text{ a vertex of } H^N\}$. By (4.1), obviously

$$\sum_{i<N} a_{ij} b_i = \sum_{i<N} a_{ji} b_i = N e_j$$

and therefore, for any vertex $V = \sum_{i<N} \epsilon_i e_i$ $(\epsilon_i = \pm 1)$ we have

$$V = \frac{1}{N} \sum_{j<N} \left(\epsilon_j \sum_{i<N} a_{ij} b_i \right) = \frac{1}{N} \sum_{i<N} \left(b_i \sum_{j<N} a_{ij} \epsilon_j \right).$$

From the definition of P it follows that

$$P(V) = \frac{1}{N} \sum_{i\in I} \left(b_i \sum_{j<N} a_{ij} \epsilon_j \right)$$

and thus, by (4.2),

$$\|P(V)\| = \frac{1}{N} \sum_{i\in I} \left| \sum_{j<N} a_{ij} \epsilon_j \right| . \tag{4.3}$$

We are interested in max $\|P(V)\|$. Since all the $(n-1)$-dimensional faces of the unit cell of l^n are equivalent, it is sufficient to determine max $\|P(V)\|$ for vertices V such that $P(V)$ belongs to the "first octant" of l^n, i.e. to $\{ \sum_{i\in I} x_i b_i;\ x_i \geqq 0 \text{ for all } i \in I\}$. For such vertices (4.3) reduces to

$$\|P(V)\| = \frac{1}{N} \sum_{j<N} \left(\sum_{i\in I} a_{ij} \right) \epsilon_j \tag{4.4}$$

which is maximal provided the ϵ_j's satisfy

$$\epsilon_j = \begin{cases} +1 & \text{if } \sum_{i\in I} a_{ij} \geqq 0, \\ -1 & \text{if } \sum_{i\in I} a_{ij} < 0. \end{cases} \tag{4.5}$$

Later, we shall have to determine explicitly those j for which the first, resp. second, inequality holds. For the determination of max $\|P(V)\|$, however, the following simple reasoning is sufficient.

Let A^* denote the n by N matrix with elements a_{ij}, $i\in I$, $0\leqq j\leqq N-1$. Then in the columns of A^* occur all the N combinations of $+1$'s and -1's which have $+1$ in the first position. Therefore there are $C_{n-1,k}$ different columns in which $+1$ occurs $n-k$ times and -1 occurs k times, for $0\leqq k \leqq n-1$. Therefore, if the ϵ_j's are determined according to (4.5), it follows from (4.4) that

$$\begin{aligned} \|P\| &= \frac{1}{N}\left\{ \sum_{k=0}^{[n/2]} (n-2k)\binom{n-1}{k} - \sum_{k>[n/2]} (n-2k)\binom{n-1}{k}\right\} \\ &= \frac{n}{N} C_{n-1,(n-1)/2}, \end{aligned}$$

as claimed. In order to establish the last equality the following elementary relations are used:

$$\sum_{k=0}^{m}\binom{2m}{k} = 2^{2m-1} + \frac{1}{2}\binom{2m}{m},$$

$$\sum_{k=0}^{m}\binom{2m+1}{k} = 2^{2m},$$

$$\sum_{k=1}^{m} k\binom{2m}{k} = m2^{2m-1},$$

$$\sum_{k=1}^{m} k\binom{2m+1}{k} = (2m+1)2^{2m-1} - \frac{m+1}{1}\binom{2m+1}{m}.$$

In the final part of the proof we shall need more precise information about the matrix A^* and about the vertices of H^N which maximize $\|P(V)\|$.

Let $\delta=(\delta_0, \delta_1, \delta_2, \delta_4, \cdots, \delta_{2^\nu}, \cdots, \delta_{2^{n-2}})$ denote a sequence of n numbers, each of which is either $+1$ or -1; let D be the set of all such sequences δ, and let δ^+ denote the sequence consisting only of $+1$'s. Then, for $x=\sum_{i\in I} x_i b_i \in l^n$ we have

$$\|x\| = \sum_{i\in I} |x_i| = \max_{\delta\in D} f^\delta(x),$$

where f^δ is the functional defined by

$$f^{\delta}(x) = \sum_{i \in I} \delta_i x_i.$$

On the other hand, let j_ν denote the coefficient of 2^ν in the binary representation $\sum_\nu j_\nu 2^\nu$ of j, so that $j_\nu = 0$ or 1. It is easily seen that the jth column of the n by N matrix $A^* = (a_{ij})$, $i \in I$, $0 \leqq j \leqq N-1$, consists of:

$+1$ in the row $i = 0$;

$+1$ (resp. -1) in the row $i = 2^k$ if $j_k = 0$ (resp. $j_k = 1$).

Therefore, when (on p. 458) we determined the maximum of

$$\|P(V)\| = \frac{1}{N} \sum_{i \in I} \left| \sum_{j<N} a_{ij} \epsilon_j \right|,$$

we took $\epsilon_j = -1$ for those j in whose binary representation more than $[n/2]$ of the j_ν's equal 1. With this determination of ϵ_j, let us denote

$$\overset{+}{V} = \sum_{j<N} \epsilon_j e_j = \sum_{k<N} \overset{+}{y_k} b_k.$$

We are interested in the vertices V^δ of H^N which maximize $\|P(V)\|$ in the other "octants" of l^n, i.e. for which

$$\|P\| = \|P(V^\delta)\| = f^\delta(P(V^\delta)). \tag{4.6}$$

In the sequel we shall deal only with those $\delta \in D$ for which $\delta_0 = +1$; we denote this set by D^*. For any $\delta \in D^*$ and any $i \notin I$ let $g(i, \delta)$ denote the number of different ν such that $\delta_{2^\nu} = 1$ while $i_\nu = 1$, where i_ν is the νth digit in the binary representation of i. In other words, $g(k, \delta) = 2^{-1} \sum_{\nu=0}^{n-2} k_\nu (1 - \delta_{2^\nu})$. As easily verified we have for each $i \in I$ and each $k \notin I$

$$\sum_{\delta \in D^*} (-1)^{g(k,\delta)} \delta_i = 0. \tag{4.7}$$

Now, putting

$$\overset{\delta}{y_i} = \begin{cases} (-1)^{g(i,\delta)} y_i^+ & \text{for } i \notin I, \\ y_i^+ & \text{for } i \in I, \end{cases}$$

and

$$\overset{\delta}{V} = \sum_{i<N} \overset{\delta}{y_i} \delta_i b_i, \qquad (\delta_i = 1 \text{ for } i \notin I), \tag{4.8}$$

it is immediate that (4.6) is satisfied.

We are now ready for the last step in the proof of Theorem 3, which parallels closely the corresponding step in the proof of Theorem 1.

Let us assume that there exists a projection $P^*: m^N \to l^n$ such that

$$\|P^*\| < \|P\|.$$

Let P^* be represented by

$$P^*(b_k) = \sum_{i \in I} \overset{k}{p}_i b_i, \qquad 0 \leqq k \leqq N-1$$

where for $k \in I$,

$$\overset{k}{p}_i = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases}$$

Then, since $\|P^*(V)\| < \|P\|$ for any vertex V of H^N we have, in particular,

$$f^\delta(P^*(V^\delta)) \leqq \|P^*(V^\delta)\| < \|P\| = \|P(V^\delta)\| = f^\delta(P(V^\delta))$$

for each $\delta \in D^*$, and thus by (4.8) and the linearity of f^δ

$$f^\delta(P^*(V^\delta) - P(V^\delta)) = f^\delta\left(P^*\left(\sum_{k \notin I} \overset{\delta}{y}_k b_k\right)\right) < 0.$$

Adding these inequalities for all $\delta \in D^*$ we obtain

$$\begin{aligned} 0 &> \sum_{\delta \in D^*} f^\delta\left(P^*\left(\sum_{k \notin I} \overset{\delta}{y}_k b_k\right)\right) \\ &= \sum_{\delta \in D^*} f^\delta\left(\sum_{k \notin I} \overset{\delta}{y}_k P^*(b_k)\right) = \sum_{i \in I} \sum_{k \notin I} \left(\sum_{\delta \in D^*} \delta_i \overset{\delta}{y}_k\right) \overset{k}{p}_i. \end{aligned}$$

Now, for any $i \in I$ and $k \notin I$ we have by the definition of $\overset{\delta}{y}_k$

$$\sum_{\delta \in D^*} \delta_i \overset{\delta}{y}_k = \overset{+}{y}_k \sum_{\delta \in D^*} \delta_i \cdot (-1)^{\sigma(k,\delta)}.$$

But, by (4.7), the last sum equals 0 and thus the contradiction $0 > 0$ is reached, which proves Theorem 3.

Remark. It is interesting to note that $\mathcal{P}(l^n)$ has the same value for $n = 2m-1$ and for $n = 2m$. On the other hand it is easy to show that if $n = 2m-1$ there is only one vertex of H^N maximizing $\|P(V)\|$ in each "octant" of l^n, while for $n = 2m$ there are

$$2^{\binom{2m-1}{m-1}}$$

different vertices in each "octant" maximizing $\|P(V)\|$.

5. **Proof of Theorem** 4. Let B^n denote the (solid) unit sphere of the n-dimensional Euclidean space E^n. Since B^n may be inscribed in a hypercube K^n in such a way that

$$n^{-1/2} K^n \subset B^n \subset K^n$$

it follows from the lemma of §3 that $\mathcal{P}(E^n) \leqq n^{1/2}$. We shall obtain the better estimate given in Theorem 4 by using the same idea but, following a suggestion of Professor A. Dvoretzky, averaging over all hypercubes K^n circumscribed about B^n.

Let S_n denote the boundary of B^n, and let $S(A)$ denote the boundary of the unit sphere of the subspace of E^n orthogonal to the span of the set $A \subset E^n$. Then each hypercube K^n circumscribed to B^n may be determined by $(n-1)$ points of contact $x_1, \cdots, x_{n-1}$ of S_n and K^n satisfying

$$(5.1) \qquad \begin{aligned} &x_1 \in S_n, \\ &x_2 \in S(x_1), \\ &\vdots \\ &x_{n-1} \in S(x_1, \cdots, x_{n-2}). \end{aligned}$$

Obviously, different sets of points $x_1, \cdots, x_{n-1}$ may determine the same K^n. The hypercube K^n determined by a set of points satisfying (5.1) shall be denoted by $K(x_1, \cdots, x_{n-1})$. We are interested in the "average" set

$$A_n = \frac{1}{s_2 \cdot s_3 \cdots s_n} \underbrace{\int\int \cdots \int}_{(x_1, \cdots, x_{n-1})} K(x_1, \cdots, x_{n-1}) d\sigma_2 \cdots d\sigma_n,$$

where s_k denotes the $(k-1)$-dimensional volume of S_k (thus $s_k = 2\pi^{k/2}/\Gamma(n/2)$), $d\sigma_j$ for $j<n$ denotes the element of volume of $S(x_1, \cdots, x_{n-j})$ and $d\sigma_n$ the element of volume of S_n; the integration being extended over all sets $(x_1, \cdots, x_{n-1})$ satisfying (5.1). (Integrals of this type may be defined either by reducing them to integrals of the support functions of the convex sets concerned (see, e.g., [2, pp. 28–29]), or, equivalently, by appropriate Riemann sums.)

By reasons of symmetry it is obvious that A_n is a sphere. Denoting its radius by r_n we shall establish that

$$(5.2) \qquad r_n = \frac{n}{\pi^{1/2}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

This statement being obvious for $n=1$, we proceed by induction. Substituting repeated integration for the multiple integral we have

$$A_{n+1} = r_{n+1} B^{n+1} = \frac{1}{s_{n+1}} \int_{S_{n+1}} \left\{ \frac{1}{s_2, \cdots, s_n} \int\int_{(x_1, \cdots, x_{n-1}) \in S(x_n);\, x_n \in S_{n+1}} \cdots \int K(x_1, \cdots, x_n) \cdot d\sigma_2, \cdots, d\sigma_n \right\} d\sigma_{n+1}.$$

The interior integral represents an orthogonal hypercylinder $C(x_n)$, whose bases are translates of $r_n S(x_n)$ and are tangent to S_{n+1} at x_n and $-x_n$. Thus $A_{n+1}=1/s_{n+1}\int_{S_{n+1}} C(x_n)d\sigma_{n+1}$.

Passing over to the support functions we have

$$r_{n+1} = \frac{1}{s_{n+1}} \int_{t_n=0}^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \int_{t_1=0}^{\pi/2} (\cos t_1 + r_n \sin t_1) \prod_{i=1}^{n} \sin^{n-i} t_i dt_i.$$

Since

$$\int_0^{\pi} \sin^{k-1} t dt = 2 \int_0^{\pi/2} \sin^{k-1} t dt = \pi^{1/2} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)},$$

there results

$$r_{n+1} = \frac{2}{n\pi^{1/2}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \left\{ 1 + r_n \pi^{1/2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right\},$$

and by using the inductive assumption on r_n, we obtain

$$r_{n+1} = \frac{n+1}{\pi^{1/2}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

which establishes (5.2).

Theorem 4 now follows immediately from the obvious remarks:

(i) If the unit cell B of a Banach space X satisfies

$$S \subset \sum_{i=1}^{k} a_i K_i \subset \mu S$$

where $a_i \geqq 0$, K_i is the unit cell of a Banach space X_i and $K_i \supset S$, then

$$\mathcal{P}(X) \leqq \mu \sum_{i=1}^{k} a_i \mathcal{P}(X_i).$$

(ii) A_n may be approximated with any desired degree of accuracy by finite sums of cubes $K(x_1, \cdots, x_{n-1})$ with appropriate weights a_i.

Thus $\mathcal{P}(E^n) \leqq r_n$ as asserted in Theorem 4.

REMARK. As it is easily shown, there exists a relationship between the numbers r_n and the projection constants of l^n spaces:

$$r_{2m} > r_{2m-1} = \mathcal{P}(l^{2m}) = \mathcal{P}(l^{2m-1}).$$

This would be especially remarkable if (as we conjecture) $\mathcal{P}(E^n) = r_n$ were true.

6. **Proof of Theorem** 5. Suppose K and S are two n-dimensional convex bodies in E^n, each having O as center of symmetry. Let $\mu(K, S)$ be the minimum of positive numbers μ having the following property: There exists an affine transform K^* of K such that

$$K^* \subset S \subset \mu K^*.$$

Obviously, $\mu(K, S) = \mu(S, K) \geqq 1$, and $\mu(K_1, K_2) \leqq \mu(K_1, K_3)\mu(K_3, K_2)$. The lemma of §3 may clearly be formulated in the following way: If K_i is the unit cell of the n-dimensional Minkowski space X_i, $i = 1, 2$, then

$$\mathcal{P}(X_1) \leqq \mu(K_1, K_2)\mathcal{P}(X_2).$$

Now, a result of John [5] may be stated as $\mu(K, B^n) \leqq n^{1/2}$, where K is any centrally symmetric n-dimensional convex body, and B^n is the unit cell of E^n. (This result may also be proved by considering the ellipsoid of minimal volume circumscribed to K.) Therefore, we have (see §5)

$$\mathcal{P}(M^n) \leqq n^{1/2}\mathcal{P}(E^n) \leqq \left(\frac{n^3}{\pi}\right)^{1/2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Applying well-known expansions of the gamma function there results $\mathcal{P}(M^n) \leqq (2/\pi)^{1/2}n + O(1/n)$.

On the other hand, Asplund [1] has recently proved that if K is any centrally symmetric convex body in the plane and S a square, then $\mu(K, S) \leqq 3/2$, with equality sign applying only if K is an affine-regular hexagon. Since the projection constant of a Minkowski plane whose unit cell is an affine-regular hexagon is $4/3$, this immediately establishes the second part of Theorem 5.

Remarks. (i) If H^n denotes a hypercube of n dimensions and K any n-dimensional centrally symmetric convex body, a theorem of Taylor [12] and Day [3] may be formulated as

$$\mu(K, H^n) \leqq n.$$

This immediately implies the estimate $\mathcal{P}(M^n) \leqq n$, slightly weaker than Theorem 5.

(ii) The determination of max $\{\mu(K, H^n), K \subset E^n\}$ seems to be extremely difficult, even in the three-dimensional case, where it possibly equals 2 (if K

is a regular octahedron, $\mu(K, H^3)=2$). Still more difficult seems to be the determination of max $\mathcal{P}(M^n)$. Probably max $\mathcal{P}(M^2)=4/3$.

(iii) There are many interesting questions connected with $\mu(K_1, K_2)$. E.g., it follows from John's theorem that max $\{\mu(K_1, K_2); K_1, K_2 \subset E^n\} \leqq n$, but the correct upper bound seems to be appreciably lower; even for $n=2$ no precise result is known. The analogy of these problems with those treated and raised by Levi [7] should be mentioned.

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